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COMPOSITE FERMIONS AS DEFORMED OSCILLATORS: WAVEFUNCTIONS AND ENTANGLEMENT [1]

Composite structure of particles somewhat modifies their statistics, compared to the pure Bose-or Fermi-ones. The spin-statistics theorem, so, is not valid anymore. Say, π-mesons, excitons, Cooper pairs are not ideal bosons, and, likewise, baryons are not pure fermions. In our preceding papers, we studied bipartite composite boson (i.e. quasiboson) systems via a realization by deformed oscillators. Therein, the interconstituent entanglement characteristics such as entanglement entropy and purity were found in terms of the parameter of deformation. Herein, we perform an analogous study of composite Fermi-type particles, and explore them in two major cases: (i) "boson + fermion" composite fermions (or cofermions, or CFs); (ii) "deformed boson + fermion" CFs. As we show, cofermions in both cases admit only the realization by ordinary fermions. Case (i) is solved explicitly, and admissible wavefunctions are found along with entanglement measures. Case (ii) is treated within few modes both for CFs and constituents. The entanglement entropy and purity of CFs are obtained via the relevant parameters and illustrated graphically.

K e y w o r d s: composite fermions (cofermions), composite bosons (cobosons, quasibosons), realization by deformed oscillators, bipartite entanglement, entanglement entropy, purity.

1. Introduction

Composite fermions play a significant role in modern quantum physics. Suffice it to mention a few instances of CFs: these include quasiparticles involved in the theory of fractional quantum Hall effect [1]; also, let us mention baryons and pentaquarks as known instances in hadron physics [2–4]. In this paper, we will focus on the entanglement properties of composite fermions basing on their fermionic oscillator realization, in two relatively simple cases: cofermion *built from* a pure fermion and a pure boson, or the cofermion as a composite *made of* a pure fermion and a deformed boson (taken in general form).

In our preceding works [5–8], we studied bipartite (two-component) composite "bosons" of two types: "fermion + fermion" and "boson + boson" with creation and annihilation operators within the typical ansatz

$$A_\alpha^\dagger = \sum_{\mu\nu} \Phi_\alpha^{\mu\nu} a_\mu^\dagger b_\nu^\dagger, \qquad A_\alpha = \sum_{\mu\nu} \overline{\Phi_\alpha^{\mu\nu}} b_\nu a_\mu, \qquad (1)$$

where the creation operators $a_\mu^\dagger$, $b_\nu^\dagger$ for the (distinguishable) constituents taken either both fermionic or both bosonic. In [5, 6], it was shown that composite bosons of a particular form (with appropriate wavefunctions $\Phi_\alpha^{\mu\nu}$) can be realized, in the operator sense, by suitable deformed bosons (deformed oscillators).

An important concept in quantum information theory, quantum communication, and teleportation [9, 10] is the notion of entanglement or quantum correlation between the constituents of a composite particle or composite system. Recently, this concept was actively studied just in the context of composite bosons [7, 11–13]. Among the measures characterizing the degree of entanglement, best known are the entanglement entropy and purity (= inverse of the Schmidt number) [9, 10]. The measures of intercomponent entanglement in a quasiboson quantify to what extent the properties of a quasiboson approach those of a true boson [11, 12, 14, 15].

For composite bosons realizable by *deformed* quantum oscillators, it is possible to directly link [7] the relevant *parameter of deformation* with the entanglement characteristics of a composite boson. Then the characteristics (or measures) of bipartite entanglement with respect to *a*- and *b*-subsystems, see (1), can be found explicitly [7] in terms of the deformation parameter: for a single composite boson, for multi-quasiboson states, and for coherent states constructed for such quasibosons.

As a very important issue, the influence of system's energy on quantum correlation and/or quantum statistics properties of the system was studied, for the quasibosons, in [8]. The energy of the quasiboson differs from the energy of a respective ideal boson, and the difference (including the energy of bound states) essentially depends [8] on the quasiboson's entanglement, clearly showing a deviation from the bosonic behavior. Such entanglement-energy relation is relevant to quantum information research, quantum communication, entanglement production [16], quantum dissociation processes [17], particle addition or subtraction [18, 19], *etc.*

Below, we explore the cofermions. Since the entanglement entropy is of primary interest, after the appropriate analysis of the realization issue, we find the entanglement entropy $S_{\rm ent}$ characterizing the composite fermion. Our treatment is performed for one-CF states (to compare, the respective results for one quasiboson states are also briefly sketched). In some analogy with the case of composite bosons, we take the cofermions as bipartite systems realized by mode-independent [2] fermionic oscillators. Another entanglement measure, purity, is considered in a special case.

We have to emphasize that our investigation of entanglement concerns not a many-cofermion system, but the states of a single (or isolated) cofermion. Accordingly, the considered entanglement and its entropy concern two constituents of the bipartite CF. These features make our approach and analysis essentially different from some recent works on the entanglement entropy of a system of free or composite fermions, see, e.g., [20, 21], where the spatial size of a subsystem plays the basic role, and entanglement entropy *was viewed in a way different from our one.*

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[2] This is understood in the fermionic, i.e. anticommuting, sense.

In Sec. 2, a sketch of the realized composite bosons is given. Sections 3–5 deal with cofermions: we perform the analysis of operator-level realization of cofermions by (deformed) fermionic oscillators. Then the entanglement entropy of the CF one-particle states is explored. Modified CFs (composed of fermion and deformed boson) are analyzed in Sec. 5(b). The purity for a CF state is considered in Sec. 5(a). The paper ends with conclusions.

