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A FIFTH-ORDER NONLINEAR SCHRÖDINGER EQUATION FOR WAVES ON THE SURFACE OF FINITE-DEPTH FLUID

We derive a high-order nonlinear Schrödinger equation with fifth-order nonlinearity for the envelope of waves on the surface of a finite-depth irrotational, inviscid, and incompressible fluid over the flat bottom. This equation includes the fourth-order dispersion, cubic-quintic nonlinearity, and cubic nonlinear dispersion effects. The coefficients of this equation are given as functions of one dimensionless parameter kh, where k is the carrier wave number, and h is the undisturbed fluid depth. These coefficients stay bounded in the infinite-depth limit.

K e y w o r d s: nonlinear Schrödinger equation, fifth-order nonlinearity, finite-depth fluid.

1. Introduction

A group of experimentalists in the field of ship structural design and analysis has recently demonstrated in Ref. [1] that, for the depths h being of the same order of magnitude as the wavelength λ (in their experiment $h = \frac{4.1}{2\pi}\lambda$, see Table 1 in [1]), the fourth-order nonlinear Schrödinger equation (NLSE4, the first two rows in Eq. (63)) derived by the present author in 2003 [2] (and then reproduced and extended to the next, i.e. fifth order, in 2005 [3]) provides an adequate model for the deterministic prediction of water waves whose amplitude and steepness are higher than those that can be described by the classical cubic nonlinear Schrödinger equation (NLSE3) derived for the wave envelope in Ref. [4].

The same experiments demonstrated that the fourth-order model becomes insufficient to describe the zone of the highest and steepest oscillations inside the envelope (it is the envelope of such oscillations that is described by NLSE). This work is called upon to add the next (fifth) order to the fourth-order model derived in Ref. [2]. The coefficients at the fifth-order terms were found by Slunyaev in Ref. [3]. However, three fifth-order coefficients turned out to be divergent as functions of the water depth h (multiplied by the wave number $k = 2\pi/\lambda$) at large kh (with two other coefficients staying bounded). We see the origin of this divergency lies in the fact of abandoning the approach proposed in Ref. [5] for using the freedom of defining the homogeneous part of the velocity potential for the sake of selecting an integration constant in the equation for the velocity potential to secure its boundedness. Such a freedom was used in a different way in Ref. [3] for the elegant record of additional contributions of the fundamental harmonic to the amplitude in the consecutive approximations. In this paper, we follow our previous work [2] (where we derived NLSE4) and use the approach proposed by Chu and Mei [5] to avoid any divergency. The consecutive contributions to the amplitude A of the first harmonic are then reconstructed by the iterative renormalization procedure yielding an equation for the renormalized total amplitude $\mathcal{A}$.

Deriving the coefficients at high-order terms in the next orders of smallness beyond the classical cubic NLSE in terms of physical parameters is of importance for the nonlinear waves of various nature in line with the technical progress toward larger amplitudes, smaller sizes, and compacted data. Indeed, the need for the compaction of data transmitted by solitons leads to a stronger dispersion of pulses and, as a consequence, requires the use of larger pulse amplitudes to compensate for this stronger dispersion. As a result, the evolution equation for such shorter pulses should include higher derivatives and high-order nonlinear and nonlinear dispersion terms. In the case of fiber optics, such high-order equations were derived by Kodama–Hasegawa [6] (fourth-order NLSE) and by Zakharov–Kuznetsov [7] (fifth-order NLSE). High-order nonlinear Schrödinger equations describing the envelope of slowly modulated wave trains governed by the nonlinear Klein–Gordon equation were derived by different methods in [8–10].

The papers describing various aspects related to NLSE4 and NLSE5 include, in particular, Refs. [11–14] published in PRE and review [15] (see also the list of references therein). In this regard, we should also mention the pioneer work of Dysthe [16], who was the first to derive NLSE4 for water waves in the deep water limit. His ideas were further developed by Debsarma [17] and other authors cited therein.

Analyzing the analytical dependence of the NLSE4 and NLSE5 coefficients on the medium parameters can yield the critical values of these parameters at which the nonlinearity is compensated for the dispersion due to the effect of high-order terms [18] that are not present in the classical NLSE3. The soliton described by Potasek and Tabor in Ref. [19] (NLSE4 soliton) is one of such examples. In the case of surface waves, the existence of such a soliton for a particular water depth kh was predicted in Ref. [20] with the use of the NLSE4 coefficients derived in Refs. [2,3]. A possibility for the existence of a Potasek–Tabor soliton in metamaterials was studied in [21]. Deriving the analytical form of high-order NLSE coefficients as functions of the medium parameters based on physical equations would allow the new solutions for solitons, breathers, lumps, freak waves [22], *etc.* that satisfy all the underlying approximations such as the smallness of amplitudes, weak dispersion, narrow spectrum, multiple oscillations inside the pulse to be separated from the solutions that attract attention only from the mathematical point of view.

Extending the results of Ref. [2] to the fifth order (NLSE5) poses a real challenge from the technical point of view. It was done in Refs. [23, 24] for a specific depth, where the cubic nonlinear term vanishes ($kh \approx 1.36$). For a wider range of depths, NLSE5 was then derived in Ref. [3]. However, some of the coefficients of this equation turned out to be divergent in the infinite-depth limit. Other versions of high-order evolution equations for finite depths were also derived in Refs. [25–31].

In this work, we derive a NLSE5 for the envelope of surface waves on a finite-depth fluid. The coefficients of this equation stay bounded in the infinite-depth limit.

This paper is organized as follows. The fundamental equations governing the wave evolution on the surface of a finite-depth irrotational, inviscid, and incompressible fluid over the flat bottom are given in Sect. II. Section III describes the method of multiple scales that is used to derive a NLSE for the wave envelope. Section IV presents the solutions to the Laplace equation and the boundary condition at the bottom. Section V gives a short overview of the NLSE3 and NLSE4 models. NLSE5 is derived in Sect. VI, and the transition to the full amplitude of the fundamental harmonic is performed is Sect. VII. Conclusions are drawn in Sect. VIII.

2. Problem Formulation

We consider the motion of an ideal incompressible irrotational homogeneous fluid in the xy plane. The coordinate x is aligned with the direction of motion, and the coordinate y is directed along the vertical upward. The fluid is assumed to be bounded by a solid flat bottom at the depth $y = -h$ and by a rapidly oscillating free surface $\eta(x, t)$ from the top, t being the temporal variable. Our primary task is to derive an evolution equation for the amplitude A of the first-harmonic envelope of this oscillating surface.

Fluid's velocity potential $\varphi(x, y, t)$ satisfies the Laplace equation

$$\varphi_{xx} + \varphi_{yy} = 0, \quad -\infty < x < \infty, \quad -h \leqslant y \leqslant \eta(x, t). \tag{1}$$

The boundary conditions at the free surface $\eta(x, t)$ are the kinematic condition implying that the fluid particles moving along the surface cannot leave it, namely

$$\eta_t - \varphi_y + \eta_x \varphi_x = 0, \quad y = \eta(x, t), \tag{2}$$

and the dynamical condition implying that, in the Cauchy–Bernoulli integral, the fluid pressure is constant on the entire surface and is equal to the atmospheric pressure, namely,

$$g\eta + \varphi_t + \frac{1}{2}\left(\varphi_x^2 + \varphi_y^2\right) = 0, \quad y = \eta(x,\, t), \tag{3}$$

where g is the acceleration due to the gravity. The vertical component of the particle velocity at the bottom is equal to zero:

$$\varphi_y = 0, \quad y = -h, \tag{4}$$

the velocity potential being a bounded function.

