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COUPLED NONLINEAR DYNAMICS IN THE THREE-MODE INTEGRABLE SYSTEM ON A REGULAR CHAIN

The article suggests the nonlinear lattice system of three dynamical subsystems coupled both in their potential and kinetic parts. Due to its essentially multicomponent structure the system is capable to model nonlinear dynamical excitations on regular quasi-one-dimensional lattices of various physical origins. The system admits a clear Hamiltonian formulation with the standard Poisson structure. The alternative Lagrangian formulation of system's dynamics is also presented. The set of dynamical equations is integrable in the Lax sense, inasmuch as it possesses a zero-curvature representation. Though the relevant auxiliary linear problem involves a spectral third-order operator, we have managed to develop an appropriate two-fold Darboux–Bäcklund dressing technique allowing one to generate the nontrivial crop solution embracing all three coupled subsystems in a rather unusual way.

Keywords: nonlinear theories and models, anharmonic lattice modes, integrable systems, Lagrangian and Hamiltonian dynamics, Darboux–Bäcklund dressing method, symmetry and conservation laws, nonlinear wave packet.

1. Introduction

More than sixty five years ago, the famous work done by E. Fermi, J. Pasta, S. Ulam and M. Tsingou [1] had established that "A one-dimensional dynamical system of 64 particles with forces between neighbors containing nonlinear terms..." demonstrates "...very little, if any, tendency toward equipartition of energy among the degrees of freedom". This unexpected result (known as the Fermi–Pasta–Ulam paradox [2] but rightly should be referred to as the Fermi–Pasta–Ulam–Tsingou paradox) inspired an avalanche of investigations concerning dynamical and stochastic aspects of various physically motivated semidiscrete (*i.e.* continuous in time and discrete in spatial coordinate) systems characterized by the pronounced nonlinear interactions between their structural elements.

The most significant among such researches were the development of a one-dimensional lattice system with the exponential nonlinearity by M. Toda [3, 4] and the subsequent discovery of its complete integrability by S.V. Manakov [5] and H. Flaschka [6]. The correct choice of boundary conditions preserving the integrability of finite systems possessing integrable infinite counterparts has been [7] and permanently remains [8] the task of a considerable interest. The same words can be said about the problem of nontrivial interactions between the nonlinear wave packets of several distinct types [9] inevitably realizable in semidiscrete nonlinear integrable systems.

From the practical point of view, the low-dimensional semidiscrete integrable systems serve as the good approximations in modeling the propagation of soliton-like excitations through the imperfect lattices [10, 11], as well as in an adequate description of the Peierls–Nabarro potential relief stumbling the motion of narrow-size solitons in real regular lattices [12, 13]. Here, it is necessary to stress that the nonintegrable Davydov–Kyslukha model of solitary excitons in one-dimensional molecular chains [14–16] has been the main driving force in our efforts [17–19] to search for its appropriate integrable twin. Moreover, the concept of the Davydov–Kyslukha model is still remained the core of investigations dealing with the launching of solitons in protein α-helix spines [20]. Among other non-integrable semidiscrete

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nonlinear models waiting for their proper integrable analogs are the one-dimensional model of nonlinear compression pulses in granular media [21–23] and the model of nonlocal solitons in a nonlinear chain of atoms [24].

In view of numerous challenging problems briefly listed above, we decided to unravel some dynamical intricacies of the three-mode integrable system on a regular chain appearing as a new prospective reduction of the earlier suggested general integrable system associated with a third-order auxiliary spectral problem [25, 26]. The present article reports the most interesting results of this investigation.

2. Nonlinear Evolution Equations for the First Prototype Integrable System

In the light of our early articles [25, 26], the general form of a semidiscrete nonlinear system relevant to our present task is given by the set of nonlinear evolution equations

$$\dot{p}_{11}(n) = F_{12}(n)G_{21}(n-1) - F_{12}(n+1)G_{21}(n), \quad (2.1)$$

$$\dot{p}_{13}(n) = F_{12}(n)G_{23}(n-1) - F_{12}(n+1)G_{23}(n), \quad (2.2)$$

$$\dot{p}_{31}(n) = F_{32}(n)G_{21}(n-1) - F_{32}(n+1)G_{21}(n), \quad (2.3)$$

$$\dot{p}_{33}(n) = F_{32}(n)G_{23}(n-1) - F_{32}(n+1)G_{23}(n), \quad (2.4)$$

$$\dot{F}_{12}(n) = p_{11}(n)F_{12}(n) + p_{13}(n)F_{32}(n), \quad (2.5)$$

$$\dot{G}_{21}(n) = -G_{21}(n)p_{11}(n) - G_{23}(n)p_{31}(n), \quad (2.6)$$

$$\dot{G}_{23}(n) = -G_{21}(n)p_{13}(n) - G_{23}(n)p_{33}(n), \quad (2.7)$$

$$\dot{F}_{32}(n) = p_{31}(n)F_{12}(n) + p_{33}(n)F_{32}(n). \quad (2.8)$$

Here, the prototype field variables $p_{11}(n)$, $p_{13}(n)$, $p_{31}(n)$, $p_{33}(n)$ and $F_{12}(n)$, $G_{21}(n)$, $G_{23}(n)$, $F_{32}(n)$ are functions of the discrete spatial coordinate n (spanning the integers from minus to plus infinity) and the continuous time τ. The over-dot stands for the derivative with respect to time. System (2.1)–(2.8) serves as a prototype system for the variety of appropriately reduced dynamical systems, inasmuch only six its field variables are proved to be truly independent.

