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REGGE CUTS AND NNLLA BFKL

In the leading and next-to-leading logarithmic approximations, QCD amplitudes with gluon quantum numbers in cross-channels and negative signature have the pole form corresponding to a reggeized gluon. The famous BFKL equation was derived using this form. In the next-to-next-to-leading approximation (NNLLA), the pole form is violated by contributions of Regge cuts. We discuss these contributions and their impact on the derivation of the BFKL equation in the NNLLA.
Keywords: gluon Reggeization, BFKL equation, Regge cuts.

1. Introduction

The equation, which is called now BFKL (Balitskii–Fadin–Kuraev–Lipatov), was first derived in non-Abelian gauge theories with spontaneously broken symmetry [1–3]. Then its applicability to QCD was shown in [4]. The derivation of the equation was based on the Reggeization of gauge bosons in non-Abelian gauge theories (gluons in QCD). The Reggeization determines the high-energy behavior of cross-sections non-decreasing, as the energy increases. In the Regge and multi-Regge kinematics in each order of perturbation theory, dominant (having the largest $\ln s$ degrees) are the amplitudes with gluon quantum numbers and negative signatures in cross-channels. They determine the s-channel discontinuites of amplitudes with the same and all other possible quantum numbers.

It is extremely important that, both in the leading logarithmic approximation (LLA) and in the next-to-leading one (NLLA), the amplitudes used in the unitarity relations are determined by the Regge pole contributions and have a simple factorized form (pole Regge form). Due to this, the Reggeization provides a simple derivation of the BFKL equation in the LLA and in the NLLA. The s-channel discontinuities are presented by Fig. 1 and symbolically can be written as $\Phi_{A'A} \otimes G \otimes \Phi_{B'B}$, where the impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ describe the transitions $A \to A'$ and $B \to B'$, G is Green's function for two interacting Reggeized gluons, $\hat{\mathcal{G}} = e^{Y\hat{\mathcal{K}}}$, $Y = \ln(s/s_0)$, $\hat{\mathcal{K}}$ is the universal (process-independent) BFKL kernel, which determines the energy dependence of scattering amplitudes and is expressed through the gluon trajectory and the Reggeon vertices. Validity of the pole Regge form is proved now in all orders of perturbation theory in the coupling constant g both in the LLA [5], and in the NLLA (see [6, 7] and references therein).

The first observation of the violation of the pole Regge form was done [8] in the high-energy limit of the results of direct two-loop calculations of the two-loop amplitudes for gg, gq, and qq scattering. Then the terms breaking the pole Regge form in two- and three-loop amplitudes of the elastic scattering were found in [9–11] using the techniques of infrared factorization.

It is worth to say that, in general, the breaking of the pole Regge form is not a surprise. It is well known that Regge poles in the complex angular momenta plane generate Regge cuts. Moreover, in amplitudes with positive signature, the Regge cuts appear already in the LLA. In particular, the BFKL Pomeron is the two-Reggeon cut in the complex angular momenta plane. But, in amplitudes with negative signature due to the signature conservation, a cut must be at least three-Reggeon one and can appear only in the NNLLA. It is natural to expect that the observed violation of the pole Regge form can be explained by their contributions.

Indeed, all known cases of breaking the pole Regge form are now explained by the three-Reggeon cuts

[12, 13]. Unfortunately, the approaches used and the explanations given in these papers are different. Their results coincide in the three-loop approach, but may diverge for more loops. It requires a further investigation.

Here, we consider the contributions of a three-Reggeon cut to the amplitudes of elastic scattering of partons (quarks and gluons) with negative signature up to four loops.

2. Three-Reggeon Cut

Since our Reggeon is the Reggeized gluon, the three-Reggeon cut first contributes to the amplitudes corresponding to the diagrams shown in Fig. 2. In contrast to the Reggeon which contribute only to amplitudes with the adjoint representation of the color group (color octet in QCD) in the t-channel, the cut can contribute to various representations. Possible representations for the quark-quark and quark-gluon scatterings are only singlet (**1**) and octet (**8**), whereas, for the gluon-gluon scattering, there are also **10**, **10***, and **27**. The account for the Bose statistics for gluons, symmetry of the representations **1** and **27**, antisymmetry **10** and **10***, and the existence of both symmetric $\mathbf{8_s}$ and antisymmetric $\mathbf{8_a}$ representations for them, gives that, in addition to the Reggeon channel, the amplitudes with negative signature are in the representations **1** for the quark-quark-scattering and in the representation **10** and **10*** for the gluon-gluon scattering. The amplitude of the process $\mathcal{A}_{AB}^{A'B'}$ depicted by the diagrams in Fig. 2 can be written as the sum over the permutations σ of products of color factors and color-independent matrix elements:

$$\mathcal{A}_{AB}^{A'B'} = \sum_{\sigma} \left(C_{AB}^{(0)\sigma}\right)^{\alpha\beta}_{\alpha'\beta'} M_{AB}^{(0)\sigma}(s,t), \tag{1}$$

where α and β (α' and β') are the color indices of an incoming (outgoing) projectile A and a target B, respectively. We use the same letters for the quark and gluon color indices; it should be remembered, however, that there is no difference between upper and lower indices (running from 1 to N_c^2-1) for gluons, whereas, for quarks, lower and upper indices (running from 1 to N_c) refer to mutually related representations.

The color factors can be decomposed into irreducible representations $\mathcal{R}$ of the color group in the t-channel:

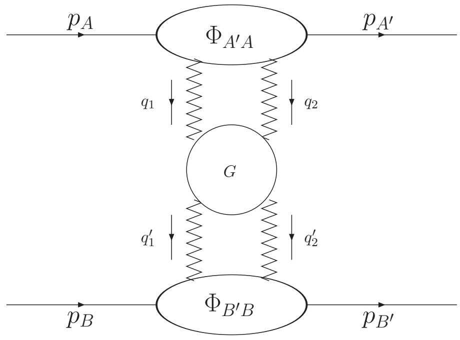


Fig. 1. Schematic representation of the s-channel discontinuites of amplitudes $A+B\to A'+B'$

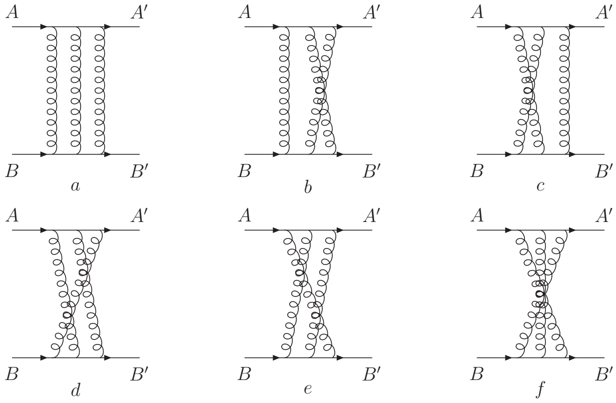


Fig. 2. Feynman diagrams of the process $A+B\to A'+B'$ with three-gluon exchanges

$$\left(C_{AB}^{(0)\sigma}\right)^{\alpha\beta}_{\alpha'\beta'} = \sum_R [\mathcal{P}_{AB}^R]^{\alpha\beta}_{\alpha'\beta'} \sum_\sigma \mathcal{G}(R)_{AB}^{(0)\sigma}, \tag{2}$$

where

$$[\mathcal{P}_{AB}^R]^{\alpha\beta}_{\alpha'\beta'} = \sum_n [\mathcal{P}_A^{R,n}]^{\alpha}_{\alpha'} [\mathcal{P}_B^{R,n}]^{\beta}_{\beta'}, \tag{3}$$

$\hat{\mathcal{P}}^{R,n}$ is the projection operator on the state n in the representation $\mathcal{R}$,

$$\mathcal{G}(R)_{AB}^{(0)\sigma} = \frac{1}{N_R\, T_A\, T_B} \left(\mathcal{T}_A^{c_1}\mathcal{T}_A^{c_2}\mathcal{T}_A^{c_3}\right)^{\alpha'}_{\alpha} \times$$
$$\times \left(\mathcal{T}_B^{c_1^\sigma}\mathcal{T}_B^{c_2^\sigma}\mathcal{T}_B^{c_3^\sigma}\right)^{\beta'}_{\beta} [\mathcal{P}_{AB}^R]^{\alpha\beta}_{\alpha'\beta'}, \tag{4}$$