3. Method of Multiple Scales

We use the method of multiple scales according to the approach elaborated for higher approximations in Refs. [2]. The variations of φ and η in time t are characterized by a superposition of the fast oscillations corresponding to time t_0 and the slow oscillations described in terms of the time τ consisting of the slow time t_1, very slow time t_2, extra slow time t_3, and exceedingly slow time t_4. Then we have

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial \tau}, \\ \frac{\partial}{\partial \tau} &= \frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2} + \varepsilon^2\frac{\partial}{\partial t_3} + \varepsilon^3\frac{\partial}{\partial t_4}. \end{aligned} \tag{5}$$

The spatial variations along the x axis are characterized by an ordinary coordinate x_0 and by a long coordinate x_1:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \varepsilon\frac{\partial}{\partial x_1}. \tag{6}$$

The velocity potential and the surface displacement (profile) are supposed to be small-value functions of the order ε, so that, taking the terms of orders up to ε^5 into account, we could write

$$\varphi = \varepsilon\varphi^{(1)} + \varepsilon^2\varphi^{(2)} + \varepsilon^3\varphi^{(3)} + \varepsilon^4\varphi^{(4)} + \varepsilon^5\varphi^{(5)}, \tag{7}$$

$$\eta = \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \varepsilon^3\eta^{(3)} + \varepsilon^4\eta^{(4)} + \varepsilon^5\eta^{(5)}. \tag{8}$$

Then we substitute expressions (5)–(8) in (1) and boundary conditions (2) and (3), wherein the functions φ_x, φ_y, and φ_t are expanded in Taylor series in powers of η at $y = 0$, and collect the terms with the like powers of ε.

4. Laplace Equation (1) and the Boundary Condition at the Bottom

4.1. Multiple-scale expansions

Since the Laplace equation (1) and the boundary condition (4) at the bottom are linear equations, the multiple-scale formalism can be set forth for them without referring to nonlinear boundary conditions (2) and (3). Substituting (6) in (1), we have

$$\frac{\partial^2\varphi}{\partial x_0^2} + \frac{\partial^2\varphi}{\partial y^2} = -2\varepsilon\frac{\partial^2\varphi}{\partial x_0\partial x_1} - \varepsilon^2\frac{\partial^2\varphi}{\partial x_1^2}. \tag{9}$$

The velocity potential φ is assumed to be weakly nonlinear, so that it could be written as expansion (7) in powers of a small parameter ε. For the five consecutive orders of ε, we have

$$\varepsilon^1\!: \; \varphi^{(1)} = \varphi_0^{(1)} + \left(\varphi_1^{(1)}\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}\right), \tag{10a}$$

$$\varepsilon^2\!: \; \varphi^{(2)} = \varphi_0^{(2)} + \left(\varphi_1^{(2)}\mathrm{e}^{\mathrm{i}\theta} + \varphi_2^{(2)}\mathrm{e}^{2\mathrm{i}\theta} + \text{c.c.}\right), \tag{10b}$$

$$\begin{aligned} \varepsilon^3\!: \; \varphi^{(3)} &= \varphi_0^{(3)} + \left(\varphi_1^{(3)}\mathrm{e}^{\mathrm{i}\theta} + \varphi_2^{(3)}\mathrm{e}^{2\mathrm{i}\theta} + \right. \\ &\left. + \varphi_3^{(3)}\mathrm{e}^{3\mathrm{i}\theta} + \text{c.c.}\right), \end{aligned} \tag{10c}$$

$$\begin{aligned} \varepsilon^4\!: \; \varphi^{(4)} &= \varphi_0^{(4)} + \left(\varphi_1^{(4)}\mathrm{e}^{\mathrm{i}\theta} + \varphi_2^{(4)}\mathrm{e}^{2\mathrm{i}\theta} + \right. \\ &\left. + \varphi_3^{(4)}\mathrm{e}^{3\mathrm{i}\theta} + \varphi_4^{(4)}\mathrm{e}^{4\mathrm{i}\theta} + \text{c.c.}\right), \end{aligned} \tag{10d}$$

$$\begin{aligned} \varepsilon^5\!: \; \varphi^{(5)} &= \varphi_0^{(5)} + \left(\varphi_1^{(5)}\mathrm{e}^{\mathrm{i}\theta} + \varphi_2^{(5)}\mathrm{e}^{2\mathrm{i}\theta} + \right. \\ &\left. + \varphi_3^{(5)}\mathrm{e}^{3\mathrm{i}\theta} + \varphi_4^{(5)}\mathrm{e}^{4\mathrm{i}\theta} + \varphi_5^{(5)}\mathrm{e}^{5\mathrm{i}\theta} + \text{c.c.}\right), \end{aligned} \tag{10e}$$

where

$$\theta = kx_0 - \omega t_0, \tag{11}$$

and c.c. stands for the complex conjugate term. Here, all the components of φ on the left-hand side depend on x_0 and x_1, and the factors at the exponents on the right-hand side depend only on x_1 and τ, with the exponential functions depending only on x_0 and t_0. The following equation (9) is written for each harmonic, and its solution is written in terms of undetermined amplitudes (which are then found from the kinematic boundary condition). Below, we show the way as to how it can be done for the zeroth and first harmonics.

4.2. Solution for the zeroth harmonic

The zeroth harmonic φ_0 represents a part of φ (terms in the first column of (10)) that does not exhibit rapid oscillations:

$$\varphi_0 = \varepsilon\varphi_0^{(1)} + \varepsilon^2\varphi_0^{(2)} + \varepsilon^3\varphi_0^{(3)} + \varepsilon^4\varphi_0^{(4)} + \varepsilon^5\varphi_0^{(5)}. \quad (12)$$

It does not depend on x_0, so that Eq. (9) could be written as

$$\frac{\partial^2\varphi_0}{\partial y^2} = -\varepsilon^2\frac{\partial^2\varphi_0}{\partial x_1^2}. \quad (13)$$

