https://doi.org/10.15407/ujpe64.12.1103

V.M. FEDORCHUK,[1,2] V.I. FEDORCHUK[2]

[1] Institute of Mathematics, Pedagogical University
(2, Podchorążych Str., Cracow 30-084, Poland; e-mail: vasyl.fedorchuk@up.krakow.pl)
[2] Ya.S. Pidstryhach Institute for Applied Problems of Mechanics and Mathematics,
Nat. Acad. of Sci. of Ukraine
(3b, Naukova Str., Lviv 79060, Ukraine)

ON THE CLASSIFICATION OF SYMMETRY REDUCTIONS AND INVARIANT SOLUTIONS FOR THE EULER–LAGRANGE–BORN–INFELD EQUATION[1]

We study a connection between the structural properties of the low-dimension ($\dim L \leq 3$) nonconjugate subalgebras of the Lie argebra of the generalized Poincaré group $P(1,4)$ and the results of symmetry reductions for the Euler–Lagrange–Born–Infeld equation. We have performed the classification of nonsingular manifolds in the space $M(1,3) \times R(u)$ invariant with respect to three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$. The results are used for the classification of symmetry reductions and invariant solutions of the Euler–Lagrange–Born–Infeld equation.

Keywords: structural properties of Lie argebras, nonsingular manifolds, classification of symmetry reductions, invariant solutions, Poincaré group $P(1,4)$, Euler–Lagrange–Born–Infeld equation.

1. Introduction

The symmetry reduction is one of the most powerful tools for the investigation of PDEs with nontrivial symmetry groups.

In what follows, we focus our attention on applications of the classical Lie method for symmetry reductions and the construction of invariant solutions of PDEs with non-trivial symmetry groups.

In 1895, S. Lie [1] considered solutions invariant with respect to groups admitted by higher-order PDEs.

According to the classical group analysis, the main classification of symmetry reductions and invariant solutions for PDEs with nontrivial symmetry groups should be performed by using ranks of nonconjugate subalgebras of Lie algebras of symmetry groups of the equations under investigations [2, 3] (see also the references therein). In this approach, the invariant solutions of PDEs with nontrivial symmetry groups are nonsingular manifolds, which are invariant with respect to those nonconjugate subalgebras. Therefore, the clasification of invariant solutions reduces to the classification of the corresponding nonsingular manifolds.

However, it turned out that the reduced equations obtained with the help of nonsingular manifolds, which are invariant with respect to nonconjugate subalgebras of the same ranks of the Lie algebras of the symmetry groups of some PDEs, were of different types. Grundland, Harnad, and Winternitz [4] were the first who pointed out and investigated the similar phenomenon. The details on this theme can be found in [5–13] (see also the references therein).

To try to explain some differences in the properties of the reduced equations and invariant solutions, which are obtained by using nonconjugate subalgebras of the same ranks of the Lie algebras of the symmetry groups of the PDEs under consideration, we recently suggested to use the structural properties of those nonconjugate subalgebras. The details on this theme can be found in [10, 12, 13] (see also the references therein).

[1] This work is based on the results presented at the XI Bolyai–Gauss–Lobachevskii (BGL-2019) Conference: Non–Euclidean, Noncommutative Geometry and Quantum Physics.

At the present time, the relationship between the structural properties of the three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ and the properties of the reduced equations for the Euler–Lagrange–Born–Infeld equation has been investigated. We obtained the following types of the reduced equations:

- identities;
- linear ordinary differential equations;
- nonlinear ordinary differential equations;
- partial differential equations.

It should be noted that, from the invariants of some nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$, it is impossible to construct the ansätze which reduce the Euler–Lagrange–Born–Infeld equation.

In this paper, we focus our attention on the reduction of the Euler–Lagrange–Born–Infeld equation to identities. More precisely, we only present the results of the symmetry reduction for those types of subalgebras, which provide us reductions to identities.

2. Lie Algebra of the Poincaré Group $P(1,4)$ and Its Nonconjugate Subalgebras

The group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. It is the smallest group which contains, as subgroups, the extended Galilei group $\widetilde{G}(1,3)$ [14] (the symmetry group of classical physics) and the Poincaré group $P(1,3)$ (the symmetry group of relativistic physics).