2. Quasibosons Formed as Two-Fermion (Two-Boson) Composites [5]

Recall the main facts about composite bosons realized [5, 6] by a set of independent modes of deformed bosons (deformed oscillators), given by the defining *deformation structure function* $\varphi(n)$ [22]. At the algebraic level, the quasiboson operators A_α, $A_\alpha^\dagger$ and the number operator N_α satisfy the same relations *on the states* as the corresponding deformed oscillator creation/annihilation and number operators:

$$A_\alpha^\dagger A_\alpha \simeq \varphi(N_\alpha), \quad [A_\alpha, A_\beta^\dagger] \simeq \delta_{\alpha\beta}\big(\varphi(N_\alpha+1)-\varphi(N_\alpha)\big),$$

$$[N_\alpha, A_\beta^\dagger] \simeq \delta_{\alpha\beta} A_\beta^\dagger, \quad [N_\alpha, A_\beta] \simeq -\delta_{\alpha\beta} A_\beta.$$

Here, $\simeq$ denotes weak equality (i.e. on the states), symbols $\delta_{\alpha\beta}$ reflect mode independence. In such realization, the structure function $\varphi(n)$ involves [5,6] the discrete deformation parameter f and is quadratic in the quasiparticles number n ($\kappa = +1$ or -1 for two bosonic or two fermionic constituents):

$$\varphi(n) = \left(1 + \kappa\frac{f}{2}\right)n - \kappa\frac{f}{2}n^2, \quad f = \frac{2}{m}, \qquad (2)$$

$m = 1, 2, \dots$. The matrices Φ_α in (1) are [5, 6]

$$\Phi_\alpha = U_1(d_a)\,{\rm diag}\Big\{0..0, \sqrt{f/2}\,U_\alpha(m), 0..0\Big\}U_2^\dagger(d_b), (3)$$

where $U_j(r)$ stands for an arbitrary unitary $r \times r$ matrices, the dimension d_a or d_b is the total number of modes for constituents with operators a_μ or b_ν.

Note that the state of one composite boson,

$$|\Psi_\alpha\rangle = \sum_{\mu\nu} \Phi_\alpha^{\mu\nu}|\mu\rangle \otimes |\nu\rangle, \quad |\mu\rangle \equiv a_\mu^\dagger|0\rangle, \quad |\nu\rangle \equiv b_\nu^\dagger|0\rangle,$$

is, in general, bipartite-entangled relative to the states of two constituent fermions (or two bosons). There are well-known measures of entanglement [9,

10]: Schmidt rank s, Schmidt number K or its inverse – purity, entanglement entropy S, and concurrence C. As proven in [7], the (internal) entanglement entropy of one composite boson

$$S_{\text{ent}} = \ln(m) = \ln(2/f). \tag{4}$$

For the multi-quasibosonic states, the respective extended results were also derived, see [7, 8].

The other known measure of entanglement [9, 10], purity is inverse to the Schmidt number: $P = 1/K$. Note that *purity* was exploited in connection with the issue of entanglement creation using scattering processes [16] (for other contexts, see [18, 23]). For *one-quasiboson entangled system*, purity is as well linked [7] with the deformation parameter $m = \frac{2}{f}$:

$$P = \sum_k \lambda_k^4 = \frac{1}{m} \tag{5}$$

and thus takes discrete values.

3. Cofermion – as Fermion *plus* Deformed Boson

From now on, we consider composite fermions which are composed of a pure (or deformed) boson and a pure fermion, from the viewpoint of their realization by deformed fermions. Like for composite bosons, the realization of CFs is required to be constructed in the way that enables to treat CFs' creation, annihilation, and number operators as the respective operators of deformed fermions. In turn, this gives the advantages compared to the standard quantum mechanical or macroscopic approach with explicit direct consideration of a composite structure, – to perform calculations, in the effective or approximate description. CFs' creation and annihilation operators are given by the same "ansatz" as in (1), but now $a_\mu^\dagger, a_\mu$ are, respectively, the creation and annihilation operators for the constituent bosons (deformed or not) and $b_\nu^\dagger, b_\nu$ – those for the constituent fermions, with standard anticommutation relations for the latter. For simplicity, we suppose that different modes of deformed bosons are independent. Then we have the following defining commutation relations for the operators of constituent bosons (deformed or not; n_μ^a is the particle number operator for deformed bosons in the μ-mode):

$$a_\mu^\dagger a_\mu = \chi(n_\mu^a), \quad [a_\mu, a_{\mu'}^\dagger] = \delta_{\mu\mu'}\big(\chi(n_\mu^a+1) - \chi(n_\mu^a)\big);$$
$$[a_\mu^\dagger, a_{\mu'}^\dagger] = 0; \quad [n_\mu^a, a_\mu^\dagger] = a_\mu^\dagger.$$

Here, the deformation structure function $\chi(n)$ means the general case of deformed constituent boson. For a non-deformed, i.e. usual, boson, $\chi(N) \equiv N$. In the case of deformation, $\chi(N)$ depends on one or more deformation parameters which admit a "no-deformation" limit. It is convenient to work with deformed boson states normalized as $|\mu\rangle = a_\mu^\dagger|0\rangle$. The same concerns CF states, obeying the normalization condition for structural matrices (wavefunctions):

$$\text{Tr}(\Phi_\beta \Phi_\alpha^\dagger) = \delta_{\alpha\beta}. \tag{6}$$

We suppose CFs to behave themselves on the states as deformed particles with a structure function $\varphi(N)$. Respective deformed fermions providing the realization are supposed to be independent (in the fermionic sense). Defining the number operator N_α for CFs as $\varphi(N_\alpha) \simeq A_\alpha^\dagger A_\alpha$, we write the realization conditions:

$$\{A_\alpha, A_\beta^\dagger\} \simeq \delta_{\alpha\beta}[\varphi(N_\alpha+1) + \varphi(N_\alpha)], \tag{7}$$

$$\{A_\alpha^\dagger, A_\beta^\dagger\} \simeq 0, \quad \alpha \neq \beta; \quad [N_\alpha, A_\beta^\dagger] \simeq \delta_{\alpha\beta} A_\beta^\dagger. \tag{8}$$

The first requirement in (8) holds automatically, and moreover, in the strict sense for all α and β,

$$\{A_\alpha^\dagger, A_\beta^\dagger\} = 0, \text{ particularly } (A_\alpha^\dagger)^2 = 0. \tag{9}$$