Now, we substitute (12) in (13). Since we introduced two spatial scales x_0 and x_1 for the coordinate x, we get a sequence of equations generated by each power of ε (due to ε^2 staying on the right-hand side of Eq. (13)), in contrast to the single equation (13) in the formulations proposed by Brinch–Nielsen–Jonsson [25] (Eq. (2)–(14) therein) and Dysthe [16] (Eq. (2.11) therein). These equations and their solutions are as follows (up to order $O(\varepsilon^5)$):

$$\frac{\partial^2\varphi_0^{(1)}}{\partial y^2} = 0 \Rightarrow \varphi_0^{(1)} = \Psi_1, \quad (14a)$$

$$\frac{\partial^2\varphi_0^{(2)}}{\partial y^2} = 0 \Rightarrow \varphi_0^{(2)} = \Psi_2, \quad (14b)$$

$$\frac{\partial^2\varphi_0^{(3)}}{\partial y^2} = -\frac{\partial^2\varphi_0^{(1)}}{\partial x_1^2} \Rightarrow$$

$$\Rightarrow \varphi_0^{(3)} = -\frac{1}{2}\frac{\partial^2\Psi_1}{\partial x_1^2}(y+C)^2 + \Psi_3, \quad (14c)$$

$$\frac{\partial^2\varphi_0^{(4)}}{\partial y^2} = -\frac{\partial^2\varphi_0^{(2)}}{\partial x_1^2} \Rightarrow$$

$$\Rightarrow \varphi_0^{(4)} = -\frac{1}{2}\frac{\partial^2\Psi_2}{\partial x_1^2}(y+C)^2 + \Psi_4, \quad (14d)$$

$$\frac{\partial^2\varphi_0^{(5)}}{\partial y^2} = -\frac{\partial^2\varphi_0^{(3)}}{\partial x_1^2} \Rightarrow \varphi_0^{(5)} = \frac{1}{24}\frac{\partial^4\Psi_1}{\partial x_1^4}(y+C)^4 -$$

$$-\frac{1}{2}\frac{\partial^2\Psi_3}{\partial x_1^2}(y+C)^2 + \Psi_5. \quad (14e)$$

The boundary condition at the bottom implies that $C = h$. Note that, for $h = \infty$, condition (4) holds true automatically, because the coefficients at parabolas in (14c), (14d), (14e) are equal to zero for the infinite depth, as shown in Ref. [2] in detail.

4.3. Solution for the first harmonic

For the first harmonic, we have similarly (omitting some details):

$$\varphi_1 = \varepsilon\varphi_1^{(1)} + \varepsilon^2\varphi_1^{(2)} + \varepsilon^3\varphi_1^{(3)} + \varepsilon^4\varphi_1^{(4)} + \varepsilon^5\varphi_1^{(5)}, \quad (15)$$

with $\varphi_1^{(1)}$–$\varphi_1^{(4)}$ being presented in [2] and

$$\varphi_1^{(5)} = \frac{1}{\sinh(kh)}\big(H\cosh\big(k(y+h)\big) +$$

$$+\frac{\mathtt{i}\omega}{24k}\frac{\partial^4 A}{\partial x_1^4}\big(-(y+h)^4\cosh\big(k(y+h)\big) +$$

$$+4h\sigma(y+h)^3\sinh(k(y+h)) -$$

$$-6h^2(2\sigma^2-1)(y+h)^2\cosh(k(y+h)) +$$

$$+4h^3\sigma(6\sigma^2-5)(y+h)\sinh(k(y+h))\big)\big), \quad (16)$$

where we introduced the notation

$$\sigma \equiv \tanh(kh). \quad (17)$$

The unknown function $A(x_1, \tau)$ should be found from the boundary conditions at the free surface. The arbitrary function H of the homogeneous solution is found from the condition of boundedness of the velocity potential at the infinite depth (as in Ref. [5]) and the calibration $\varphi_1^{(i)}|_{y=0} = 0$, $i = \overline{1, 5}$. Then we have

$$H = \frac{\mathtt{i}\omega}{24k}\frac{\partial^4 A}{\partial x_1^4}h^4(24\sigma^4 - 28\sigma^2 + 5).$$

For the infinite depth, we get

$$\lim_{kh\to\infty}\varphi_1^{(5)} = \frac{\mathtt{i}\omega}{24k}\frac{\partial^4 A}{\partial x_1^4}y^4\exp(ky),$$

i.e. $\varphi_1^{(5)}$ is bounded. Note that when the boundedness conditions are not met at lower orders of ε, the right-hand side of the kinematic condition in the next order of ε may become divergent, by producing unbounded coefficients in the evolution equation at higher order.

4.4. Solutions for higher harmonics

The solutions to Eq. (9) with the boundary condition (4) at the bottom for higher harmonics up to the order $O(\varepsilon^5)$ are found in the same way [2]. We do not present them here for brevity.

With the solutions to the Laplace equation found in the first five orders of ε, we proceed to the dynamical and kinematic boundary conditions at the free surface. The corresponding equations in each order of ε will generate evolution equations for the unknown function $A(x_1,\tau)$ and the induced mean flow described by the functions $\Psi_1(x_1,\tau)$, $\Psi_2(x_1,\tau)$, ... In the first three orders of ε, these equations generate a NLSE3 for the amplitude A, which was first derived in Ref. [4]. The fourth-order equations generate a NLSE4, which was derived in Ref. [2]. Below, we reproduce some of those results to preserve the integrity of our presentation. Then we use the fifth-order equations to derive a NLSE5.

5. NLSE3 and NLSE4 Approximations

Summarizing the results obtained earlier in Ref. [2], we calculated four consecutive approximations to the unknown functions by the method of multiple scales. The surface displacement was written as a sum of four terms, namely,

$$\eta = \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \varepsilon^3\eta^{(3)} + \varepsilon^4\eta^{(4)},$$

with

$$\begin{aligned}
&\varepsilon^1\!:\ \eta^{(1)} = \eta_0^{(1)} + (\eta_1^{(1)}e^{\mathtt{i}\,\theta} + \text{c.c.}),\\
&\varepsilon^2\!:\ \eta^{(2)} = \eta_0^{(2)} + (\eta_1^{(2)}e^{\mathtt{i}\,\theta} + \eta_2^{(2)}e^{\mathtt{i}\,2\theta} + \text{c.c.}),\\
&\varepsilon^3\!:\ \eta^{(3)} = \eta_0^{(3)} + (\eta_1^{(3)}e^{\mathtt{i}\,\theta} + \eta_2^{(3)}e^{\mathtt{i}\,2\theta} +\\
&+\eta_3^{(3)}e^{\mathtt{i}\,3\theta} + \text{c.c.}),\\
&\varepsilon^4\!:\ \eta^{(4)} = \eta_0^{(4)} + (\eta_1^{(4)}e^{\mathtt{i}\,\theta} + \eta_2^{(4)}e^{\mathtt{i}\,2\theta} + \eta_3^{(4)}e^{\mathtt{i}\,3\theta} +\\
&+\eta_4^{(4)}e^{\mathtt{i}\,4\theta} + \text{c.c.}).
\end{aligned} \tag{18}$$

The amplitudes $\eta_n^{(m)}$ were all calculated in terms of the complex first-harmonic amplitude $A(x,\, t_1,\, t_2,\, t_3)$. The corresponding expressions can be found in Ref. [2]. In particular, for the first harmonic, we have [see formulas (20), (22), and (49) in [2]]:

$$\begin{aligned}
&\varepsilon^1\!:\ \eta_1^{(1)} = \frac{1}{2}A,\\
&\varepsilon^2\!:\ \eta_1^{(2)} = \frac{1}{2}\mathtt{i}\alpha A_x,\\
&\varepsilon^3\!:\ \eta_1^{(3)} = \frac{1}{2}(\beta A_{xx} + \gamma A^2\overline{A}\ ),\\
&\varepsilon^4\!:\ \eta_1^{(4)} = \frac{1}{2}\mathtt{i}(\chi\, A_{xxx} + \xi\, A\,\overline{A}\, A_x + \zeta\, A^2\, \overline{A}_x),
\end{aligned} \tag{19}$$

where the subscripts x and τ denote the partial derivatives with respect to these variables and

$$\alpha = -\frac{\omega'}{\omega}, \quad \beta = -\frac{1}{2}\frac{\omega''}{\omega},$$

$$\gamma = -\frac{k^2}{16\sigma^4\nu}((9\,\sigma^4 - 8\,\sigma^2 - 3)\,(\sigma^2-1)^2\,k^2\,h^2 - \\ -2\,\sigma\,(\sigma^2+1)\,(\sigma^4-3)\,k\,h + \sigma^2\,(\sigma^4 - 16\,\sigma^2 - 3)).$$