System (2.1)–(2.8) permits the zero-curvature representation

$$\dot{L}(n|z) = A(n+1|z)L(n|z) - L(n|z)A(n|z) \quad (2.9)$$

with the spectral $L(n|z)$ and evolution $A(n|z)$ operators specified by the matrices

$$L(n|z) = \\ = \begin{pmatrix} p_{11}(n) + \lambda(z) & F_{12}(n) & p_{13}(n) \\ G_{21}(n) & 0 & G_{23}(n) \\ p_{31}(n) & F_{32}(n) & p_{33}(n) + \lambda(z) \end{pmatrix}, \quad (2.10)$$

$$A(n|z) = \\ = \begin{pmatrix} 0 & -F_{12}(n) & 0 \\ -G_{21}(n-1) & \lambda(z) & -G_{23}(n-1) \\ 0 & -F_{32}(n) & 0 \end{pmatrix}. \quad (2.11)$$

Therefore, it acquires the status of a system integrable in the Lax sense [27, 28]. Here, both the functional spectral parameter $\lambda(z)$ and the rationalized spectral parameter z are assumed being time- and coordinate-independent. Due to the simplest admissible choice (2.11) of the evolution operator $A(n|z)$, system (2.1)–(2.8) should be referred to as the first prototype nonlinear integrable system in an infinite hierarchy generatible by the adopted spectral operator (2.10).

3. Generation of Local Conservation Laws

It is well known that any nonlinear integrable system on an infinite regular chain possesses the infinite number of local conservation laws. The most straightforward way to isolate some of them is based upon the universal local conservation law

$$\frac{d}{d\tau}\ln[\det L(n|z)] = \mathrm{Sp}\,A(n+1|z) - \mathrm{Sp}\,A(n|z) \quad (3.1)$$

appearing as a simple contraction of system's zero-curvature representation (2.9).

Inasmuch as the determinant $\det L(n|z)$ of the spectral matrix $L(n|z)$ depends on two distinct powers of the spectral parameter $\lambda(z)$, and the expression $\mathrm{Sp}A(n+1|z) - \mathrm{Sp}A(n|z)$ is equal to zero, the universal local conservation law (3.1) produces two unicellular conservation laws imposing two natural constraints onto the set of prototype field variables. The explicit record of natural constraints depends on a particular choice of boundary conditions for the prototype field variables and on an expected physical sense of reduced field variables. Assuming the underlying lattice being spatially uniform and pinned to the immovable frame of reference, we take the natural constraints in the following form:

$$F_{12}(n)G_{21}(n) + F_{32}(n)G_{23}(n) = -1, \tag{3.2}$$

$$p_{11}(n)F_{32}(n)G_{23}(n) - p_{31}(n)F_{12}(n)G_{23}(n) + \\ + p_{33}(n)F_{12}(n)G_{21}(n) - p_{13}(n)F_{32}(n)G_{21}(n) = 0. \tag{3.3}$$

These constraints (3.2) and (3.3) are consistent with the following boundary conditions for the prototype fields: $\lim_{n\to-\infty} F_{12}(n) \to F_{12}$, $\lim_{n\to-\infty} G_{21}(n) \to G_{21}$, $\lim_{n\to-\infty} G_{23}(n) \to G_{23}$, $\lim_{n\to-\infty} F_{32}(n) \to F_{32}$, $\lim_{n\to-\infty} p_{11}(n) \to 0$, $\lim_{n\to-\infty} p_{13}(n) \to 0$, $\lim_{n\to-\infty} p_{33}(n) \to 0$, $\lim_{n\to-\infty} p_{31}(n) \to 0$. As a consequence, the limiting eigenvalue problem

$$L(z)X(z) = X(z)\zeta(z) \tag{3.4}$$

(with $L(z) = \lim_{n\to-\infty} L(n|z)$) prescribes the functional relationship $\lambda(z) = z + 1/z$ (establishing a particular realization of the Zhukovskiy transformation [29, 30]) in view of very simple resulting expressions

$$\zeta_1(z) = z, \tag{3.5}$$

$$\zeta_2(z) = \lambda(z) \equiv z + 1/z, \tag{3.6}$$

$$\zeta_3(z) = 1/z \tag{3.7}$$

for the eigenvalues $\zeta_j(z)$.

The capability of the universal local conservation law (3.1) in generating system's local conservation laws is seen to be restricted only by two specimens.

In contrast, there exists the generalized procedure [18, 19, 31, 32] permitting to develop an infinite set of local conservation laws recursively without references to auxiliary spectral data. By definition, any local conservation law associated with some semidiscrete system given on an infinite quasi-one-dimensional lattice can be written in the form

$$\dot\rho(n) = J(n|n-1) - J(n+1|n), \tag{3.8}$$

where the quantities $\rho(n)$ and $J(n+1/2|n-1/2)$ are referred to as the local density and the local current, respectively. Bearing in mind this general definition (3.8), we must find the recursive presentation (*i.e.* presentation in powers of z or $1/z$) for the auxiliary quantities $\Gamma_{jk}(n|z)$ governed by the following set of spatial Riccati equations:

$$\Gamma_{jk}(n+1|z)\sum_{i=1}^{3} L_{ki}(n|z)\Gamma_{ik}(n|z) = \\ = \sum_{i=1}^{3} L_{ji}(n|z)\Gamma_{ik}(n|z) \tag{3.9}$$

with the restrictions

$$\Gamma_{ji}(n|z)\Gamma_{ik}(n|z) = \Gamma_{jk}(n|z) \tag{3.10}$$

being taken into account. The obtained series should be substituted into the collection of three ($j = 1, 2, 3$) generating equations

$$\frac{d}{d\tau} \ln M_{jj}(n|z) = B_{jj}(n+1|z) - B_{jj}(n|z). \tag{3.11}$$

Here, the composite functions

$$M_{jj}(n|z) = \sum_{i=1}^{3} L_{ji}(n|z)\Gamma_{ij}(n|z) \tag{3.12}$$

and

$$B_{jj}(n|z) = \sum_{i=1}^{3} A_{ji}(n|z)\Gamma_{ij}(n|z) \tag{3.13}$$

serve to generate the hierarchy of local densities and the hierarchy of local currents, respectively. In so doing, the quantities $L_{jk}(n|z)$ and $A_{jk}(n|z)$ denote the matrix elements of the spectral $L(n|z)$ and evolution $A(n|z)$ operators, respectively. Collecting terms with the same powers of the spectral parameter in each of three ($j = 1, 2, 3$) generating series (3.11), it is possible to recover any required number of local conservation laws from their infinite series.