The Lie algebra of the group $P(1,4)$ is generated by 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}(\mu,\nu = 0,1,2,3,4)$ and $P_\mu(\mu = 0,1,2,3,4)$ which satisfy the commutation relations

$$[P_\mu, P_\nu] = 0,$$

$$[M_{\mu\nu}, P_\sigma] = g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu,$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$.

Here, we consider the following representation [15] of the Lie algebra of the group $P(1,4)$:

$$P_0 = \frac{\partial}{\partial x_0},\ P_1 = -\frac{\partial}{\partial x_1},\ P_2 = -\frac{\partial}{\partial x_2},\ P_3 = -\frac{\partial}{\partial x_3},$$

$$P_4 = -\frac{\partial}{\partial u},\ M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu,\ x_4 \equiv u.$$

Below, we will use the following basis elements:

$$G = M_{04},\ L_1 = M_{23},\ L_2 = -M_{13},\ L_3 = M_{12},$$

$$P_a = M_{a4} - M_{0a},\ C_a = M_{a4} + M_{0a},\ X_0 = \frac{1}{2}(P_0 - P_4),$$

$$X_k = P_k,\quad X_4 = \frac{1}{2}(P_0 + P_4),\quad (a,k = 1,2,3).$$

The nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ have been described in works [16–18].

The Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ is generated by the basis elements

$$L_1,\ L_2,\ L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4.$$

The classification of all nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ of dimensions ≤ 3 was performed in [19].

3. On the Classification of Symmetry Reductions for the Euler–Lagrange–Born–Infeld Equation

The Born–Infeld equations in the spaces of various dimensions and various types have many applications in the fluid dynamics, theory of continuous medium, general theory of relativity, field theory, theory of minimal surfaces, nonlinear electrodynamics, theory of conservation laws, *etc.* More details on this theme can be found in [20–24] (see also the references therein).

Let us consider the Euler–Lagrange–Born–Infeld equation of the form

$$\Box u\,(1 - u_\nu u^\nu) + u^\mu u^\nu u_{\mu\nu} = 0,$$

where

$$u = u(x),\quad x = (x_0, x_1, x_2, x_3) \in M(1,3),$$

$$u_\mu \equiv \frac{\partial u}{\partial x^\mu},\quad u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x^\mu \partial x^\nu},\quad u^\mu = g^{\mu\nu}u_\nu,$$

and

$$g_{\mu\nu} = (1,-1,-1,-1)\delta_{\mu\nu},\quad \mu,\nu = 0,1,2,3,$$

$\Box$ is the d'Alembert operator.

In 1984, Fushchich and Serov [15] studied the symmetry properties and constructed some classes of exact solutions for the multidimensional nonlinear Euler–Lagrange equation. It follows from [15] that the Lie algebra of the symmetry group of the Euler–Lagrange–Born–Infeld equation contains, as subalgebra, the Lie algebra of the Poincaré group $P(1,4)$. As

ISSN 2071-0194. Ukr. J. Phys. 2019. Vol. 64, No. 12

we wrote earlier, the results of the classification of all the low-dimensional ($\dim L \leq 3$) nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ can be found in [19].

In order to classify symmetry reductions and invariant solutions for the Euler–Lagrange–Born–Infeld equation, we need the classification of nonsingular manifolds in the space $M(1,3) \times R(u)$ invariant with respect to nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$.

Till now, we have performed the classification of nonsingular manifolds in the space $M(1,3) \times R(u)$ invariant with respect to three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ and have used the results for the classification of symmetry reductions and invariant solutions for the Euler–Lagrangev–Born–Infeld equation. As we wrote above, we will focus our attention on the reduction of the Euler–Lagrange–Born–Infeld equation to identities. Therefore, we only present the results of the symmetry reduction for those types of subalgebras which provide us reductions to identities.

Now, we present some of the results obtained.

3.1. Lie algebras of the type $3A_1$

Taking the invariants of nine subalgebras into account, we constructed the ansätze which reduce the Euler–Lagrange–Born–Infeld equation to identities.

Below, we present some of the results obtained.