In each approximation except for the first one, we found the evolution equation for A in terms of the corresponding slow time, with the first approximation yielding the linear dispersion relation:

$$\varepsilon^1\!:\ \omega^2 = gk\sigma, \tag{20}$$

$$\varepsilon^2\!:\ A_{t_1} + \omega' A_x = 0, \tag{21}$$

$$\varepsilon^3\!:\ \mathtt{i}A_{t_2} + \frac{1}{2}\,\omega'' A_{xx} + \omega k^2 q_3\,\overline{A}\,A^2 = 0, \tag{22}$$

$$\varepsilon^4\!:\ A_{t_3} - \frac{1}{6}\,\omega''' A_{xxx} + \omega k(q_{41}\,\overline{A}\,A\,A_x + q_{42}\,A^2\,\overline{A}_x) = 0, \tag{23}$$

where ω', ω'', and ω''' are the derivatives of the linear dispersion relation with respect to k.

In the linear approximation $O(\varepsilon^1)$, the envelope A of rapid oscillations is constant with respect to the fast time t_0.

In the $O(\varepsilon^2)$ approximation, the envelope slowly evolves with the group speed

$$V_g \equiv \frac{\partial\omega}{\partial k} = \frac{\omega}{2k}\left(1 + \frac{1-\sigma^2}{\sigma}kh\right). \tag{24}$$

In the $O(\varepsilon^3)$ approximation (NLSE3, see [4]), the envelope can have the property of a soliton and propagate pertaining its shape (when the dispersion described by the term $\frac{1}{2}\,\omega'' A_{xx}$ is compensated for the nonlinearity introduced by the term $q_3\,\overline{A}\,A^2$). In the fourth approximation (NLSE4), this compensation mechanism and soliton's shape can be different as compared to NLSE3 due to other nonlinear dispersive terms.

The coefficient q_3 was found in Ref. [4], and the coefficients q_{41}, q_{42} are expressed in terms of the dimensionless parameter kh:

$$q_{41} = \widehat{q}_{41} - \delta, \quad \widehat{q}_{41} = \widetilde{q}_{41} - \frac{\mu}{\nu}q_{40}, \tag{25}$$

$$q_{42} = \widehat{q}_{42} + \delta, \quad \widehat{q}_{42} = \widetilde{q}_{42} + \frac{\mu}{\nu}q_{40}, \tag{26}$$

$$\mu \equiv 4\sigma\left(1 - \frac{k(\sigma^2-1)}{2\omega}V_g\right) = (\sigma^2-1)^2\,kh - \sigma(\sigma^2-5), \tag{27}$$

$$\nu \equiv \frac{4k\sigma}{g}(V_g^2 - gh) = ((\sigma+1)^2\, kh-\sigma)((\sigma-1)^2\, kh-\sigma), \tag{28}$$

and the correction

$$\delta = \frac{(\sigma^2-1)\mu}{8\sigma\nu}\frac{\omega'' k^2}{\omega} \tag{29}$$

was found in Ref. [3].

Equations (22) and (23) are written in the form with the excluded components of the zero harmonic Ψ_1 and Ψ_2. These components were found from the kinematic boundary condition and are expressed in terms of the amplitude A as

$$\frac{\partial\Psi_1}{\partial x_1} = \frac{\omega k\mu}{2\sigma\nu}A\overline{A}, \tag{30}$$

$$\frac{\partial\Psi_2}{\partial x_1} = -\frac{4\mathrm{i}\omega\sigma}{\nu}q_{40}\left(A\frac{\partial\overline{A}}{\partial x_1} - \overline{A}\frac{\partial A}{\partial x_1}\right). \tag{31}$$

Since the product $A\overline{A}$ evolves with a group speed as the amplitude A, according to Eq. (21) and its complex conjugate, we used an ansatz implying that the zeroth harmonic amplitudes Ψ_1 and Ψ_2 evolve with the group speed

$$\frac{\partial\,\Psi_1}{\partial t_1} = -V_g\frac{\partial\Psi_1}{\partial x_1}, \quad \frac{\partial\Psi_2}{\partial t_1} = -V_g\frac{\partial\Psi_2}{\partial x_1}. \tag{32}$$

Figure 3 in Sect. VII demonstrates that the difference between the coefficients $\hat{q}_{41}$ and q_{41}, as well as between the coefficients $\hat{q}_{42}$ and q_{42}, is not significant (see also a discussion in Ref. [18]). By this reason, the contribution δ was not taken into account in Ref. [2]. It is absolutely insignificant for the depth $kh = 4.1$ at which the experiment of Ref. [1] was performed. Nevertheless, this contribution was fairly taken into account in Ref. [3], and we include it into our consideration here to avoid any misunderstanding.

6. NLSE5

6.1. Dynamical boundary condition

The dynamical boundary condition of order $O(\varepsilon^5)$ has the following form:

$$\varphi_{t_0}^{(5)} + \frac{\omega^2}{k\,\sigma}\eta^{(5)} = -\frac{1}{2}\,\varphi_{yyt_0}^{(1)}\,(2\,\eta^{(1)}\,\eta^{(3} + (\eta^{(2)})^2) -$$
$$-\frac{1}{2}(\varphi_{yyt_0}^{(3)} + \varphi_{yyt_1}^{(2)} + \varphi_{yyt_2}^{(1)})\,(\eta^{(1)})^2 -$$
$$-(\varphi_{yyt_0}^{(2)} + \varphi_{yyt_1}^{(1)})\,\eta^{(1)}\,\eta^{(2)} - \frac{1}{24}\,\varphi_{yyyyt_0}^{(1)}(\eta^{(1)})^4 -$$
$$-\varphi_{yt_0}^{(1)}\,\eta^{(4)} - \varphi_{t_4}^{(1)} - \varphi_{t_3}^{(2)} - \varphi_{t_2}^{(3)} - \varphi_{t_1}^{(4)} -$$
$$-(\varphi_{yt_0}^{(4)} + \varphi_{yt_1}^{(3)} + \varphi_{yt_2}^{(2)} + \varphi_{yt_3}^{(1)})\,\eta^{(1)} - (\varphi_{yt_0}^{(2)} + \varphi_{yt_1}^{(1)})\,\eta^{(3)} -$$
$$-(\varphi_{yt_0}^{(3)} + \varphi_{yt_1}^{(2)} + \varphi_{yt_2}^{(1)})\,\eta^{(2)} - \frac{1}{2}\,\varphi_{yyyt_0}^{(1)}(\eta^{(1)})^2\,\eta^{(2)} -$$
$$-\frac{1}{6}(\varphi_{yyyt_0}^{(2)} + \varphi_{yyyt_1}^{(1)})(\eta^{(1)})^3 -$$
$$-\varphi_y^{(1)}\Big[\varphi_y^{(4)} + \varphi_{yy}^{(1)}\,\eta^{(3)} + \varphi_{yy}^{(2)}\eta^{(2)} + \varphi_{yy}^{(3)}\eta^{(1)} +$$
$$+\varphi_{yyy}^{(1)}\eta^{(1)}\,\eta^{(2)} + \frac{1}{2}\,\varphi_{yyy}^{(2)}\,(\eta^{(1)})^2 + \frac{1}{6}\,\varphi_{yyyy}^{(1)}(\eta^{(1)})^3\Big] -$$
$$-(\varphi_y^{(2)} + \varphi_{yy}^{(1)}\,\eta^{(1)})\Big[\varphi_y^{(3)} + \varphi_{yy}^{(1)}\,\eta^{(2)} + \varphi_{yy}^{(2)}\eta^{(1)} +$$
$$+\frac{1}{2}\,\varphi_{yyy}^{(1)}\,(\eta^{(1)})^2\Big] - \varphi_0^{(1)}\Big[\varphi_{yx_0}^{(1)}\,\eta^{(3)} + (\varphi_{yx_0}^{(2)} +$$
$$+\varphi_{yx_1}^{(1)})\,\eta^{(2)} + (\varphi_{yx_0}^{(3)} + \varphi_{yx_1}^{(2)})\,\eta^{(1)} + \varphi_{yyx_0}^{(1)}\eta^{(1)}\,\eta^{(2)} +$$
$$+\frac{1}{2}(\varphi_{yyx_0}^{(2)} + \varphi_{yyx_1}^{(1)})\eta^{(1)^2} + \varphi_{x_0}^{(4)} + \varphi_{x_1}^{(3)} + \frac{1}{6}\varphi_{yyyx_0}^{(1)}\eta^{(1)^3}\Big] -$$
$$-(\varphi_{x_0}^{(2)} + \varphi_{yx_0}^{(1)}\eta^{(1)} + \varphi_{x_1}^{(1)})\Big[\varphi_{yx_0}^{(1)}\,\eta^{(2)} + (\varphi_{yx_0}^{(2)} +$$
$$+\varphi_{yx_1}^{(1)})\,\eta^{(1)} + \varphi_{x_0}^{(3)} + \frac{1}{2}\,\varphi_{yyx_0}^{(1)}(\eta^{(1)})^2 + \varphi_{x_1}^{(2)}\Big]. \tag{33}$$

We substitute the expansion

$$\eta^{(5)} = \eta_5^{(0)} + \Big(\eta_5^{(1)}\mathrm{e}^{\mathrm{i}\theta} + \eta_5^{(2)}\mathrm{e}^{2\mathrm{i}\theta} +$$
$$+\eta_5^{(3)}\mathrm{e}^{3\mathrm{i}\theta} + \eta_5^{(4)}\mathrm{e}^{4\mathrm{i}\theta} + \eta_5^{(5)}\mathrm{e}^{5\mathrm{i}\theta} + \mathrm{c.c.}\Big) \tag{34}$$

and the expressions for $\eta^{(1)}$, $\varphi^{(1)}$, $\eta^{(2)}$, $\varphi^{(2)}$, $\eta^{(3)}$, $\varphi^{(3)}$, $\eta^{(4)}$, $\varphi^{(4)}$ found in the previous iterations in Eq. (33) and account for expression (10e) for $\varphi^{(5)}$ consisting of $\varphi_0^{(5)}$ given by Eq. (14e), $\varphi_1^{(5)}$ given by Eq. (16), and yet undetermined functions $\varphi_2^{(5)}$, $\varphi_3^{(5)}$, $\varphi_4^{(5)}$, and $\varphi_5^{(5)}$. By equating the nonexponential term and the coefficients at the like powers of exponential functions to zero, we find the expressions for $\eta_0^{(5)}$ and $\eta_1^{(5)}$ in terms of A, Ψ_1, Ψ_2, Ψ_3 and for $\eta_2^{(5)}$, $\eta_3^{(5)}$, $\eta_4^{(5)}$, and $\eta_5^{(5)}$ in terms of yet undetermined amplitudes $\phi_2^{(5)}$, $\phi_3^{(5)}$, $\phi_4^{(5)}$, and $\phi_5^{(5)}$ related to the velocity potential.

The differential equations for A and Ψ_3, as well as the algebraic expressions for $\phi_2^{(5)}$, $\phi_3^{(5)}$, $\phi_4^{(5)}$, and $\phi_5^{(5)}$, are found from the kinematic boundary condition for the zeroth, first, second, third, fourth, and fifth harmonics, respectively.

6.2. Kinematic boundary condition

The kinematic boundary condition of order $O(\varepsilon^5)$ has the following form:

$$\eta^{(5)}_{t_0} - \varphi^{(5)}_y = \frac{1}{2}\,\varphi^{(1)}_{yyy}(2\,\eta^{(1)}\,\eta^{(3)} + (\eta^{(2)})^2) +$$
$$+\frac{1}{2}\,\varphi^{(3)}_{yyy}\,(\eta^{(1)})^2 + \varphi^{(2)}_{yyy}\,\eta^{(1)}\,\eta^{(2)} - \eta^{(4)}_{t_1} - \eta^{(1)}_{t_4} - \eta^{(2)}_{t_3} -$$
$$-\,\eta^{(3)}_{t_2} + \frac{1}{2}\,\varphi^{(1)}_{yyyy}\,(\eta^{(1)})^2\,\eta^{(2)} + \frac{1}{6}\,\varphi^{(2)}_{yyyy}\,(\eta^{(1)})^3 +$$
$$+\frac{1}{24}\,\varphi^{(1)}_{yyyyy}\,(\eta^{(1)})^4 - \eta^{(1)}_{x_0}\Big[(\varphi^{(1)}_{x_0y}\,\eta^{(3)} +$$
$$+(\varphi^{(2)}_{x_0y} + \varphi^{(1)}_{x_1y})\,\eta^{(2)} + \varphi^{(3)}_{x_0y} + \varphi^{(2)}_{x_1y})\,\eta^{(1)} +$$
$$+\varphi^{(1)}_{x_0yy}\,\eta^{(1)}\,\eta^{(2)} + \frac{1}{2}\,(\varphi^{(2)}_{x_0yy} + \varphi^{(1)}_{yyx_1})\,(\eta^{(1)})^2 + \varphi^{(4)}_{x_0} +$$
$$+\varphi^{(3)}_{x_1} + \frac{1}{6}\,\varphi^{(1)}_{x_0yyy}\,(\eta^{(1)})^3\Big] - (\eta^{(4)}_{x_0} + \eta^{(3)}_{x_1})\,\varphi^{(1)}_{x_0} -$$
$$-\,(\eta^{(2)}_{x_0} + \eta^{(1)}_{x_1})\Big[(\varphi^{(1)}_{x_0y}\,\eta^{(2)} + (\varphi^{(2)}_{x_0y} + \varphi^{(1)}_{x_1y})\,\eta^{(1)} +$$
$$+\varphi^{(3)}_{x_0} + \frac{1}{2}\varphi^{(1)}_{x_0yy}\,(\eta^{(1)})^2 + \varphi^{(2)}_{x_1})\Big] -$$
$$-\,(\eta^{(3)}_{x_0} + \eta^{(2)}_{x_1})\,\Big[\varphi^{(2)}_{x_0} + \varphi^{(1)}_{x_0y}\,\eta^{(1)} + \varphi^{(1)}_{x_1}\Big] + \varphi^{(1)}_{yy}\eta_4 +$$
$$+\varphi^{(4)}_{yy}\,\eta^{(1)} + \varphi^{(2)}_{yy}\,\eta^{(3)} + \varphi^{(3)}_{yy}\,\eta^{(2)}. \qquad (35)$$