The most productive is the second ($j = 2$) of the generating series (3.11). To develop the second generating series, it is sufficient to consider two auxiliary functions $\Gamma_{12}(n|z)$ and $\Gamma_{32}(n|z)$, since $\Gamma_{jj}(n|z) \equiv 1$ in view of properties (3.10). Due to the evident symmetry $\lambda(z) = \lambda(1/z)$ of the functional spectral parameter $\lambda(z)$, we restrict ourselves only to serial expansions at the center $|z| \to 0$ of a rationalized complex spectral plane and seek the auxiliary functions $\Gamma_{12}(n|z)$ and $\Gamma_{32}(n|z)$ in the following way:

$$\Gamma_{12}(n|z) = z\sum_{j=0}^{\infty} \gamma_{12}(n|m)z^m, \tag{3.14}$$

$$\Gamma_{32}(n|z) = z\sum_{j=0}^{\infty}\gamma_{32}(n|m)z^m. \qquad (3.15)$$

Then, for the generating function $\ln M_{22}(n|z)$ written up to the second power in the spectral parameter z, we obtain

$$\ln M_{22}(n|z) = \ln z - [p_{11}(n) + p_{33}(n)]z +$$
$$- [G_{21}(n)F_{12}(n+1) + G_{23}(n)F_{32}(n+1) + 1]z^2 +$$
$$+ [p_{11}^2(n)/2 + p_{11}^2(n)/2 + p_{13}(n)p_{31}(n)]z^2. \qquad (3.16)$$

By virtue of the second generating equation (*i.e.* Eq. (3.11) taken at $j = 2$), the quantities

$$p(n) = p_{11}(n) + p_{33}(n) \qquad (3.17)$$

and

$$h(n) = p_{11}^2(n)/2 + p_{11}^2(n)/2 + p_{13}(n)p_{31}(n) - 1 -$$
$$- G_{21}(n)F_{12}(n+1) - G_{23}(n)F_{32}(n+1) \qquad (3.18)$$

acquire the status of local densities.

The comprehensive analysis shows that the former (3.17) of these local densities should be identified with the density of system's momentum, whereas the latter one (3.18) – with the density of system's energy.

4. Lagrangian and Hamiltonian Formulations of the Reduced Three-Mode Nonlinear Integrable System

As we have already mentioned in Section 2, the natural constraints (3.2) and (3.3) are empowered to reduce eight prototype field variables to six actual ones. In so doing, the dynamics of the reduced system will be governed by three nonlinear Lagrange equations or, alternatively, by six nonlinear Hamilton equations. The key idea to introduce appropriate dynamical variables is to invent parametrization formulas converting both of the natural constraints (3.2) and (3.3) into identities. A particular realization of such parametrizations is not unique [25, 26]. Here, we consider the following one:

$$\sqrt{2}\ F_{12}(n) = +\exp[+q_-(n)]\sqrt{1+\mathrm{i}s(n)}, \qquad (4.1)$$

$$\sqrt{2}\ G_{21}(n) = -\exp[-q_-(n)]\sqrt{1+\mathrm{i}s(n)}, \qquad (4.2)$$

$$\sqrt{2}\ G_{23}(n) = -\exp[-q_+(n)]\sqrt{1-\mathrm{i}s(n)}, \qquad (4.3)$$

$$\sqrt{2}\ F_{32}(n) = +\exp[+q_+(n)]\sqrt{1-\mathrm{i}s(n)}, \qquad (4.4)$$

$$p_{11}(n) = \frac{1}{4}[\,\dot q_-(n) - \dot q_+(n)][1+s^2(n)] +$$
$$+ \frac{1}{2}\dot q_-(n)[1+\mathrm{i}s(n)], \qquad (4.5)$$

$$p_{13}(n) = \frac{\exp[\,q_-(n) - q_+(n)]}{4\sqrt{1+s^2(n)}}\times$$
$$\times\Big\{\dot q_+(n)[1+s^2(n)][1+\mathrm{i}s(n)] +$$
$$+ \dot q_-(n)[1+s^2(n)][1-\mathrm{i}s(n)] + 2\mathrm{i}s(n)\Big\}, \qquad (4.6)$$

$$p_{31}(n) = \frac{\exp[\,q_+(n) - q_-(n)]}{4\sqrt{1+s^2(n)}}\times$$
$$\times\Big\{\dot q_-(n)[1+s^2(n)][1-\mathrm{i}s(n)] +$$
$$+ \dot q_+(n)[1+s^2(n)][1+\mathrm{i}s(n)] - 2\mathrm{i}s(n)\Big\}, \qquad (4.7)$$

$$p_{33}(n) = \frac{1}{4}[\dot q_+(n) - \dot q_-(n)][1+s^2(n)] + \frac{1}{2}\dot q_+(n)[1-\mathrm{i}s(n)]. \qquad (4.8)$$

Though these parametrization formulas (4.1)–(4.8) are distinguished by a rather uncommon structure, they admit the complex conjugate symmetries $q_-(n) = q_+^*(n)$ and $q_+(n) = q_-^*(n)$ between the field variables $q_-(n)$ and $q_+(n)$ supplemented by the reality $s(n) = s^*(n)$ of field variable $s(n)$.