1. $\langle P_1 - \gamma X_3, \gamma > 0\rangle \oplus \langle P_2 - X_2 - \delta X_3, \delta \neq 0\rangle \oplus \langle X_4\rangle$:

Ansatz

$$x_3(x_0+u)^2 - (\gamma x_1 + x_2\delta - x_3)(x_0+u) - \gamma x_1 = \varphi(\omega),$$
$$\omega = x_0 + u.$$

A solution of the Euler–Lagrange–Born–Infeld equation has the form

$$x_3(x_0+u)^2 - (\gamma x_1 + x_2\delta - x_3)(x_0+u) - \gamma x_1 = \varphi(x_0+u),$$

where φ is an arbitrary smooth function.

2. $\langle P_1 - \gamma X_3, \gamma > 0\rangle \oplus \langle P_2 - X_2\rangle \oplus \langle X_4\rangle$:

Ansatz

$$x_3(x_0+u)^2 - (\gamma x_1 - x_3)(x_0+u) - \gamma x_1 = \varphi(\omega),$$
$$\omega = x_0 + u.$$

A solution of the Euler–Lagrange–Born–Infeld equation has the form

$$x_3(x_0+u)^2 - (\gamma x_1 - x_3)(x_0+u) - \gamma x_1 = \varphi(x_0+u),$$

where φ is an arbitrary smooth function.

3. $\langle P_1 - X_3\rangle \oplus \langle P_2\rangle \oplus \langle X_4\rangle$:

Ansatz

$$x_3 - \frac{x_1}{x_0+u} = \varphi(\omega), \quad \omega = x_0 + u.$$

A solution of the Euler–Lagrange–Born–Infeld equation has the form

$$x_3 - \frac{x_1}{x_0+u} = \varphi(x_0+u),$$

where φ is an arbitrary smooth function.

As we see in the above-presented cases, ansätze (1)–(3) (nonsingular manifolds invariant with respect to the corresponding subalgebras) are the solutions of the Euler–Lagrange–Born–Infeld equation.

It should be noted that subalgebras (1)–(3) belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

3.2. Lie algebras of the type $A_{3.1}$

Taking the invariants of ten nonconjugate subalgebras into account, we constructed the ansätze which reduced the the Euler–Lagrange–Born–Infeld equation to identities.

Below, we present some of the results obtained.

1. $\langle 4X_4,\ P_1 - X_2 - \gamma X_3,\ P_2 + X_1 - \mu X_2 - \delta X_3,\ \gamma > 0,\ \delta \neq 0,\ \mu > 0\rangle$:

Ansatz

$$x_3(x_0+u)^2 - (\gamma x_1 + x_2\delta - \mu x_3)(x_0+u) + (\delta - \gamma\mu)x_1 - x_2\gamma + x_3 = \varphi(\omega), \quad \omega = x_0 + u.$$

A solution of the Euler–Lagrange–Born–Infeld equation has the form

$$x_3(x_0+u)^2 - (\gamma x_1 + x_2\delta - \mu x_3)(x_0+u) + (\delta - \gamma\mu)x_1 - x_2\gamma + x_3 = \varphi(x_0+u),$$

where φ is an arbitrary smooth function.

2. $\langle 4X_4, P_1 - X_2 - \gamma X_3, P_2 + X_1 - \mu X_2, \gamma > 0, \mu > 0\rangle$:

Ansatz

$$x_3(x_0+u)^2 - (\gamma x_1 - \mu x_3)(x_0+u) - \gamma\mu x_1 - x_2\gamma +$$

$+ x_3 = \varphi(\omega), \quad \omega = x_0 + u.$

A solution of the Euler–Lagrange–Born–Infeld equation has the form

$$x_3(x_0+u)^2 - (\gamma x_1 - \mu x_3)(x_0+u) - \gamma\mu x_1 - x_2\gamma + $$
$$+ x_3 = \varphi(x_0+u),$$

where φ is an arbitrary smooth function.

3. $\langle 2\mu X_4,\ P_3 - X_2,\ X_1 + \mu X_3, \mu > 0\rangle$:

Ansatz

$$x_2 - \frac{x_3 - \mu x_1}{x_0 + u} = \varphi(\omega), \quad \omega = x_0 + u.$$

A solution of the Euler–Lagrange–Born–Infeld equation has the form

$$x_2 - \frac{x_3 - \mu x_1}{x_0 + u} = \varphi(x_0 + u),$$

where φ is an arbitrary smooth function.