The latter, fermionic nilpotency, and (7) considered on vacuum and one-CF states, along with the second equation in (8), yield the usual *fermionic structure function* $\varphi(N)$:

$$\varphi(0) = \varphi(2) = ... = 0, \quad \varphi(1) = \chi(1) = 1. \tag{10}$$

We analyze requirement (7), proceeding like in [5], but alternate (interchange) the commutator and anticommutator of the l.h.s. of (7) with $A_{\gamma_i}^\dagger$. Introducing the notation

$$\Delta^k \chi(n_\mu^a) \equiv \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \chi(n_\mu^a+l), \quad k = 0, 1, ...,$$

with the first terms

$$\Delta^0 \chi(n_\mu^a) = \chi(n_\mu^a), \quad \Delta^1 \chi(n_\mu^a) = \chi(n_\mu^a+1) - \chi(n_\mu^a),$$
$$\Delta^2 \chi(n_\mu^a) = \chi(n_\mu^a+2) - 2\chi(n_\mu^a+1) + \chi(n_\mu^a),$$

for the anticommutator $\{A_\alpha, A_\beta^\dagger\}$ in (7), we have

$$\{A_\alpha, A_\beta^\dagger\} = \sum_\mu (\Phi_\beta \Phi_\alpha^\dagger)^{\mu\mu} \Delta^1 \chi(n_\mu^a) +$$

$$+\sum_{\mu\mu'}(\Phi_\beta\Phi_\alpha^\dagger)^{\mu'\mu}a_{\mu'}^\dagger a_\mu - \sum_{\mu\nu\nu'}\overline{\Phi_\alpha^{\mu\nu}}\Phi_\beta^{\mu\nu'}\Delta^1\chi(n_\mu^a)\,b_{\nu'}^\dagger b_\nu.$$

This will be exploited below. For a *nondeformed constituent boson* ($\chi(n)\equiv n$), the latter with the use of (6) reduces to

$$\{A_\alpha,A_\beta^\dagger\}=\delta_{\alpha\beta}+\sum_{\mu\mu'}(\Phi_\beta\Phi_\alpha^\dagger)^{\mu'\mu}a_{\mu'}^\dagger a_\mu-\sum_{\nu\nu'}(\Phi_\alpha^\dagger\Phi_\beta)^{\nu\nu'}b_{\nu'}^\dagger b_\nu.$$

Next, we calculate the commutator

$$\left[\{A_\alpha,A_\beta^\dagger\},A_\gamma^\dagger\right]=\sum_{\mu\mu_1\nu_1}\Big[(\Phi_\beta\Phi_\alpha^\dagger)^{\mu_1\mu}\Phi_\gamma^{\mu\nu_1}-$$
$$-(\Phi_\gamma\Phi_\alpha^\dagger)^{\mu_1\mu}\Phi_\beta^{\mu\nu_1}\Big]a_{\mu_1}^\dagger b_{\nu_1}^\dagger\cdot\left(\Delta^1\chi(n_\mu^a)+\delta_{\mu\mu_1}\Delta^2\chi(n_\mu^a)\right)+$$
$$+\sum_{\mu\nu\nu_1\nu_2}\overline{\Phi_\alpha^{\mu\nu}}\Phi_\beta^{\mu\nu_1}\Phi_\gamma^{\mu\nu_2}a_\mu^\dagger b_{\nu_1}^\dagger\Delta^2\chi(n_\mu^a)b_{\nu_2}^\dagger b_\nu. \tag{11}$$

A nondeformed analog (at $\chi(n)\equiv n$) of the latter is

$$[\{A_\alpha,A_\beta^\dagger\},A_\gamma^\dagger]=\sum_{\mu\nu}\left(\Phi_\beta\Phi_\alpha^\dagger\Phi_\gamma-\Phi_\gamma\Phi_\alpha^\dagger\Phi_\beta\right)^{\mu\nu}a_\mu^\dagger b_\nu^\dagger.$$

So, the validity of (7) on one-CF states $|\gamma\rangle$ yields the following relation of basic importance for the wavefunctions:

$$(\Phi_\beta\Phi_\alpha^\dagger\Phi_\gamma)^{\mu\nu}-(\Phi_\gamma\Phi_\alpha^\dagger\Phi_\beta)^{\mu\nu}+$$
$$+\left(\chi(2)-2\right)\left[(\Phi_\beta\Phi_\alpha^\dagger)^{\mu\mu}\Phi_\gamma^{\mu\nu}-(\Phi_\gamma\Phi_\alpha^\dagger)^{\mu\mu}\Phi_\beta^{\mu\nu}\right]=0. \tag{12}$$

Note that, in the case of non-deformed constituent boson, this relation yields $[\{A_\alpha,A_\beta^\dagger\},A_\gamma^\dagger]=0$. That leads to a closed set of realization conditions on the matrices Φ_α, see (6),(7), namely:

$$\Phi_\beta\Phi_\alpha^\dagger\Phi_\gamma-\Phi_\gamma\Phi_\alpha^\dagger\Phi_\beta=0,\quad \mathrm{Tr}(\Phi_\beta\Phi_\alpha^\dagger)=\delta_{\alpha\beta}. \tag{13}$$

However, for a nontrivial deformation, i.e. $\chi(N)\not\equiv N$, the double and higher commutators (or anticommutators) are significant and must be taken into account (we drop them).