By dint of the adopted parametrizations, the prototype nonlinear evolution equations (2.1)–(2.8) provide the dynamics of an actual nonlinear dynamical system to be governed by the set of Lagrange equations

$$\frac{d}{d\tau}[\partial\mathcal{L}/\partial\dot q_+(n)] = \partial\mathcal{L}/\partial q_+(n), \qquad (4.9)$$

$$\frac{d}{d\tau}[\partial\mathcal{L}/\partial\dot s(n)] = \partial\mathcal{L}/\partial s(n), \qquad (4.10)$$

$$\frac{d}{d\tau}[\partial\mathcal{L}/\partial\dot q_-(n)] = \partial\mathcal{L}/\partial q_-(n) \qquad (4.11)$$

with the Lagrange function given by the expression

$$\mathcal{L} = \frac{1}{4}\sum_{m=-\infty}^{\infty}[1-\mathrm{i}s(m)]\dot q_+^2(m) +$$
$$+ \frac{1}{4}\sum_{m=-\infty}^{\infty}[1+\mathrm{i}s(m)]\dot q_-^2(m) +$$
$$+ \frac{1}{8}\sum_{m=-\infty}^{\infty}[1+s^2(m)][\dot q_+(m) - \dot q_-(m)]^2 +$$

$$+\frac{1}{4}\sum_{m=-\infty}^{\infty}\frac{\dot{s}^2(m)}{1+s^2(m)}-$$
$$-\frac{1}{2}\sum_{m=-\infty}^{\infty}\exp\bigl[+q_+(m+1)-q_+(m)\bigr]\times$$
$$\times\sqrt{[1-\mathrm{i}s(m+1)][1-\mathrm{i}s(m)]}-$$
$$-\frac{1}{2}\sum_{m=-\infty}^{\infty}\exp\bigl[+q_-(m+1)-q_-(m)\bigr]\times$$
$$\times\sqrt{[1+\mathrm{i}s(m+1)][1+\mathrm{i}s(m)]}+\sum_{m=-\infty}^{\infty}1. \qquad (4.12)$$

On the other hand, the relevant Hamiltonian formulation of the reduced nonlinear system in question is based on the set of Hamilton equations

$$\dot{q}_+(n)=\partial\mathcal{H}/\partial p_+(n), \qquad (4.13)$$
$$\dot{s}(n)=\partial\mathcal{H}/\partial r(n), \qquad (4.14)$$
$$\dot{q}_-(n)=\partial\mathcal{H}/\partial p_-(n), \qquad (4.15)$$
$$\dot{p}_+(n)=-\partial\mathcal{H}/\partial q_+(n), \qquad (4.16)$$
$$\dot{r}(n)=-\partial\mathcal{H}/\partial s(n), \qquad (4.17)$$
$$\dot{p}_-(n)=-\partial\mathcal{H}/\partial q_-(n) \qquad (4.18)$$

with the Hamilton function given by the expression

$$\mathcal{H}=\frac{1}{2}\sum_{m=-\infty}^{\infty}\frac{p_+^2(m)}{1-\mathrm{i}s(m)}+\frac{1}{2}\sum_{m=-\infty}^{\infty}\frac{p_-^2(m)}{1+\mathrm{i}s(m)}+$$
$$+\frac{1}{4}\sum_{m=-\infty}^{\infty}[p_+(m)+p_-(m)]^2+$$
$$+\sum_{m=-\infty}^{\infty}[1+s^2(m)]r^2(m)+$$
$$+\frac{1}{2}\sum_{m=-\infty}^{\infty}\exp\bigl[q_+(m+1)-q_+(m)\bigr]\times$$
$$\times\sqrt{[1-\mathrm{i}s(m+1)][1-\mathrm{i}s(m)]}+$$
$$+\frac{1}{2}\sum_{m=-\infty}^{\infty}\exp\bigl[q_-(m+1)-q_-(m)\bigr]\times$$
$$\times\sqrt{[1+\mathrm{i}s(m+1)][1+\mathrm{i}s(m)]}-\sum_{m=-\infty}^{\infty}1. \qquad (4.19)$$

The standard Hamiltonian form (4.13)–(4.18) of above dynamical equations points out on the standardness of related fundamental Poisson brackets.

In either of its incarnations (4.9)–(4.12) or (4.13)–(4.19), the reduced nonlinear dynamical system comprises three dynamical subsystems coupled both in their kinetic and potential parts, and it can be referred to as the integrable three-subsystem nonlinear lattice model with the combined inter-mode couplings.

Thus, in the Hamiltonian formulation (4.13)–(4.19), the two constituent subsystems are described by the field variables $p_+(n)$, $q_+(n)$ and $p_-(n)$, $q_-(n)$. These subsystems can be treated as the subsystems of nonlinear vibrations related to the two complementary sorts of structural elements of a quasi-one-dimensional lattice. The subsystems of nonlinear vibrations are seen to be the subsystems of the complex-valued Toda-like type. In accordance with the Hamiltonian representation, the field variables $p_+(n)$ and $p_-(n)$ should be considered as the momentum functions conjugate to the coordinate field variables $q_+(n)$ and $q_-(n)$, respectively. As for the field variables $r(n)$ and $s(n)$, they specify the intermediate subsystem and have the meaning of its momentum and coordinate fields, respectively. The field variables of this intermediate subsystem is seen to be essentially mixed with the field variables of both Toda-like subsystems serving as some gluing subsystem between them. Following the terminology of solid state physics [35], the discrete spatial coordinate n serves to mark the ordinal position of a unit cell in a regular quasi-one-dimensional lattice.

Evidently, the subdivision into three coupled subsystems can be properly reformulated also in the case of system's Lagrange representation (4.9)–(4.12), inasmuch as the Hamiltonian and Lagrangian descriptions are always related via the Legendre transformation [36].

5. Fundamentals of System's Integration by the Darboux–Bäcklund Dressing Scheme

Among the numerous general methods of searching for the nontrivial solutions of integrable nonlinear systems, the Darboux–Bäcklund dressing approach seems to be the most simple and universal one. The advantages of the Darboux–Bäcklund dressing technique are especially valuable for any system associated with an auxiliary spectral problem of the third or more higher order. According to the Caudrey classification [33] (see also [25, 26]), our general nonlinear system (2.1)–(2.8) is associated with the auxiliary spectral problem of the third order, inasmuch as the relevant limiting eigenvalue problem (3.4) possesses three *distinct* eigenvalues (3.5)–(3.7).

In this section, we formulate the key ideas allowing to grasp the essence of the Darboux–Bäcklund dressing scheme applicable to the integration of both the prototype nonlinear system (2.1)–(2.8) in general and the reduced nonlinear three-mode system (given by formulas (4.9)–(4.12) or (4.13)–(4.19)) in particular.