As we see, in the above-presented cases, ansätze (1)–(3) (nonsingular manifolds invariant with respect to the corresponding subalgebras) are the solutions of the Euler–Lagrange–Born–Infeld equation.

It should be noted that subalgebras (1)–(3) belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

3.3. Lie algebras of the type $A_{3,6}$

Taking the invariants of two nonconjugate subalgebras into account, we constructed ansätze which reduced the Euler–Lagrange–Born–Infeld equation to identities.

Below, we present some of the result obtained.

$\langle X_1,\ -X_2,\ -(L_3 + 2X_4)\rangle$:

Ansatz

$$x_0 + u = \varphi(\omega), \quad \omega = x_3.$$

A solution of the Euler–Lagrange–Born–Infeld equation has the form

$$x_0 + u = \varphi(x_3),$$

where φ is an arbitrary smooth function.

As we see in the above-presented case, the ansatz (nonsingular manifold, invariant with respect to the corresponding subalgebra) is a solution of the Euler–Lagrange–Born–Infeld equation.

It should be noted that the subalgebra belongs to the Lie algebra of the extended Galilei group $\widetilde{G}(1.3) \subset \subset P(1.4)$.

4. Conclusions

We have performed the classification of nonsingular manifolds in the space $M(1,3) \times R(u)$ invariant with respect to three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ and have used the results for the classification of symmetry reductions and invariant solutions for the Euler–Lagrange–Born–Infeld equation.

We have focused our attention on the classification of symmetry reductions of the Euler–Lagrange–Born–Infeld equation to identities. More precisely, we have presented only the results of the symmetry reduction for those types of three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ which give reductions to identities. It is known [19] that the Lie algebra of the group $P(1,4)$ contains three-dimensional nonconjugate subalgebras of the following types: $3A_1$, $A_2 \oplus A_1$, $A_{3.1}$, $A_{3.2}$, $A_{3.3}$, $A_{3.4}$, $A_{3.6}$, $A^a_{3.7}$, $A_{3.8}$, and $A_{3.9}$.

From the results obtained, it follows that all above-presented symmetry reductions of the Euler–Lagrange–Born–Infeld equation to identities can be obtained using some subalgebras of the following types: $3A_1$, $A_{3.1}$, $A_{3.6}$.

1. S. Lie. Zur allgemeinen Theorie der partiellen Differentialgleichungen beliebiger Ordnung. *Leipz. Berichte*, I. 53 (Reprinted in S. Lie. *Gesammelte Abhandlungen*, **4**, Paper IX) (1895).
2. L.V. Ovsiannikov. *Group Analysis of Differential Equations* (Academic Press, 1982) [ISBN: 0-12-531680-1].
3. P.J. Olver. *Applications of Lie Groups to Differential Equations* (Springer, 1986).
4. A.M. Grundland, J. Harnad, P. Winternitz. Symmetry reduction for nonlinear relativistically invariant equations. *J. Math. Phys.* **25**, 791 (1984).
5. V.M. Fedorchuk, I.M. Fedorchuk, O.S. Leibov. Reduction of the Born–Infeld, the Monge–Ampere and the eikonal equation to linear equations. *Dokl. Akad. Nauk Ukr.*, No. 11, 24 (1991).
6. V. Fedorchuk. Symmetry reduction and exact solutions of the Euler–Lagrange–Born–Infeld, the multidimensional Monge–Ampère and the eikonal equations. *J. Nonlinear Math. Phys.* **2**, 329 (1995).
7. V.M. Fedorchuk. Symmetry reduction and some exact solutions of a nonlinear five-dimensional wave equation. *Ukr. Math. J.* **48**, 636 (1996).
8. A.G. Nikitin, O. Kuriksha. Group analysis of equations of axion electrodynamics. In: *Group Analysis of Differential Equations and Integrable Systems* (University of Cyprus, 2011), pp. 152–163.