4. Composite Quasifermions with Non-Deformed Constituent Boson

Consider the realization of CFs formed of a usual boson and a fermion. The respective wavefunctions of CFs realized by usual fermions satisfy Eqs. (13) which will be solved below. Let us choose a matrix $\Phi_\alpha=\Phi_{\alpha_1}$ with maximal rank and perform the singular value (SVD- or Schmidt-) decomposition:

$$\Phi_{\alpha_1}=U_1 D_{\alpha_1}V_1^\dagger,$$

$$D_{\alpha_1}=\mathrm{diag}\{\lambda_i^{\alpha_1},i=\kappa_1,...,\kappa_r,...\}=$$
$$=\mathrm{diag}\{\lambda_{\kappa_1}^{\alpha_1}E_{m_1},\lambda_{\kappa_2}^{\alpha_1}E_{m_2},...,\lambda_{\kappa_r}^{\alpha_1}E_{m_r}\} \tag{14}$$

with real non-negative $\lambda_{\kappa_l}^{\alpha_1}$ put in descending order [3], $\lambda_{\kappa_1}^{\alpha_1}>...>\lambda_{\kappa_r}^{\alpha_1}\geq 0$, obeying $\sum_l m_l(\lambda_{\kappa_l}^{\alpha_1})^2=1$, and unitary matrices U_1, V_1. For remaining matrices Φ_γ, $\gamma\neq\alpha_1$, we make replacement $\Phi_\gamma\to\tilde\Phi_\gamma$:

$$\Phi_\gamma=U_1\tilde\Phi_\gamma V_1^\dagger,\quad \gamma\neq\alpha_1. \tag{15}$$

The first equation in (13) at $\alpha=\beta=\alpha_1$ now reads

$$D_{\alpha_1}^2\tilde\Phi_\gamma-\tilde\Phi_\gamma D_{\alpha_1}^2=0\ \ \text{or}\ \ \left((\lambda_i^{\alpha_1})^2-(\lambda_j^{\alpha_1})^2\right)(\tilde\Phi_\gamma)_{ij}=0.$$

If $\lambda_i^{\alpha_1}\neq\lambda_j^{\alpha_1}$, we have $(\tilde\Phi_\gamma)_{ij}=0$. According to the block-diagonal form of D_{α_1}, see (14), the other matrices $\tilde\Phi_\gamma$ also take block-diagonal form:

$$\tilde\Phi_\gamma=\mathrm{diag}\{\tilde\Phi_{\gamma,1},\tilde\Phi_{\gamma,2},...,\tilde\Phi_{\gamma,r}\},\quad \gamma\neq\alpha_1.$$

The dimensions of unit matrices E_{m_k} and square matrices $\tilde\Phi_{\alpha,k}$ are equal to the multiplicities of singular values $\lambda_{\kappa_k}^{\alpha_1}$. Now, the first equation in (13) reduces to the set of r independent systems, $k=1,2,..,r$:

$$\lambda_{\kappa_k}^{\alpha_1}(\tilde\Phi_{\beta,k}^\dagger\tilde\Phi_{\gamma,k}-\tilde\Phi_{\gamma,k}\tilde\Phi_{\beta,k}^\dagger)=0, \tag{16}$$
$$\lambda_{\kappa_k}^{\alpha_1}(\tilde\Phi_{\beta,k}\tilde\Phi_{\gamma,k}-\tilde\Phi_{\gamma,k}\tilde\Phi_{\beta,k})=0, \tag{17}$$
$$\tilde\Phi_{\beta,k}\tilde\Phi_{\alpha,k}^\dagger\tilde\Phi_{\gamma,k}-\tilde\Phi_{\gamma,k}\tilde\Phi_{\alpha,k}^\dagger\tilde\Phi_{\beta,k}=0,\quad \alpha,\beta,\gamma\neq\alpha_1. \tag{18}$$

From (16)–(17), we infer that the matrices $\tilde\Phi_{\gamma,k}$, $\gamma\neq\alpha_1$, constitute the set of commuting normal matrices (commuting with their Hermitian conjugate ones) at fixed $k=1,2,..,r-1$ and possibly at fixed $k=r$, if $\lambda_{\kappa_r}^{\alpha_1}\neq 0$. Under such premises, as known [24], there is a fixed unitary matrix $\tilde U_k$ such that $\tilde\Phi_{\gamma,k}=\tilde U_k\tilde D_{\gamma,k}\tilde U_k^\dagger$ with diagonal one $\tilde D_{\gamma,k}$, $\gamma\neq\alpha_1$. If $\lambda_{\kappa_r}^{\alpha_1}\neq 0$, Eqs. (18) are automatically satisfied, and we have

$$\Phi_\alpha=UD_\alpha V^\dagger,\ \ D_\alpha\big|_{\alpha\neq\alpha_1}=\mathrm{diag}\{\tilde D_{\alpha,1},...,\tilde D_{\alpha,r}\} \tag{19}$$

with $U=U_1\,\mathrm{diag}\{\tilde U_k\}$, $V=V_1\,\mathrm{diag}\{\tilde U_k\}$, $k=\overline{1,r}$.

If $\lambda_{\kappa_r}^{\alpha_1}=0$, Eqs. (18) at $k=\overline{1,r-1}$ are satisfied, while the remaining one, at $k=r$, for $\tilde\Phi_{\alpha,r}$, $\alpha\neq\alpha_1$, can be solved like above for Φ_α. Thus, by induction on the number $\#\alpha\equiv m_{CF}$ of matrices (modes) Φ_α, we

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[3] Physical meaning of indices κ_l may be the relative momentum of constituents in the c.m. system plus other quantum numbers.
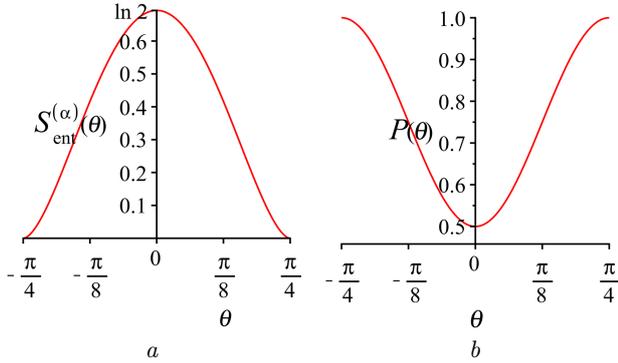
Fig. 1. Entanglement entropy $S_{\text{ent}}^{(\alpha)}$ (a) and purity P versus the parameter θ (b) for each mode $\alpha = 1, 2$

can show that (19) with arbitrary U and V presents the general solution of (13) [4].