To proceed with this task, let us start with the general definition of Darboux transformation [34]:

$$ {}^{c}X(n|z) = {}^{cs}D(n|z)\,{}^{s}X(n|z). \tag{5.1} $$

The Darboux transformation (5.1) connects the seed (*a priori* known) ${}^{s}X(n|z)$ and crop (required) ${}^{c}X(n|z)$ solutions of the auxiliary linear problem

$$ X(n+1|z) = L(n|z)X(n|z), \tag{5.2} $$

$$ \dot{X}(n|z) = A(n|z)X(n|z) \tag{5.3} $$

via the Darboux matrix ${}^{cs}D(n|z)$ which should be properly chosen in accordance with the particular realizations (2.10) and (2.11) of auxiliary spectral $L(n|z)$ and evolution $A(n|z)$ operators. To be appropriate, the Darboux matrix ${}^{cs}D(n|z)$ must obey the set of matrix equations

$$ {}^{cs}D(n+1|z)\,{}^{s}L(n|z) = {}^{c}L(n|z)\,{}^{cs}D(n|z), \tag{5.4} $$

$$ {}^{cs}\dot{D}(n|z) = {}^{c}A(n|z)\,{}^{cs}D(n|z) - {}^{cs}D(n|z)\,{}^{s}A(n|z) \tag{5.5} $$

the first of which, (5.4), implements the implicit Bäcklund transformation between the seed ${}^{s}p_{11}(n)$, ${}^{s}p_{13}(n)$, ${}^{s}p_{31}(n)$, ${}^{s}p_{33}(n)$, ${}^{s}F_{12}(n)$, ${}^{s}G_{21}(n)$, ${}^{s}G_{23}(n)$, ${}^{s}F_{32}(n)$, and crop ${}^{c}p_{11}(n)$, ${}^{c}p_{13}(n)$, ${}^{c}p_{31}(n)$, ${}^{c}p_{33}(n)$, ${}^{c}F_{12}(n)$, ${}^{c}G_{21}(n)$, ${}^{c}G_{23}(n)$, ${}^{c}F_{32}(n)$ solutions for the prototype (and, consequently, for the actual dynamical) fields. The second one (5.5) taken in its contracted form

$$ \frac{d}{d\tau}\ln[\det {}^{cs}D(n|z)] = \mathrm{Sp}\,{}^{c}A(n|z) - \mathrm{Sp}\,{}^{s}A(n|z) \tag{5.6} $$

allows us to uncover the crucial spectral properties of the Darboux matrix sufficient to restore explicitly its components indispensable for the development of the whole Darboux–Bäcklund dressing integration scheme.

In our calculations, we have tested two diverse forms of one-fold ($c = s+1$) ansätze for the Darboux matrix expanded as the simplest truncated Taylor series in the spectral parameter $\lambda(z)$. Unfortunately, both the ansatz with the leading powers reduplicating those of the evolution matrix (2.11) and the extended ansatz expanded akin to the spectral matrix (2.10) are unable to generate nontrivial spatially finite solutions (from the vacuum one) embracing all three subsystems of the whole dynamical system (4.13)–(4.19). However, the two-fold ($c = s+2$) ansatz

$$ {}^{cs}D(n|z) = \begin{pmatrix} {}^{cs}K_{11}(n) & {}^{cs}C_{12}(n)\lambda(z) + {}^{cs}T_{12}(n) & {}^{cs}K_{13}(n) \\ {}^{cs}E_{21}(n)\lambda(z) + {}^{cs}V_{21}(n) & \lambda^2(z) + {}^{cs}D_{22}(n)\lambda(z) + {}^{cs}U_{22}(n) & {}^{cs}E_{23}(n)\lambda(z) + {}^{cs}V_{23}(n) \\ {}^{cs}K_{31}(n) & {}^{cs}C_{32}(n)\lambda(z) + {}^{cs}T_{32}(n) & {}^{cs}K_{33}(n) \end{pmatrix} \tag{5.7} $$

turns out to be fruitful. It can be shown that this ansatz (5.7) is consistent with the set of governing matrix equations (5.4) and (5.5) for the Darboux matrix ${}^{cs}D(n|z)$.

Then, in view of the identity

$$ \mathrm{Sp}\,{}^{c}A(n|z) - \mathrm{Sp}\,{}^{s}A(n|z) \equiv 0, \tag{5.8} $$

the contracted equation (5.6) yields

$$ \det {}^{cs}D(n|z) = $$

$$ = [\lambda(z) - \lambda({}^{cs}z_{+})][\lambda(z) - \lambda({}^{cs}z_{-})]\,{}^{cs}W(n), \tag{5.9} $$

where the quantity ${}^{cs}W(n)$ and the spectral data ${}^{cs}z_{+}$, ${}^{cs}z_{-}$ are proved to be time-independent ${}^{cs}\dot{W}(n) = 0$, ${}^{cs}\dot{z}_{+} = 0$, ${}^{cs}\dot{z}_{-} = 0$. Evidently, $\det {}^{cs}D(n|{}^{cs}z_{\pm}) = 0$, and, hence, the Darboux transformation (5.1) yields $\det {}^{c}X(n|,{}^{cs}z_{\pm}) = 0$ implying that

$$ \sum_{k=1}^{3} {}^{c}X_{jk}(n|{}^{cs}z_{\pm})\;{}^{cs}\varepsilon_{k}({}^{cs}z_{\pm}) = 0 \tag{5.10} $$

or, in more details,

$$ \sum_{i=1}^{3}\sum_{k=1}^{3} {}^{cs}D_{ji}(n|{}^{cs}z_{\pm})\,{}^{s}X_{ik}(n|{}^{cs}z_{\pm})\,{}^{cs}\varepsilon_{k}({}^{cs}z_{\pm}) = 0. \tag{5.11} $$

Here, the functions ${}^{cs}D_{ji}(n|z)$ and ${}^{s}X_{ik}(n|z)$ stand for the elements of respective matrices ${}^{cs}D(n|z)$ and ${}^{s}X(n|z)$, while the time- and space-independent parameters ${}^{cs}\varepsilon_k({}^{cs}z_\pm)$ serve as the spectral data. The invariability of the spectral parameters ${}^{cs}\varepsilon_k({}^{cs}z_\pm)$ in time and space has a status of a rigorously proved theorem.