6.2.1. Zeroth harmonic

When being set equal to zero, the nonexponential term of the kinematic boundary condition results in the evolution equation for the Ψ_3 amplitude of the zeroth harmonic:

$$-\frac{k\sigma}{\omega^2}\frac{\partial^2\Psi_3}{\partial t_1^2} + h\frac{\partial^2\Psi_3}{\partial x_1^2} + \omega k^2 q^{(a)}_{50}\frac{\partial}{\partial x_1}(A\bar{A})^2 +$$
$$+\frac{\omega}{k^2}q^{(b)}_{50}\frac{\partial}{\partial x_1}\left(A\frac{\partial^2\bar{A}}{\partial x_1^2} + \bar{A}\frac{\partial^2 A}{\partial x_1^2}\right) +$$
$$+\frac{\omega}{k^2}q^{(c)}_{50}\frac{\partial}{\partial x_1}\left(\frac{\partial A}{\partial x_1}\frac{\partial\bar{A}}{\partial x_1}\right) = 0. \qquad (36)$$

The explicit expressions for the coefficients of this equation are given in Appendix A.

6.2.2. First harmonic

When being set equal to zero, the $\exp(\mathtt{i}\theta)$ term of the kinematic boundary condition results in the evolution equation for the amplitude A with respect to time t_4:

$$\mathtt{i}\frac{\partial A}{\partial t_4} - \frac{\omega''''}{24}\frac{\partial^4 A}{\partial x_1^4} + \omega k^4\,\widetilde{q}_{51}A^3\bar{A}^{\,2} +$$
$$+\,\omega\,\widetilde{q}_{52}A\bar{A}\,\frac{\partial^2 A}{\partial x_1^2} + \omega\,\widetilde{q}_{53}A^2\frac{\partial^2\bar{A}}{\partial x_1^2} + \omega\,\widetilde{q}_{54}A\,\frac{\partial A}{\partial x_1}\frac{\partial\bar{A}}{\partial x_1} +$$
$$+\,\omega\,\widetilde{q}_{55}\bar{A}\left(\frac{\partial A}{\partial x_1}\right)^2 - kA\,\frac{\partial\Psi_3}{\partial x_1} +$$
$$+\,\frac{k^2(\sigma^2-1)}{2\omega}A\left(\frac{\partial\Psi_3}{\partial t_1} + \frac{\partial\Psi_1}{\partial t_3} + \frac{\partial\Psi_2}{\partial t_2}\right) = 0. \qquad (37)$$

The explicit expressions for the coefficients of this equation are given in Appendix A. Our further task is to express the derivatives $\frac{\partial\Psi_3}{\partial t_1}$, $\frac{\partial\Psi_1}{\partial t_3}$, and $\frac{\partial\Psi_2}{\partial t_2}$ in terms of A.

(I) Derivative $\frac{\partial\Psi_3}{\partial t_1}$. Following the general approach introduced by Chu and Mei in Ref. [5], the amplitude Ψ_3 of the homogeneous solution is found from the boundedness condition for $\varphi^{(3)}_0$ at $kh\to\infty$. In view of Eq. (14c) for the function $\varphi^{(3)}_0$, this boundedness condition implies that

$$\Psi_3 = -\frac{\omega}{4}\frac{\partial(A\bar{A})}{\partial x_1}h. \qquad (38)$$

Then the asymptotic behavior of $\varphi^{(3)}_0$ at the infinite depth is evaluated as follows:

$$\lim_{kh\to\infty}\varphi^{(3)}_0 = \frac{\omega}{2}\frac{\partial(A\bar{A})}{\partial x_1}\Big(y + \frac{1}{8k}\Big), \qquad (39)$$

which turns out to be finite.

To find the derivative $\frac{\partial\Psi_3}{\partial t_1}$, we need to use the similar ansatz as in formulas (32) for the derivatives $\frac{\partial\Psi_1}{\partial t_1}$ and $\frac{\partial\Psi_2}{\partial t_1}$, where the functions Ψ_1 and Ψ_2 were assumed to evolve with the group speed of linear waves. Since there is condition (38) imposed on the function Ψ_3, any additional assumptions should comply with this condition. Thus, we need to introduce an additional matching correction Δ in the ansatz for $\frac{\partial\Psi_3}{\partial t_1}$:

$$\frac{\partial\Psi_3}{\partial t_1} = -V_g\frac{\partial\Psi_3}{\partial x_1} + \Delta. \qquad (40)$$

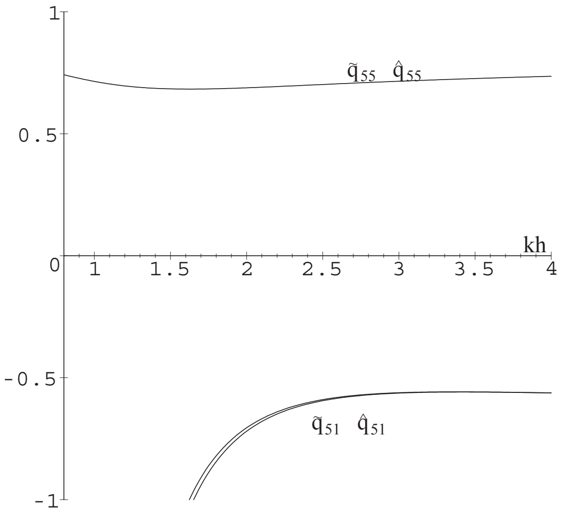

Fig. 1. Coefficients $\hat{q}_{51}$ and $\hat{q}_{55}$ as compared to $\tilde{q}_{51}$ and $\tilde{q}_{55}$

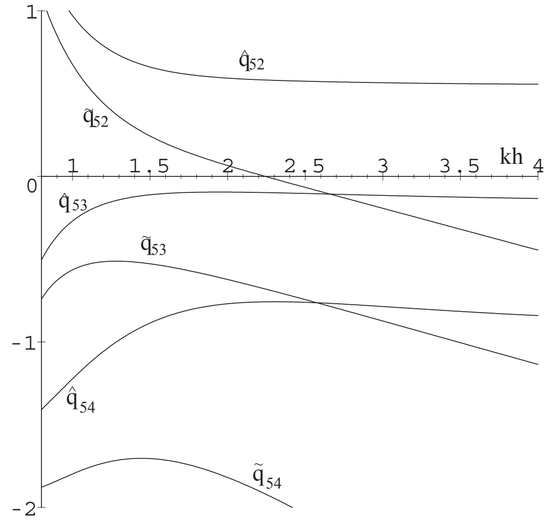

Fig. 2. Coefficients $\hat{q}_{52}$, $\hat{q}_{53}$, and $\hat{q}_{54}$ as compared to $\tilde{q}_{52}$, $\tilde{q}_{53}$, and $\tilde{q}_{54}$