N_R is the dimension of the representation R, $\mathcal{T}^a$ are the color group generators in the corresponding representations, $[\mathcal{T}^a,\mathcal{T}^b] = if_{abc}\mathcal{T}^c$; $(\mathcal{T}^a)^{\alpha'}_{\alpha} = -if_{\alpha'\alpha}$ for

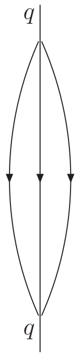
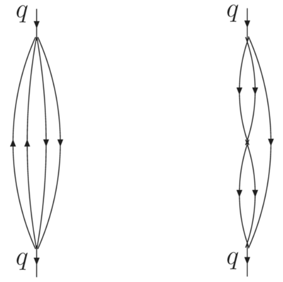


Fig. 3. Schematic representation of $A_2(q_\perp)$



a *b*

Fig. 4. Schematic representation of $A_3^a(q_\perp)$ and $A_3^b(q_\perp)$

gluons and $(\mathcal{T}^a)^{\alpha'}_{\alpha} = (t^a)^{\alpha'}_{\alpha}$ for quarks; $\mathrm{Tr}(\mathcal{T}_i^a \mathcal{T}_i^b) = T_i \delta_{ab}$, $T_q = 1/2$, $T_g = N_c$.

In [12] the Reggeon channel (R=8) was considered. It was discovered that that the terms violating the pole factorization in $\mathcal{G}(8)^{(0)\sigma}_{AB}$ do not depend on σ (let us call them $\mathcal{G}(8)^{(0)}_{AB}$), so that the momentum-dependent factors for them are summed up to the eikonal amplitude

$$\sum_\sigma M^{(0)\sigma}_{AB} = A^{eik} = g^6 \frac{s}{t}\left(\frac{-4\pi^2}{3}\right)\mathbf{q}^{\,2} A_2(q_\perp), \tag{5}$$

where $A_2(q_\perp)$ is depicted by the diagram presented in Fig. 3 and is written as

$$A_2(q_\perp) = \int \frac{d^{2+2\epsilon} l_1 d^{2+2\epsilon} l_2}{(2\pi)^{2(3+2\epsilon)} \mathbf{l}_1^2 \mathbf{l}_2^2 (\mathbf{q} - \mathbf{l}_1 - \mathbf{l}_2)^2}. \tag{6}$$

Note that we use the "infrared" ϵ, $\epsilon = (D-4)/2$, D is the space-time dimension.

This result is very important, because the contribution of the cut must be gauge-invariant, whereas $M^{(0)\sigma}_{AB}$ taken separately are gauge-dependent.

In [14], other channels with possible cut contributions were considered. It was shown that, for them, the color coefficients $\mathcal{G}(R)^{(0)\sigma}_{AB}$ do not depend on σ,

$$\mathcal{G}(10+\bar{1}0)^{(0)}_{gg} = \frac{-3}{4} N_c, \quad \mathcal{G}(1)^{(0)}_{qq} = \frac{(N_c^2-4)(N_c^2-1)}{16 N_c^3}, \tag{7}$$

so that the momentum-dependent factors for them are also summed up to the eikonal amplitude (5).

The separation of the pole and cut contributions in the octet channel is impossible in the two-loop approximation, because of the ambiguity of the allocation of parts of the amplitudes violating the factorization. The separation becomes possible for higher loops, due to the different energy dependences of the pole and cut contributions. The energy dependence of the pole contribution is determined by the Regge factor of a Reggeized gluon $\exp(\omega(t)\ln s)$, where $\omega(t)$ is the gluon trajectory, whereas, for the three-Reggeon cut, it is

$$e^{[(\hat{\omega}_1 + \hat{\omega}_2 + \hat{\omega}_3 + \hat{\mathcal{K}}_r(1,2) + \hat{\mathcal{K}}_r(1,3) + \hat{\mathcal{K}}_r(2,3)) \ln s]}, \tag{8}$$

where $\hat{\mathcal{K}}_r(m,n)$ is the real part of the BFKL kernel describing the interaction between Reggeons m and n.