Insofar as some elements of the Darboux matrix ${}^{cs}D(n|z)$ are always *a priori* known, the obtained formula (5.11) should be treated as the set of six nonuniform linear equations to restore unknown elements of the Darboux matrix. In so doing, the seed solution ${}^{s}X(n|z)$ to the auxiliary linear problem (5.2) and (5.3) must be found beforehand. Once the necessary elements of the Darboux matrix have been restored, the proper equations taken among the implicit Bäcklund transformation (5.4) allow us to obtain explicit expressions for the prototype field functions and, hence, to find out the respective solution to the nonlinear dynamical system under study (4.9)–(4.12) (see also (4.13)–(4.19)).

N.B. The integration of an integrable nonlinear dynamical system in the framework of the Darboux–Bäcklund transformation approach should be treated as a sort of inverse problem, inasmuch as its key idea consists in inverting the auxiliary spectral data of a purely linear problem into the solution of associated nonlinear dynamical equations. In this sense, the Darboux–Bäcklund dressing technique is affined with the method of inverse-scattering transform [6, 25, 26, 28, 33].

6. Explicit Solution Generated by the Two-Fold Darboux–Bäcklund Transformation

The parametrization formulas (4.1)–(4.8) for the prototype field variables prompt us to restrict calculations concerning the solution to the reduced nonlinear system (4.9)–(4.12) only by the expressions for the coordinate field functions $q_+(n)$, $s(n)$, $q_-(n)$. To put it differently, we must concentrate entirely on the calculations of prototype functions $F_{12}(n)$, $G_{21}(n)$, $G_{23}(n)$, $F_{32}(n)$.

Following the two-fold Darboux–Bäklund integration scheme presented in the previous sections, we have dressed the prototype seed solution ${}^{s}p_{11}(n) = 0$, ${}^{s}p_{13}(n) = 0$, ${}^{s}p_{31}(n) = 0$, ${}^{s}p_{33}(n) = 0$, ${}^{s}F_{12}(n) = {}^{0}F_{12}$, ${}^{s}G_{21}(n) = {}^{0}G_{21}$, ${}^{s}G_{23}(n) = {}^{0}G_{23}$, ${}^{s}F_{32}(n) = {}^{0}F_{32}$ to find the prototype crop functions ${}^{2}F_{12}(n)$, ${}^{2}G_{21}(n)$, ${}^{2}G_{23}(n)$ ${}^{2}F_{32}(n)$ in the form

$$
\begin{aligned}
{}^{2}F_{12}(n) ={}^{0} F_{12}\ & \frac{{}^{0}R(n|{}^{20}z_+)\,{}^{0}R(n-1|{}^{20}z_-) -{}^{0} R(n|{}^{20}z_-)\,{}^{0}R(n-1|{}^{20}z_+)}{{}^{0}R(n-1|{}^{20}z_+)\,{}^{0}R(n-2|{}^{20}z_-) -{}^{0} R(n-1|{}^{20}z_-)\,{}^{0}R(n-2|{}^{20}z_+)} - \\
- \mathrm{i}\,{}^{0}G_{23}\ & \frac{{}^{0}S(n|{}^{20}z_+)\,{}^{0}R(n-1|{}^{20}z_-) -{}^{0} S(n|{}^{20}z_-)\,{}^{0}R(n-1|{}^{20}z_+)}{{}^{0}R(n-1|{}^{20}z_+)\,{}^{0}R(n-2|{}^{20}z_-) -{}^{0} R(n-1|{}^{20}z_-)\,{}^{0}R(n-2|{}^{20}z_+)},
\end{aligned}
\tag{6.1}
$$

$$
{}^{2}G_{21}(n) ={}^{0} G_{21}\ \frac{{}^{0}R(n-1|{}^{20}z_+)\,{}^{0}R(n-2|{}^{20}z_-) -{}^{0} R(n-1|{}^{20}z_-)\,{}^{0}R(n-2|{}^{20}z_+)}{{}^{0}R(n|{}^{20}z_+)\,{}^{0}R(n-1|{}^{20}z_-) -{}^{0} R(n|{}^{20}z_-)\,{}^{0}R(n-1|{}^{20}z_+)},
\tag{6.2}
$$

$$
\begin{aligned}
{}^{2}F_{32}(n) ={}^{0} F_{32}\ & \frac{{}^{0}R(n|{}^{20}z_+)\,{}^{0}R(n-1|{}^{20}z_-) -{}^{0} R(n|{}^{20}z_-)\,{}^{0}R(n-1|{}^{20}z_+)}{{}^{0}R(n-1|{}^{20}z_+)\,{}^{0}R(n-2|{}^{20}z_-) -{}^{0} R(n-1|{}^{20}z_-)\,{}^{0}R(n-2|{}^{20}z_+)} + \\
+\mathrm{i}\,{}^{0}G_{21}\ & \frac{{}^{0}S(n|{}^{20}z_+)\,{}^{0}R(n-1|{}^{20}z_-) -{}^{0} S(n|{}^{20}z_-)\,{}^{0}R(n-1|{}^{20}z_+)}{{}^{0}R(n-1|{}^{20}z_+)\,{}^{0}R(n-2|{}^{20}z_-) -{}^{0} R(n-1|{}^{20}z_-)\,{}^{0}R(n-2|{}^{20}z_+)},
\end{aligned}
\tag{6.3}
$$

$$
{}^{2}G_{23}(n) ={}^{0} G_{23}\ \frac{{}^{0}R(n-1|{}^{20}z_+)\,{}^{0}R(n-2|{}^{20}z_-) -{}^{0} R(n-1|{}^{20}z_-)\,{}^{0}R(n-2|{}^{20}z_+)}{{}^{0}R(n|{}^{20}z_+)\,{}^{0}R(n-1|{}^{20}z_-) -{}^{0} R(n|{}^{20}z_-)\,{}^{0}R(n-1|{}^{20}z_+)}.
\tag{6.4}
$$