To find the correction Δ, we first differentiate (40) with respect to t_1 and use the anzatz $\frac{\partial\Delta}{\partial t_1} = -V_g\frac{\partial\Delta}{\partial x_1}$, so that

$$\frac{\partial^2\Psi_3}{\partial t_1^2} = V_g^2\frac{\partial^2\Psi_3}{\partial x_1^2} - 2V_g\frac{\partial\Delta}{\partial x_1}. \tag{41}$$

By substituting (41) in (36) and integrating over x_1, we get

$$\frac{\nu}{4\sigma k}\frac{\partial\Psi_3}{\partial x_1} = \omega k^2 q_{50}^{(a)}(A\bar{A})^2 + + \frac{\omega}{k^2}\,q_{50}^{(b)}\left(A\,\frac{\partial^2\bar{A}}{\partial x_1^2} + \bar{A}\,\frac{\partial^2 A}{\partial x_1^2}\right) + + \frac{\omega}{k^2}\,q_{50}^{(c)}\,\frac{\partial A}{\partial x_1}\frac{\partial\bar{A}}{\partial x_1} + \frac{2\sigma k}{\omega^2}\,V_g\Delta, \tag{42}$$

where ν is given by relation (28). On the other hand, by differentiating (38) with respect to t_1, we obtain another expression for $\frac{\partial\Psi_3}{\partial x_1}$:

$$\frac{\partial\Psi_3}{\partial x_1} = -\frac{\omega}{4k}\left(A\,\frac{\partial^2\bar{A}}{\partial x_1^2} + 2\frac{\partial A}{\partial x_1}\frac{\partial\bar{A}}{\partial x_1} + \bar{A}\frac{\partial^2 A}{\partial x_1^2}\right)kh. \tag{43}$$

Comparing (42) and (43), we finally find a relation for Δ:

$$\frac{2\sigma k}{\omega^2}\,V_g\Delta = -q_{50}^{(a)}\omega k^2(A\bar{A})^2 - - (q_{50}^{(b)} + c)\frac{\omega}{k^2}\left(\bar{A}\frac{\partial^2 A}{\partial {x_1}^2} + A\frac{\partial^2\bar{A}}{\partial {x_1}^2}\right) - - (q_{50}^{(c)} + 2c)\frac{\omega}{k^2}\frac{\partial\bar{A}}{\partial x_1}\frac{\partial A}{\partial x_1}, \tag{44}$$

where $c = \frac{\nu kh}{16\sigma}$. Since the correction Δ is nonzero, the component Ψ_3 evolves with a speed that is slightly different from the group speed V_g, in contrast to the components Ψ_1 and Ψ_2. Then, substituting expression (40) for $\frac{\partial\Psi_3}{\partial t_1}$, expression (43) for $\frac{\partial\Psi_3}{\partial x_1}$, and Δ given by formula (44) in Eq. (37), we finally get:

$$\mathtt{i}\frac{\partial A}{\partial t_4} - \frac{\omega''''}{24}\frac{\partial^4 A}{\partial x_1^4} + \omega k^4\hat{q}_{51}A^3\bar{A}^{\,2} + \omega\hat{q}_{52}A\bar{A}\,\frac{\partial^2 A}{\partial x_1^2} + + \omega\hat{q}_{53}A^2\frac{\partial^2\bar{A}}{\partial x_1^2} + \omega\hat{q}_{54}A\frac{\partial A}{\partial x_1}\frac{\partial\bar{A}}{\partial x_1} + \omega\hat{q}_{55}\bar{A}\left(\frac{\partial A}{\partial x_1}\right)^2 - - (\sigma^2-1)\frac{k^2}{2\omega}A\left(\frac{\partial\Psi_2}{\partial t_2} + \frac{\partial\Psi_3}{\partial t_1}\right) = 0, \tag{45}$$

where

$$\begin{aligned}
&\hat{q}_{51} = \tilde{q}_{51} + (\sigma^2-1)\frac{1}{4\sigma\frac{k}{\omega}V_g}q_{50}^{(a)},\\
&\hat{q}_{52} = \tilde{q}_{52} + \frac{\mu}{16\sigma}kh + (\sigma^2-1)\frac{1}{4\sigma\frac{k}{\omega}V_g}(q_{50}^{(b)} + c),\\
&\hat{q}_{53} = \tilde{q}_{53} + \frac{\mu}{16\sigma}kh + (\sigma^2-1)\frac{1}{4\sigma\frac{k}{\omega}V_g}(q_{50}^{(b)} + c),\\
&\hat{q}_{54} = \tilde{q}_{54} + \frac{\mu}{8\sigma}kh + (\sigma^2-1)\frac{1}{4\sigma\frac{k}{\omega}V_g}(q_{50}^{(c)} + 2c),\\
&\hat{q}_{55} = \tilde{q}_{55},
\end{aligned} \tag{46}$$

and μ is given by relation (27).

Figures 1 and 2 show the coefficients $\hat{q}_{51}$, $\hat{q}_{52}$, $\hat{q}_{53}$, $\hat{q}_{54}$, and $\hat{q}_{55}$ calculated by formulas (46) as compared to the coefficients $\tilde{q}_{51}$, $\tilde{q}_{52}$, $\tilde{q}_{53}$, $\tilde{q}_{54}$, and $\tilde{q}_{55}$. It can be seen that, owing to the second terms in formulas (46), the coefficients $\tilde{q}_{52}$, $\tilde{q}_{53}$, and $\tilde{q}_{54}$ that are divergent at large kh were transformed into the bounded coefficients $\hat{q}_{52}$, $\hat{q}_{53}$, and $\hat{q}_{54}$. At the same time, since the coefficients $\hat{q}_{51}$ and $\hat{q}_{55}$ do not contain the terms proportional to kh in expressions (46), the coefficients $\tilde{q}_{51}$ and $\tilde{q}_{55}$ are also bounded. The boundedness of the coefficients $\hat{q}_{52}$, $\hat{q}_{53}$, and $\hat{q}_{54}$ at large kh is the MAIN distinction and advantage of the present work. This progress was achieved by accounting for the boundedness conditions for φ at $kh \to \infty$ in the orders $O(\varepsilon)$ through $O(\varepsilon^4)$ (as was earlier proposed in Refs. [2,5]).