Conversely: all the matrices (19) with $\text{Tr}(D_\beta \overline{D_\alpha}) = \delta_{\alpha\beta}$ satisfy system (13) and thus give its general solution. The diagonal elements of D_α are components of the orthonormal vectors λ^α in the complex space:

$$D_\alpha = \text{diag}\{\lambda_1^\alpha, \lambda_2^\alpha, ...\}, \quad (\lambda^\alpha, \lambda^\beta) \equiv \sum_i \lambda_i^\alpha \overline{\lambda_i^\beta} = \delta_{\alpha\beta}. \quad (20)$$

So, the "boson-fermion" entanglement entropy for realized CFs in the αth mode is given along with the orthogonality constraint as

$$S_{\text{entang}}^{(\alpha)} = -\sum_i |\lambda_i^\alpha|^2 \ln |\lambda_i^\alpha|^2, \quad (\lambda^\alpha, \lambda^\beta) = \delta_{\alpha\beta}. \quad (21)$$

Note that, for a parametrization of solution (19), the one of suitable $SU(n)$ can be used: this concerns unitary matrices U, V and diagonal matrices D_α, since their elements (λ_i^α) constitute rows (columns) of a unitary matrix, see (20).

Remark 1. While for the realized composite bosons, the block $U_\alpha(m)$ in (3) is associated with the αth mode, yielding $m^2 - 1$ free *real* parameters in wavefunctions (maximally, $m^2/2$-dimensional *complex* space of states), the realized CFs admit more general wavefunctions. Indeed, the *particular* orthonormal CF wavefunctions

$$\Phi_\alpha = U \, \text{diag}\Big\{0..0, \underbrace{U_\alpha(m) D_m^{(\alpha)} V_\alpha^\dagger(m)}_{GL(m)}, 0..0\Big\} V^\dagger$$

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[4] Indeed, if $\#\alpha = 1$, it is just SVD. Let (19) be valid for $(\#\alpha) - 1$ modes. Then, at $\lambda_{\kappa_r}^{\alpha_1} = 0$, the induction assumption is to be applied to (18).

already have avg. $m^2 - 1$ free *complex* parameters per mode.

5. Cofermions in Low-Mode Cases

Consider, in more details, low-mode subcases with CFs in two modes $\alpha = 1, 2$, and constituents – upto three modes. Recall that the general solution in the case of nondeformed constituent boson was given in Sec. 4.

5(a). Two or three modes of a constituent fermion + usual boson

First, let both constituents be in two modes, i.e. $\mu, \nu = \overline{1,2}$. The solution for realized wavefunctions reads

$$\Phi_\alpha = e^{-i\tilde{\alpha}\psi} U \begin{pmatrix} \cos(\tilde{\alpha}+\theta)e^{i\phi} & 0 \\ 0 & \sin(\tilde{\alpha}+\theta)e^{-i\phi} \end{pmatrix} V^\dagger, \quad (22)$$

where $\tilde{\alpha} = \frac{\pi}{2}(\alpha - \alpha_c) = \mp\frac{\pi}{4}$, $\alpha_c = 3/2$. Due to a low dimensionality of matrices, it is convenient to use the angle parametrization of $SU(2)$. There can be other parametrization as well.

The *entanglement entropy within the CF*, realized by a usual fermion, for each of the two modes is as follows:

$$S_{\text{ent}}^{(\alpha)} \equiv -\sum_k (\lambda_k^\alpha)^2 \ln(\lambda_k^\alpha)^2 \Big|_{\alpha=1,2} = S_2\Big(\frac{\pi}{4}+\theta\Big), \quad (23)$$

where $S_2(x) \equiv -\sin^2 x \ln \sin^2 x - \cos^2 x \ln \cos^2 x$. This result is visualized in Fig. 1, a. The maximum $S_{\text{ent}} = \ln 2$ may correspond to the most tightly bound state of a realized CF. Clearly, $S_{\text{ent}} = 0$ means the opposite, i.e. the most loosely bound one. Another entanglement measure of a CF state, *purity P*, is

$$P|_{\alpha=1,2} \equiv \sum_k (\lambda_k^\alpha)^4 = \frac{1}{4}(3 - \cos 4\theta).$$

The purity ranges from $P = 1/2$ (at $\theta = 0$) to $P = 1$ (at $\theta = \pm\frac{\pi}{4}$), see Fig. 1 b.

Remark 2. Nonseparable CF states with *fixed* intermediate ($0 < S_{\text{ent}} < \ln 2$) entanglement entropy and two respective wavefunctions $\Phi_\alpha^{\mu\nu}$ are parametrized, in total, by 6 independent real parameters.

Non-deformed constituents in three modes

In this case, we take $\alpha = \overline{1,2}$, $\mu, \nu = \overline{1,3}$ so that $\Phi_\alpha^{\mu\nu}$ are some 3×3-matrices. The general solution (19) reads

$$\Phi_\alpha = e^{-i\psi(\alpha)} U \, \text{diag}\{\lambda_1^\alpha, \lambda_2^\alpha, \lambda_3^\alpha\} V^\dagger, \quad U, V \in SU(3),$$

with *complex* λ_k^α, $k = \overline{1,3}$, satisfying the orthonormality conditions (20), and U, V not depending on α.