The calculations of the first logarithmic correction to the cut contribution in the octet channel was performed in [12, 14, 15] and, in the other channels, in [14]. In the latter case, the correction is

$$\mathcal{G}(10+\bar{1}0)^{(0)}_{gg}\, g^6 \frac{s}{t}\left(\frac{-4\pi^2}{3}\right)\mathbf{q}^{\,2} g^2 N_c \times \\ \times \ln s \left(-\frac{1}{2} A_3^a(q_\perp) - \frac{1}{2} A_3^b(q_\perp)\right), \tag{9}$$

$$\mathcal{G}(1)^{(0)}_{qq}\, g^6 \frac{s}{t}\left(\frac{-4\pi^2}{3}\right)\mathbf{q}^{\,2} g^2 N_c \times \\ \times \ln s \left(\frac{3}{2} A_3^a(q_\perp) - \frac{3}{2} A_3^b(q_\perp)\right), \tag{10}$$

and in the first case as

$$\mathcal{G}(8)^{(0)}_{AB}\, g^6 \frac{s}{t}\left(\frac{-4\pi^2}{3}\right)\mathbf{q}^{\,2} g^2 N_c \times \\ \times \ln s \left(\frac{1}{2} A_3^a(q_\perp) - A_3^b(q_\perp)\right), \tag{11}$$

where $A_3^a(q_\perp)$ and $A_3^b(q_\perp)$ are depicted by the diagrams presented in Figs. 4, *a* and 4, *b*, respectively,

$$A_3^a(q_\perp) = \int \frac{d^{2+2\epsilon} l_1\, d^{2+2\epsilon} l_2 d^{2+2\epsilon} l_3}{(2\pi)^{3(3+2\epsilon)} \mathbf{l}_1^2 \mathbf{l}_2^2 \mathbf{l}_3^2 (\mathbf{q} - \mathbf{l}_1 - \mathbf{l}_2 - \mathbf{l}_3)^2}, \tag{12}$$

$$A_3^b(q_\perp) = = \int \frac{d^{2+2\epsilon}l_1\, d^{2+2\epsilon}l_2 d^{2+2\epsilon}l_3 (\mathbf{q}-l_1)^2}{(2\pi)^{3(3+2\epsilon)}\mathbf{l}_1^2\mathbf{l}_2^2\mathbf{l}_3^2(\mathbf{q}-\mathbf{l}_1-\mathbf{l}_2)^2(\mathbf{q}-\mathbf{l}_1-\mathbf{l}_3)^2}. \quad (13)$$

It was shown in [12, 14, 15] that the violation of the pole Regge form, analyzed in this approximation in [9]–[11] with the help of the infrared factorization, can be explained by the pole and cut contributions. In other words, the restrictions imposed by the infrared factorization on the parton scattering amplitudes with the adjoint representation of the color group in the t-channel and negative signature can be fulfilled in the NNLLA with two and three loops, if, in addition to the Regge pole contribution, there is the Regge cut contribution. It should be noted that this result is limited to three loops and cannot be considered as a proof that, in the NNLLA, the only singularities in the J plane are the Regge pole and the three-Reggeon cut. Moreover, the explanation of the violation of the pole Regge form given in [13] differs from that described above. In this paper, in addition to the cut with the vertex of interaction with particles i having the color structure

$$(C^{(0)c})^{\alpha}_{\alpha'} = (\mathcal{T}^c)^{\alpha}_{\alpha'}\frac{1}{3!}\,\mathrm{Tr}\sum_{\sigma}\left(\mathcal{T}_i^{c_1^\sigma}\mathcal{T}_i^{c_2^\sigma}\mathcal{T}_i^{c_3^\sigma}\mathcal{T}_i^{c}\right), \quad (14)$$

the Reggeon-cut mixing is introduced. Actually, in the three-loop approximation, the mixing is not required.

Whether the mixing is necessary can be verified in the four-loop approximation.

The four-loop calculations should answer the questions whether the existence of a pole and a cut is sufficient in this approximation, with or without mixing.

In the four-loop approximation, there are three types of corrections. The first (simplest) ones come from the account for the Regge factors of each of three Reggeons. The second type of the corrections is given by the products of the trajectories and real parts of the BFKL kernels, and the third one comes from the account for Reggeon–Reggeon interactions. All types of corrections are expressed through the integrals over the transverse momentum space corresponding to the diagrams in Fig. 5:

$$I_i = \int \frac{d^{2+2\epsilon}l_1\, d^{2+2\epsilon}l_2\, d^{2+2\epsilon}l_3}{(2\pi)^{3(3+2\epsilon)}\mathbf{l}_1^2\mathbf{l}_2^2\mathbf{l}_3^2} F_i \delta^{2+2\epsilon}(\mathbf{q}-\mathbf{l}_1-\mathbf{l}_2-\mathbf{l}_3), \quad (15)$$