9. A.G.Nikitin, O. Kuriksha. Invariant solutions for equations of axion electrodynamics. *Commun. Nonlinear Sci. Numer. Simul.* **17**, 4585 (2012).
10. V. Fedorchuk, V. Fedorchuk. On classification of symmetry reductions for the eikonal equation. *Symmetry* **8** (6), 51 (2016).
11. A.M. Grundland, A. Hariton. Algebraic aspects of the supersymmetric minimal surface equation. *Symmetry* **9** (12), 318 (2017).
12. V. Fedorchuk, V. Fedorchuk. On classification of symmetry reductions for partial differential equations. In: *Collection of the Scientific Works Dedicated to the 80th Anniversary of B.J. Ptashnyk* (Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of NAS of Ukraine, 2017), pp. 241-255 [ISBN 978-966-02-8315-2].
13. V. Fedorchuk, V. Fedorchuk. *Classification of Symmetry Reductions for the Eikonal Equation* (Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of National Academy of Sciences of Ukraine, 2018) [ISBN: 978-966-02-8468-5].
14. W.I. Fushchich, A.G. Nikitin. Reduction of the representations of the generalized Poincaré algebra by the Galilei algebra. *J. Phys. A: Math. and Gen.* **13**, 2319 (1980).
15. V.I. Fushchich, N.I. Serov. Some exact solutions of the multidimensional nonlinear Euler–Lagrange equation. *Dokl. Akad. Nauk SSSR* **278**, 847 (1984) (in Russian).
16. V.M. Fedorchuk. Splitting subalgebras of the Lie algebra of the generalized Poincaré group $P(1,4)$. *Ukr. Math. J.* **31**, 554 (1979).
17. V.M. Fedorchuk. Nonsplitting subalgebras of the Lie algebra of the generalized Poincaré group $P(1,4)$. *Ukr. Math. J.* **33**, 535 (1981).
18. W.I. Fushchich, A.F. Barannik, L.F. Barannik, V.M. Fedorchuk. Continuous subgroups of the Poincarré group $P(1,4)$. *J. Phys. A: Math. and Gen.* **18**, 2893 (1985).
19. V.M. Fedorchuk, V.I. Fedorchuk. On classification of the low-dimension nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$. *Proc. of the Inst. of Math. of NAS of Ukraine* **3**, 302 (2006).
20. M. Born. On the quantum theory of electromagnetic field. *Proc. Royal Soc. A* **143**, 410 (1934).
21. M. Born, L. Infeld. Foundations of the new field theory. *Proc. Royal Soc. A* **144**, 425 (1934).
22. N.A. Chernikov. Born–Infeld equations in Einstein's unified field theory. *Probl. Teor. Gravit. Élement. Chast.*, 130 (1978) (in Russian).
23. M. Kõiv, V. Rosenhaus. Family of two-dimensional Born–Infeld equations and a system of conservation laws. *Eesti NSV Tead. Akad. Toimetised Füüs. – Mat.(Izv. Akad. Nauk Est. SSR. Fizika, Matematika)* **28**, 187 (1979) (in Russian).
24. N.S. Shavokhina. Minimal surfaces and nonlinear electrodynamics. In: *Selected Topics in Statistical Mechanics* (World Sci. Publ., 1990).

Received 10.09.19

В.М. Федорчук, В.І. Федорчук

ПРО КЛАСИФІКАЦІЮ СИМЕТРІЙНИХ РЕДУКЦІЙ ТА ІНВАРІАНТНИХ РОЗВ'ЯЗКІВ РІВНЯННЯ ОЙЛЕРА–ЛАГРАНЖА–БОРНА–ІНФЕЛЬДА

Р е з ю м е

Вивчається зв'язок між структурними властивостями низьковимірних (dim$L \leq 3$) неспряжених підалгебр алгебри Лі узагальненої групи Пуанкаре $P(1,4)$ і результатами симетрійних редукцій для рівняння Ойлера–Лагранжа–Борна–Інфельда. Проведено класифікацію несингулярних многовидів в просторі $M(1,3)\times R(u)$, інваріантних відносно тривимірних неспряжених підалгебр алгебри Лі групи $P(1,4)$, і отримані результати використано для класифікації симетрійних редукцій та інваріантних розв'язків рівняння Ойлера–Лагранжа–Борна–Інфельда.