Here,

$$
\begin{aligned}
{}^{0}R(n|{}^{20}z_\pm) &= ({}^{20}z_\pm)^n\ \exp(\tau/{}^{20}z_\pm)\,{}^{20}\varepsilon_1({}^{20}z_\pm) + \\
&+ (1/{}^{20}z_\pm)^n\ \exp(\tau\,{}^{20}z_\pm)\,{}^{20}\varepsilon_3({}^{20}z_\pm),
\end{aligned}
\tag{6.5}
$$

$$
{}^{0}S(n|{}^{20}z_\pm) = {}^{20}\varepsilon_2({}^{20}z_\pm)\,({}^{20}z_\pm + 1/{}^{20}z_\pm)^n.
\tag{6.6}
$$

The symmetry relations $q_-(n) = q_+^*(n)$, $s(n) = s^*(n)$, $q_+(n) = q_-^*(n)$ between the field variables are proved to support the symmetry relations between the spectral data elucidated by the following parametrization formulas:

$$^{20}z_\pm = \exp(\pm\mu + \mathrm{i}k), \tag{6.7}$$

$$^{20}\varepsilon_1(^{20}z_\pm) = \exp(\pm\alpha + \mathrm{i}\beta), \tag{6.8}$$

$$^{20}\varepsilon_2(^{20}z_\pm) = g\exp(\pm\mathrm{i}\delta), \tag{6.9}$$

$$^{20}\varepsilon_3(^{20}z_\pm) = \exp(\mp\alpha - \mathrm{i}\beta), \tag{6.10}$$

where k, α, β, g, and δ are arbitrary real parameters, and μ is a real positive parameter. However, in order to suppress systematic divergences of the crop field functions at spatial and temporal infinities, we are obligated to impose severe constraints,

$$k = \sigma\pi/2, \tag{6.11}$$

$$\mu > \ln(\sqrt{2}), \tag{6.12}$$

onto the parameters k and μ, with the coefficient σ being defined by the equality

$$\sigma^2 = 1. \tag{6.13}$$

Peripeteia of calculations prompted us to introduce the shorthand notations

$$\Phi(n) = 2\sigma\sinh(2\mu n - \mu + 2\alpha) + 2\sinh(\mu)\sin[\sigma\pi n - \sigma\pi/2 + 2\beta - 2\tau\sigma\cosh(\mu)], \tag{6.14}$$

$$\Theta(n) = g\,\exp(+\mu n - \mu + \alpha)\,[2\sinh(\mu)]^n \times \sin[\sigma\pi/2 + \delta - \beta + \tau\sigma\exp(-\mu)] + g\,\exp(-\mu n + \mu - \alpha)[2\sinh(\mu)]^n \times \sin[\sigma\pi n - \sigma\pi/2 + \delta + \beta - \tau\sigma\exp(+\mu)] \tag{6.15}$$

for the typical functional combinations, as well as to invoke the obvious parametrization formulas

$$^0F_{12}\sqrt{2} = +\exp(+q_-)\sqrt{1+\mathrm{i}s}, \tag{6.16}$$

$$^0G_{21}\sqrt{2} = -\exp(-q_-)\sqrt{1+\mathrm{i}s}, \tag{6.17}$$

$$^0G_{23}\sqrt{2} = -\exp(-q_+)\sqrt{1-\mathrm{i}s}, \tag{6.18}$$

$$^0F_{32}\sqrt{2} = +\exp(+q_+)\sqrt{1-\mathrm{i}s} \tag{6.19}$$

for the prototype seed quantities $^0F_{12}$, $^0G_{21}$, $^0G_{23}$, and $^0F_{32}$.

The result for the crop solution to the integrable nonlinear lattice system of our interest (4.9)–(4.12) obtainable in the framework of the two-fold Darboux–Bäcklund dressing technique reads

$$q_+(n) = q_+ + \frac{1}{2}\ln\left[\frac{\Phi^2(n)}{\Phi^2(n-1)}\right] - \frac{\mathrm{i}}{2}\arctan\left[\frac{\exp(-q_+ - q_-)\,\Theta(n)}{\sqrt{1+s^2}\;\Phi(n) + s\exp(-q_- - q_+)\,\Theta(n)}\right] + \frac{1}{4}\ln\left\{\left[1 + s\,\frac{\exp(-q_+ - q_-)}{\sqrt{1+s^2}}\,\frac{\Theta(n)}{\Phi(n)}\right]^2 + \frac{\exp(-2q_+ - 2q_-)}{1+s^2}\,\frac{\Theta^2(n)}{\Phi^2(n)}\right\}, \tag{6.20}$$

$$s(n) = s + \exp(-q_+ - q_-)\,\sqrt{1+s^2}\,\frac{\Theta(n)}{\Phi(n)}, \tag{6.21}$$

$$q_-(n) = q_- + \frac{1}{2}\ln\left[\frac{\Phi^2(n)}{\Phi^2(n-1)}\right] + \frac{\mathrm{i}}{2}\arctan\left[\frac{\exp(-q_+ - q_-)\,\Theta(n)}{\sqrt{1+s^2}\Phi(n) + s\exp(-q_- - q_+)\,\Theta(n)}\right] + \frac{1}{4}\ln\left\{\left[1 + s\,\frac{\exp(-q_+ - q_-)}{\sqrt{1+s^2}}\,\frac{\Theta(n)}{\Phi(n)}\right]^2 + \frac{\exp(-2q_+ - 2q_-)}{1+s^2}\,\frac{\Theta^2(n)}{\Phi^2(n)}\right\}. \tag{6.22}$$

Here, the upper front indices in the field functions $^2q_+(n)$, $^2s(n)$, and $^2q_-(n)$ have been omitted for a stylistic reason.