A parametrization of two orthonormal vectors (λ_k^1) and (λ_k^2) follows from the one of $SU(3)$, since the rows/colums of matrices from $SU(3)$ separately form orthonormal vectors. Indeed, using the parametrization from [25], we obtain $(\alpha = 1 \leftrightarrow \alpha = 2)$ symmetric unified expressions:

$$(\lambda_k^\alpha) = $$
$$= \begin{pmatrix} e^{i\phi_1}\big(\sin\theta_1\cos\theta_2\cos(\tilde\alpha + \theta_3) - \\ - \sin\theta_2\sin(\tilde\alpha + \theta_3)e^{i\phi_2}\big), \\ \cos\theta_1\cos(\tilde\alpha + \theta_3), \\ e^{-i\phi_3}\big(\sin\theta_1\sin\theta_2\cos(\tilde\alpha + \theta_3) + \\ + \cos\theta_2\sin(\tilde\alpha + \theta_3)e^{i\phi_2}\big) \end{pmatrix}, \quad (24)$$

where $0 \leq \theta_1, \theta_2, \theta_3 + \pi/4 \leq \pi/2$, $0 \leq \phi_{1,2,3} \leq 2\pi$. The *entanglement entropy* $S_{\text{ent}}^{(\alpha)}(\theta_1, \theta_2, \theta_3, \phi_2)$ *of a CF* in the αth mode stems from (21) with "symmetrized" squared absolute values of Schmidt coefficients:

$$\big|\lambda_{1,3}^\alpha\big|^2 = \frac{1}{4}(1 + \sin^2\theta_1) \pm \frac{1}{4}\frac{\cos^4\theta_1}{1 + \sin^2\theta_1}\frac{\sin 2(\theta_3^+ - \theta_3^-)}{\sin 2(\theta_3^+ + \theta_3^-)} - $$
$$- \frac{1}{2}\frac{\sin 2\theta_3^\pm}{\sin 2(\theta_3^+ + \theta_3^-)}\cos^2\theta_1\cos 2\big(\theta_3 \mp \theta_3^\mp + \tilde\alpha\big),$$
$$|\lambda_2^\alpha|^2 = \frac{1}{2}(1 - \sin^2\theta_1) + \frac{1}{2}\cos^2\theta_1\cos 2(\theta_3 + \tilde\alpha), \quad (25)$$

where two "shift" angles $\theta_3^\pm$ replace θ_2, ϕ_2, while the upper or lower sign corresponds to the first or, respectively, third Schmidt coefficient. The transition $(\theta_2, \phi_2) \to (\theta_3^-, \theta_3^+)$ is given by the unified formula

$$\text{tg}\, 2\theta_3^\pm = \pm\frac{2\sin\theta_1\,\text{tg}(\theta_2 + \frac{\pi}{2}\delta_{\mp 1,1})\cos\phi_2}{1 - \sin^2\theta_1\,\text{tg}^2(\theta_2 + \frac{\pi}{2}\delta_{\mp 1,1})}.$$

Note that the "shift" is counted from the parameter θ_3 which has a special role being directly *linked* with the mode number α, see (24).

Remark 3. Parametrization *asymmetric* under $(\alpha = 1) \leftrightarrow (\alpha = 2)$, see [25], of the modes can also play a role. From the physics viewpoint, it is possible, when the realization is applied to a system with *ad hoc* asymmetry, for example, due to an applied external asymmetric field or like for *s*- vs. *p*-levels/modes of CF.

Thus, the *CF entanglement entropies* $S_{\text{ent}}^{(1)}$ and $S_{\text{ent}}^{(2)}$ are parametrized by three angles and one phase. Unlike the constituents in the two-mode $\mu, \nu = \overline{1,2}$ case
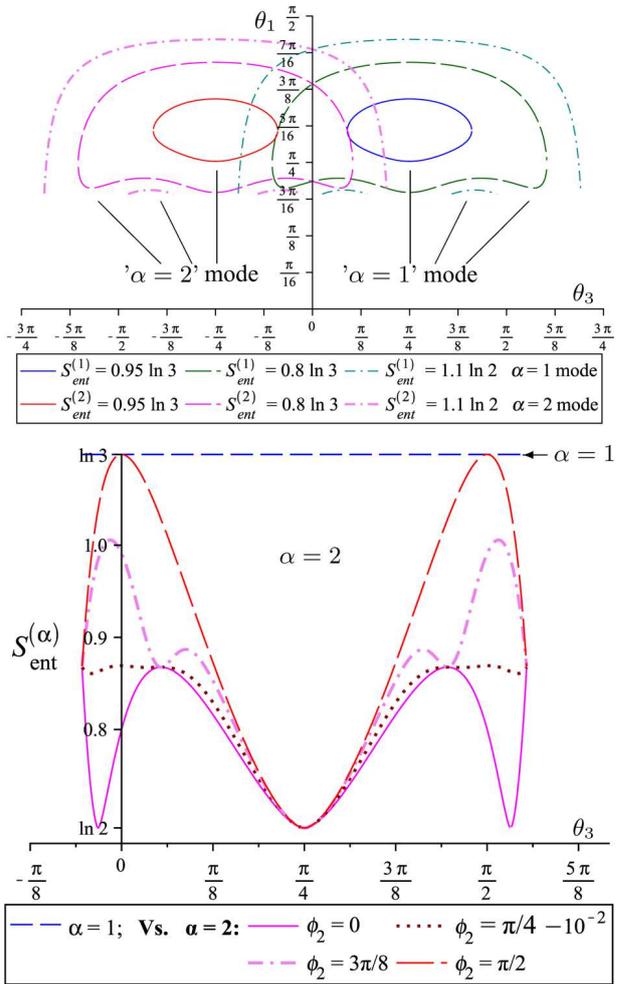
Fig. 2. Upper: Equientropic curves ($S_{\text{ent}}^{(\alpha)} = \text{const}$) versus angles θ_1 and θ_3 at fixed $\theta_3^\mp = \frac{\pi}{3}$. Lower: Entanglement entropy $S_{\text{ent}}^{(2)}(\theta_3, \phi_2)$ for a cofermion in the $\alpha = 2$ mode at the fixed entanglement entropy $S_{\text{ent}}^{(1)} = \max = \ln 3$ for an $\alpha = 1$ cofermion

where $S_{\text{ent}}^{(1)} - S_{\text{ent}}^{(2)} = 0$ and $0 \leq S_{\text{ent}}^{(\alpha)} \leq \ln 2$, now, in the $\mu, \nu = \overline{1,3}$ case, we have $|S_{\text{ent}}^{(1)} - S_{\text{ent}}^{(2)}| \leq \ln 2$, while $0 \leq S_{\text{ent}}^{(\alpha)} \leq \ln 3$. The limits for above differences impose a restriction on the realizable states. To illustrate the dependences $S_{\text{ent}}^{(\alpha)} = S_{\text{ent}}^{(\alpha)}(\theta_1, \theta_3, \theta_3^\mp)$ at C_3-symmetric choice $\theta_3^\mp = \frac{\pi}{3}$, for both modes $\alpha = 1, 2$, Fig. 2 (upper) shows the equientropic curves versus θ_1-, θ_3-angles. The somewhat similar behavior, though for the entropy of mixing [26] within a three-level system, was given in the context of the $SU(3)$ parametrization of qutrits (see [26]).