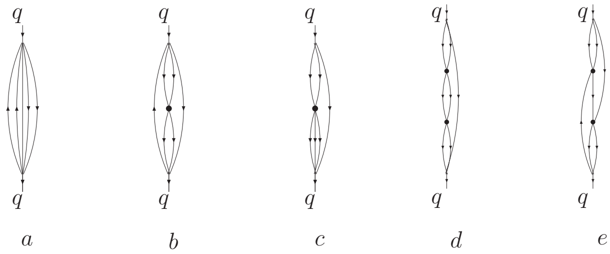


Fig. 5. Four-loop diagrams

$$F_a = f_1(\mathbf{l}_1)f_1(\mathbf{l}_2),\quad F_b = f_1(\mathbf{l}_1)f_1(\mathbf{l}_1),\quad F_c = f_2(\mathbf{l}_1+\mathbf{l}_2),$$
$$F_d = f_1(\mathbf{l}_1+\mathbf{l}_2)f_1(\mathbf{l}_1+\mathbf{l}_2),\quad F_e = f_1(\mathbf{q}-\mathbf{l}_1)f_1(\mathbf{q}-\mathbf{l}_3), \quad (16)$$

$$f_1(\mathbf{k}) = \mathbf{k}^2 \int \frac{d^{2+2\epsilon}l}{(2\pi)^{(3+2\epsilon)}\mathbf{l}^2(\mathbf{l}-\mathbf{k})^2},$$
$$f_2(\mathbf{k}) = \int \frac{d^{2+2\epsilon}l f_1(\mathbf{l})}{(2\pi)^{(3+2\epsilon)}\mathbf{l}^2(\mathbf{l}-\mathbf{k})^2}. \quad (17)$$

These integrals enter the total four-loop correction with different color factors in the approaches with or without Reggeon-cut mixing. The question of whether the four-loop amplitudes of the elastic scattering in QCD are given by the Regge pole and cut contributions, with or without mixing, can be solved by comparing these corrections with the result obtained with the use of the infrared factorization.

3. Discussion

The gluon Reggeization is the basis of the BFKL approach. The BFKL equation was derived assuming the pole Regge form of amplitudes with gluon quantum numbers in cross channels and negative signature. It is proved now in all orders of perturbation theory that this form is valid both in the leading and in the next-to-leading logarithmic approximations. However, this form is violated in the NNLLA.

Currently, there are two evidences of the violation. First, it was discovered, using the results of direct calculations of parton (gg, gq and qq) scattering amplitudes in the two-loop approximation, that the non-logarithmic terms (the lowest terms of the NNLLA) do not agree with the pole Regge form of the amplitudes. Second, it was shown using the techniques of infrared factorization that there are single-logarithmic terms with three loops wihich can not be attributed to the Regge pole contribution.

It was shown that the observed violation can be explained by the three-Reggeon cuts [12, 13]. But the assertion that the QCD amplitudes with gluon quantum numbers in cross-channels and negative signature are given in the NNLLA by the contributions of the Regge pole and the three-Reggeon cut is only a hypotheses. Since there is no general proof of it, it should be checked in each order of perturbation theory. In addition, the approaches used and the explanations given in [12] and [13] are different. Their results coincide in the three-loop case but may diverge for more loops.

The calculations of the cut contributions presented here aim to prove this hypothesis in the four-loop case. Unfortunately, direct calculations in that order in the NNLLA do not exist, and there is no hope for that they will be done in the foreseeable future. But it seems possible to obtain the corresponding results using the infrared factorization. The comparison of the results should answer the questions whether the existence of a pole and a cut is sufficient with or without mixing.

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РЕДЖЕВСЬКІ РОЗРІЗИ
І BFKL У НАБЛИЖЕННІ NNLLA

Р е з ю м е

У головному та наступному логарифмічному наближеннях КХД амплітуди з глюонними квантовими числами в крос-каналі та від'ємною сигнатурою мають полюсну форму, яка відповідає реджезованому глюону. За допомогою цієї форми виводиться знамените рівняння BFKL. В наближенні NNLLA полюсна форма порушена внесками реджевських розрізів. Ми обговорюємо ці внески та їх вплив на отримання рівняння BFKL у наближенні NNLLA.