Bearing in mind the obtained solutions (6.20)–(6.22) for the coordinate field components and relying upon the standard definitions

$$p_+(n) = \partial\mathcal{L}/\partial\dot{q}_+(n) = \frac{1}{2}\,[1 - \mathrm{i}s(n)]\,\dot{q}_+(n) + \frac{1}{4}\,[1 + s^2(n)]\,[\dot{q}_+(n) - \dot{q}_-(n)], \tag{6.23}$$

$$r(n) = \partial\mathcal{L}/\partial\dot{s}(n) = \frac{1}{2}\,\frac{\dot{s}(n)}{1+s^2(n)}, \qquad (6.24)$$

$$p_-(n) = \partial\mathcal{L}/\partial\dot{q}_-(n) = \frac{1}{2}\left[1+\mathrm{i}s(n)\right]\dot{q}_-(n) + \\ + \frac{1}{4}\left[1+s^2(n)\right]\left[\dot{q}_-(n)-\dot{q}_+(n)\right] \qquad (6.25)$$

of momentum field components, we can conclude that all three constituent subsystems are characterized by the essentially nonzero excitations of a rather sophisticated mathematical structure.

In order to understand the character of the obtained solutions (6.20)–(6.22) for the coordinate field components $q_+(n)$, $s(n)$, $q_-(n)$, let us analyze the spatial and temporal behavior of their two determinative functions $\Phi^2(n)/\Phi^2(n-1)$ and $\Theta(n)/\Phi(n)$. In view of the earlier imposed conditions (6.11) and (6.12), each of determinative functions is bounded at both spatial infinities $n \to \pm\infty$. However, if the parameter α has been chosen improperly, expression (6.14) for $\Phi(n)$ as a function of time τ can change its sign even at some admissible (*i.e.* integer) values of the spatial coordinate n. In such spatial points, the expression $\Theta(n)/\Phi(n)$ as a function of time undergoes infinite discontinuities. To exclude such an unphysical scenario, it is sufficient to impose the strict condition

$$\alpha = -\mu l \qquad (6.26)$$

onto the parameter α, with the parameter l being taken as an arbitrary integer. Indeed, though quantity (6.14) under the adjusting condition (6.26) possesses two vulnerable spatial points $n = l$ and $n = l+1$, it can never change its sign as a function of time τ at either of them. Hence, the expression $1/\Phi(n)$ as a function of time τ can demonstrate the infinite splashes at two neighboring spatial points points $n = l$ and $n = l+1$ arising periodically in time with the cyclic frequency equal to $2\cosh(\mu)$. The expression $\Theta(n)/\Phi(n)$ in turn is responsible for the modulated splashes. Modulating cyclic frequencies are equal to $\exp(+\mu)$ and $\exp(-\mu)$.

Bearing in mind the above analysis and considering formulas (6.20)–(6.25), we conclude that both of Toda-like subsystems can support the superposition of the pulson and modulated pulson nonlinear excitations of a rather complicated form. In contrast, the form of modulated pulson nonlinear excitations related to the mediated system turns out to be simpler. In so doing, the excitations related to the Toda-like subsystems are located mainly in vicinities of three neighboring spatial points $n = l$, $n = l+1$, $n = l+2$, while the excitations related to the mediated system are located mainly in vicinities of two neighboring spatial points $n = l$, $n = l+1$.

7. Conclusions

The aim of this work was to introduce the integrable nonlinear lattice system comprising three dynamical subsystems with the combined inter-mode couplings as a conceivable model for the analytic investigation of nonlinear excitations on regular quasi-one-dimensional lattices of various physical origins. The presently suggested system is emerged as a particular reduction of a prototype nonlinear system associated with the semidiscrete auxiliary linear problem of the third order proposed in our previous works [25, 26].

The considered reduction is not unique, though it is inspired by the so-called natural constraints inherent to the prototype nonlinear system. The suggested reduced system permits both Lagrangian and Hamiltonian representations in terms of three canonical subsystems. Two subsystems of the Toda-like type are related by the symmetry of complex conjugation, while a single intermediate subsystem serves as an additional coupler between the Toda-like ones. Due to the specific symmetries between the field variables, the Hamiltonian function arises as an essentially real quantity $\mathcal{H} = \mathcal{H}^*$. In view of this fact, there exists a principal opportunity to rearrange the complex symmetric Toda-like subsystems into two real-valued subsystems characterized by the translational and orientational degrees of freedom separately. In such a reduction, the system could serve as a sort of a toy model to describe the translational and orientational subsystems of long macromolecules.

Relying upon the prototype auxiliary linear problem, we have developed the two-fold Darboux–Bäcklund dressing scheme in the most general terms suitable for the integration of any reduced nonlinear system compatible with two natural constraints. In the framework of this technique, we have found the explicit solution to the reduced semidiscrete nonlinear integrable system of our main interest. The solution embraces all three coupled nonlinear subsystems and manifests a pronounced modulated pulson character.

The main local conserved densities related to the prototype nonlinear system have been found explicitly in the framework of the generalized direct recurrent approach.

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ЗВ'ЯЗАНА НЕЛІНІЙНА ДИНАМІКА ТРИМОДОВОЇ ІНТЕҐРОВНОЇ СИСТЕМИ НА РЕГУЛЯРНОМУ ЛАНЦЮЖКУ

Запропоновано нелінійну ґратчасту систему трьох динамічних підсистем, зв'язаних як в потенціальній, так і в кінетичній частинах. Завдяки своїй суттєво багатокомпонентній будові система здатна моделювати нелінійні динамічні збудження на квазіодновимірних ґратках різноманітної фізичної природи. Система має чітке Гамільтонове формулювання зі стандартною Пуассоновою структурою. Подано також і альтернативне Лаґранжове формулювання динаміки системи. Динамічні рівняння системи є інтеґровними в сенсі Лакса, оскільки допускають представлення нульової кривини. Складність доречної допоміжної лінійної задачі зі спектральним оператором третього порядку не стала на заваді у побудові техніки подвійного одягання Дарбу–Беклунда, прийнятної для зґенерування нетривіального розв'язку, що охоплює усі три зв'язані підсистеми доволі незвичним чином.

Ключові слова: нелінійні теорії та моделі, ангармонійні ґраткові моди, інтеґровні системи, Лаґранжова та Гамільтонова динаміки, метод одягання Дарбу–Беклунда, симетрія та закони збереження, нелінійний хвильовий пакет.