Observe that, at $\theta_3^{\pm} = \frac{\pi}{3}$ and $\sin^2\theta_1 = \frac{1}{3}$, the entropy for the αth mode acquires a C_3-cyclically symmetric form:

$$S_{\text{ent}}^{(\alpha)} = \sum_{l=-1}^{1} S_1\left(\frac{2}{3}\cos^2\left(\theta_3 + \frac{\pi}{3}l + \tilde{\alpha}\right)\right), \qquad (26)$$

where $S_1(x) \equiv -x\ln x$.

5(b). CFs composed of a fermion plus a deformed boson

Consider the two-mode case ($\alpha = 1, 2$) of CF composed of a usual fermion and a χ-*deformed* boson. The specificity of two modes implies that it is sufficient to consider the realization conditions (7)–(8) on the ground and one-CF states. Then, for the single non-zero two-CF state $A_1^{\dagger}A_2^{\dagger}|0\rangle$, the realization conditions will be automatically satisfied. Denoting $\delta\chi_2 \equiv \chi(2) - 2$, requirement (12) on respective "one-CF states" reduces to two independent equations:

$$L(\Phi_1, \Phi_2) \equiv \Phi_1\Phi_1^{\dagger}\Phi_2 - \Phi_2\Phi_1^{\dagger}\Phi_1 +$$

$$+ \delta\chi_2\left[\text{diag}\{(\Phi_1\Phi_1^{\dagger})^{\mu\mu}\}\Phi_2 - \text{diag}\{(\Phi_2\Phi_1^{\dagger})^{\mu\mu}\}\Phi_1\right] = 0, \qquad (27)$$

$$-L(\Phi_2, \Phi_1) = 0. \qquad (28)$$

Using substitution (15), restricting to $\mu, \nu = 1, 2$, taking

$$U_1 = U(u, v) \equiv \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \in SU(2), \ |u|^2 + |v|^2 = 1,$$

and applying the identity

$$U^{\dagger}\,\text{diag}\{(UXU^{\dagger})^{\mu\mu}\}U = \frac{1}{2}X + \frac{1}{2}RXR,$$

with the Hermitian

$$R = \begin{pmatrix} |u|^2 - |v|^2 & 2\overline{u}v \\ 2u\overline{v} & |v|^2 - |u|^2 \end{pmatrix},$$

we arrive at the matrix equations

$$\chi(2)(D_1^2\tilde{\Phi}_2 - \tilde{\Phi}_2 D_1^2) + \delta\chi_2 R(D_1^2 R\tilde{\Phi}_2 - \tilde{\Phi}_2 D_1 R D_1) = 0, \qquad (29)$$

$$\chi(2)(D_1\tilde{\Phi}_2^{\dagger}\tilde{\Phi}_2 - \tilde{\Phi}_2\tilde{\Phi}_2^{\dagger}D_1) +$$

$$+ \delta\chi_2(R D_1\tilde{\Phi}_2^{\dagger}R\tilde{\Phi}_2 - R\tilde{\Phi}_2\tilde{\Phi}_2^{\dagger}R D_1) = 0. \qquad (30)$$

In view of three-dimensionality of the space of $\tilde{\Phi}_2^{\dagger}$ satisfying the orthogonality condition $\text{Tr}(D_1\tilde{\Phi}_2^{\dagger}) = 0$, we present them as a linear combination:

$$\tilde{\Phi}_2 = x_1\begin{pmatrix} \lambda_2^{(1)} & 0 \\ 0 & -\lambda_1^{(1)} \end{pmatrix} + x_2\begin{pmatrix} 0 & \varkappa\lambda_1^{(1)} \\ \overline{\varkappa}\lambda_2^{(1)} & 0 \end{pmatrix} +$$

$$+ x_3\begin{pmatrix} 0 & -\varkappa\lambda_2^{(1)} \\ \overline{\varkappa}\lambda_1^{(1)} & 0 \end{pmatrix},$$

where $\varkappa = e^{i(\arg v - \arg u)}$. Equation (30) reduces to three linear (in x_1, x_2, x_3) equations, the associated determinant should be zero:

$$\det(...) = -\chi(2)(\chi(2) - 2)|u|^2|v|^2\left(|\lambda_1^1|^2 - |\lambda_2^1|^2\right)^2 = 0.$$

This is possible in the following cases:

a) $\chi(2) = 0$. The solution of (27)–(28) reads

$$\Phi_{\alpha} = U(u, (-1)^{\alpha-1}v)\begin{pmatrix} \cos(\tilde{\alpha} + \theta) & 0 \\ 0 & \sin(\tilde{\alpha} + \theta) \end{pmatrix}V^{\dagger}. \quad (31)$$

b) $\chi(2) = 1$. We find two classes of solutions:

$$\Phi_{\alpha} = \text{diag}\{\cos(\tilde{\alpha} + \theta), \sin(\tilde{\alpha} + \theta)\}V^{\dagger}, \qquad (32)$$

$$\Phi_{\alpha} \equiv (\Phi_{\alpha}^{\mu\nu}) = \left(\delta_{\mu\mu_0}\overline{V_{\nu\alpha}}\right), \quad \text{fixed } \mu_0 = 1 \text{ or } 2. \quad (33)$$

c) $\chi(2) = 2$, i.e. $\delta\chi_2 = 0$ – non-deformed one, see (22).

d) At $\chi(2) \neq 0, 1, 2$, the solution is identical to (32).

So, the *entanglement entropy of the cofermion* containing a deformed boson is either given by the general parameter-dependent expression, see (23), or the constant $S_{\text{ent}} = 0$ in the special case $\chi(2) = 1$, for (33). The deformations $\chi(2) = 0$, 1, or 2 are disjoint from the continuous set. How the deformation parameter $\chi(2)$ is reflected in physical quantities will be analyzed in the subsequent paper.

6. Conclusions

After we settled the problem of realization of composite fermions (CFs) by usual fermions, we explored the bipartite entanglement (inside the CF) measured by the entanglement entropy of CF. The analysis has been performed in relatively simpler cases: *i)* CFs with a non-deformed *constituent boson*, for which we have considered the examples of two and three modes for the both constituents, *ii)* CF containing a *deformed* boson, both in two modes. The resulting expressions are given in (21) in the case of non-deformed

constituents, (23) for CFs and constituents being in two modes, and finally by (24), (26) for two-mode CFs with three-mode constituents.

As is found, for the entanglement entropy of realized CFs of the type "fermion + deformed boson", the constituent boson deformation does not manifest itself in the explicit formulas for the entanglement entropy and purity of CF within each fixed subcase in these one- and two-mode cases. At first sight, this is *in some contrast* to the earlier studied entanglement entropy of quasibosons [7], where the major issue was just the dependence on the deformation parameter f. Therein, all other parameters of the states [like those from $U_1(d_a)$, $U_2(d_b)$, $U_\alpha(m)$ in (3)] did not enter the entanglement measures. In the present case of CFs, the situation is different: the CFs are realized by *nondeformed* fermions, so, the analogous deformation parameter corresponding to CF as a whole is absent. The quantity $\chi(2)$, being related to the deformation parameter(s), in the case of CF has different origin, since it concerns the constituent of CF. Nevertheless, there appear additional parameters present in the matrix Φ of ansatz (1) which, along with $\chi(2)$, define the entanglement entropy of CFs. Thus, these parameters determine the form of CF states (their wavefunctions). This dependence on the involved parameter is shown in Fig. 1. Also noteworthy are the properties of CF entanglement entropy shown in Fig. 2. In that case, the behavior is apparently richer.

Let us note again that this paper presents explicit formulas for the entanglement entropy *inside an individual composite fermion* (i.e. for the interconstituent entanglement), see also Introduction. In contrast, the authors in [20, 21] explored the (statistical) entanglement entropy of *many-fermion systems* that occupy a certain space region. In particular, the efficient numerical methods were applied in [21] to the system of 37 composite fermions, and the linear size of a subsystem entered the final result for the entanglement entropy. While the results of [20] depend explicitly on the space region dimensionality, we operate here with one or more modes irrespectively of a particular space dimensionality.

What about the role of a deformation parameter f in the situation with quasibosons? In that case, we had [7,8] quite natural feature: the entanglement entropy was rising with decreasing values of f, i.e. with approaching the truly bosonic behavior, either for the Fock states at a fixed mode or for the

coherent states. In the present case of CFs, the physical meaning of the parameter(s) which the entanglement entropy and purity depend upon is not clear enough, and that issue deserves the further study. Nevertheless, concerning the considered cases of 2 or 3 modes for the constituents, we may remark the following. Since the only parameters affecting the intercomponent entanglement of CF are θ (the 2-mode case, see (23)) or θ_i, $i = \overline{1,3}$, and ϕ_2 (the 3-mode case, see (24)), they should correspond to internal quantum numbers of CF like spin, parameter(s) of the binding energy of CF, *etc.*

Remark also that the above parameters (like θ) of the realized states can be related to such (rather unexpected) parameters as CF constituents' mass ratio or reduced mass. That concerns, e.g., the trion CF composed of an exciton, modeled by a deformed boson, and an electron or a hole. It is motivated, say, by Fig. 2 in [27], where the trion binding energy depends on the reduced mass of the electron-hole pair, while the extent of bipartite entanglement usually is related to the binding energy of a composite particle. We intend to explore such entanglement-energy relation and other implications elsewhere.

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СКЛАДЕНІ ФЕРМІОНИ
ЯК ДЕФОРМОВАНІ ОСЦИЛЯТОРИ:
ХВИЛЬОВІ ФУНКЦІЇ ТА ЗАПЛУТАНІСТЬ

Р е з ю м е

Складена структура частинок дещо змінює їх статистику порівняно із класичними бозе- та фермістатистиками. Теорема про зв'язок спіну зі статистикою, отже, не виконується. Скажімо, π-мезони, екситони, куперівські пари не є ідеальними бозонами і, подібним чином, баріони не є простими ферміонами. У попередніх статтях ми вивчали двочастинкові *складені бозони* (тобто квазібозони) за допомогою реалізації їх через деформовані осцилятори. Були знайдені такі характеристики міжкомпонентної заплутаності як ентропія заплутаності та чистота (*purity*) в термінах параметра деформації. У цій роботі ми виконуємо аналогічний розгляд складених частинок *фермі-типу* та досліджуємо їх у двох основних випадках: (i) складені ферміони (чи коферміони, чи СФ-и) типу "бозон + ферміон"; (ii) СФ-и типу "деформований бозон + ферміон". Як ми показуємо, коферміони, в обох випадках, допускають реалізацію лише звичайними ферміонами. Випадок (i) розглянуто повністю та знайдено хвильові функції разом із мірами заплутаності. Випадок (ii) розглянуто в межах декількох мод, як для СФ-ів так і складових. Ентропію заплутаності та "п'юріті" визначено через задіяні параметри і проілюстровано графічно.