(II) Derivatives $\frac{\partial \Psi_2}{\partial t_2}$ and $\frac{\partial \Psi_1}{\partial t_3}$. The derivative of the amplitude Ψ_1 with respect to time t_3 is expressed as

$$\frac{\partial \Psi_1}{\partial t_3} = \frac{\partial}{\partial t_3}\int \frac{\partial \Psi_1}{\partial x_1} dx_1, \tag{47}$$

with $\frac{\partial \Psi_1}{\partial x_1}$ given by expression (30) and $\frac{\partial A}{\partial t_3}$ given by Eq. (23). Then we have

$$\frac{\partial \Psi_1}{\partial t_3} = \frac{\omega k \mu}{2\sigma\nu}\Bigg(\frac{\omega'''}{6}\left(\frac{\partial^2 A}{\partial x_1^2}\bar{A} + \frac{\partial^2 \bar{A}}{\partial x_1^2}A - \frac{\partial A}{\partial x_1}\frac{\partial \bar{A}}{\partial x_1}\right) - \\ - \frac{1}{2}\omega k(\tilde{q}_{41} + \tilde{q}_{42})(A\bar{A})^2\Bigg). \tag{48}$$

The derivative of the amplitude Ψ_2 with respect to time t_2 is expressed as

$$\frac{\partial \Psi_2}{\partial t_2} = \frac{\partial}{\partial t_2}\int \frac{\partial \Psi_2}{\partial x_1} \mathrm{d}x_1, \tag{49}$$

with $\frac{\partial \Psi_2}{\partial x_1}$ given by expression (31) and $\frac{\partial A}{\partial t_2}$ given by Eq. (22). Then we have

$$\frac{\partial \Psi_2}{\partial t_2} = \frac{4\omega\sigma}{\nu}q_{40}\Bigg(\frac{\omega''}{2}\left(2\frac{\partial A}{\partial x_1}\frac{\partial \bar{A}}{\partial x_1} - \frac{\partial^2 A}{\partial {x_1}^2}\bar{A} - \frac{\partial^2 \bar{A}}{\partial {x_1}^2}A\right) - \\ - q_3\omega k^2(A\bar{A})^2\Bigg). \tag{50}$$

Substituting (48) and (50) in (45), we get the final equation for A with respect to time t_4:

$$\mathtt{i}\frac{\partial A}{\partial t_4} - \frac{\omega''''}{24}\frac{\partial^4 A}{\partial x_1^4} + \omega k^4 q_{51}A^3\bar{A}^{\,2} + \\ + \omega q_{52}A\bar{A}\frac{\partial^2 A}{\partial x_1^2} + \omega q_{53}A^2\frac{\partial^2 \bar{A}}{\partial x_1^2} + \\ + \omega q_{54}A\frac{\partial A}{\partial x_1}\frac{\partial \bar{A}}{\partial x_1} + \omega k q_{55}\bar{A}\left(\frac{\partial A}{\partial x_1}\right)^2 = 0, \tag{51}$$

where

$$q_{51} = \hat{q}_{51} + (\sigma^2 - 1)\frac{2\sigma}{\nu}\tilde{q}_{40}q_3 + \\ + (\sigma^2 - 1)\frac{\mu}{8\sigma\nu}(\tilde{q}_{41} + \tilde{q}_{42}), \\ q_{52} = \hat{q}_{52} + (\sigma^2 - 1)\frac{\sigma}{\nu}\tilde{q}_{40}\omega''\frac{k^2}{\omega} - \\ - (\sigma^2 - 1)\frac{\mu}{\sigma\nu}\frac{\omega'''}{24}\frac{k^3}{\omega}, \\ q_{53} = \hat{q}_{53} + (\sigma^2 - 1)\frac{\sigma}{\nu}\tilde{q}_{40}\omega''\frac{k^2}{\omega} - \\ - (\sigma^2 - 1)\frac{\mu}{\sigma\nu}\frac{\omega'''}{24}\frac{k^3}{\omega}, \\ q_{54} = \hat{q}_{54} - 2(\sigma^2 - 1)\frac{\sigma}{\nu}\tilde{q}_{40}\omega''\frac{k^2}{\omega} + \\ + (\sigma^2 - 1)\frac{\mu}{\sigma\nu}\frac{\omega'''}{24}\frac{k^3}{\omega}, \\ q_{55} = \hat{q}_{55}. \tag{52}$$

It can be seen from the above formulas that the terms with derivatives $\frac{\partial \Psi_1}{\partial t_3}$ and $\frac{\partial \Psi_2}{\partial t_2}$ introduce small nonzero corrections to the coefficients q_{51}, q_{52}, q_{53}, and q_{54} as compared to the corresponding coefficients $\hat{q}_{51}$, $\hat{q}_{52}$, $\hat{q}_{53}$, and $\hat{q}_{54}$ (the last coefficients q_{55} and $\hat{q}_{55}$ coincide).

6.3. Equation in terms of the integral slow time τ

To derive the evolution equation for A in the integral slow time τ, we sum up Eq. (21) multiplied by $\mathtt{i}$, Eq. (22) multiplied by ε, Eq. (23) multiplied by $\mathtt{i}\varepsilon^2$, and Eq. (51) multiplied by ε^3. Then, taking relation (5 into account), we get the evolution equation in terms of the integral slow time τ:

$$\mathtt{i}(A_\tau + \omega' A_x) + \varepsilon\Big(\frac{\omega''}{2}A_{xx} + \omega k^2 q_3\, A^2\bar{A}\Big) + \\ + \mathtt{i}\varepsilon^2\Big(-\frac{\omega'''}{6}A_{xxx} + \omega k q_{41}\, A\bar{A}A_x + \omega k q_{42}\, A^2\bar{A}_x\Big) + \\ + \varepsilon^3\Big(-\frac{\omega''''}{24}A_{xxxx} + \omega k^4 q_{51}\, A^3\bar{A}^{\,2} + \omega q_{52}\, A\bar{A}A_{xx} + \\ + \omega q_{53}\, A^2\bar{A}_{xx} + \omega q_{54}\, A\bar{A}_x\, A_x + \omega q_{55}\, \bar{A}A_x^2\Big) = 0. \tag{53}$$

7. NLSE5 for the Total Amplitude $\mathcal{A}$

It follows from Eq. (19) that the total first-harmonic amplitude $\mathcal{A}$ contains contributions of higher approximations besides the amplitude A introduced in the first approximation:

$$\mathcal{A} = A + \mathtt{i}\varepsilon\alpha\, A_x + \varepsilon^2\left(\beta\, A_{xx} + \gamma\, A^2\,\bar{A}\right) +$$

$$+\,\mathtt{i}\,\varepsilon^3\,(\chi\,A_{xxx}+\xi\,A\,\bar{A}\,A_x+\zeta\,A^2\,\bar{A}_x). \tag{54}$$

It is the amplitude $\mathcal{A}$ that is an observable in the experiment, and it is advantageous to derive the evolution equation for $\mathcal{A}$ instead of Eq. (53) for A. To this end, we rewrite relation (54) as follows:

$$A=\mathcal{A}-\mathtt{i}\varepsilon\alpha\,A_x-\varepsilon^2\,(\beta\,A_{xx}+\gamma\,A^2\,\bar{A})-$$
$$-\,\mathtt{i}\,\varepsilon^3\,(\chi\,A_{xxx}+\xi\,A\,\bar{A}\,A_x+\zeta\,A^2\,\bar{A}_x). \tag{55}$$

Then we use the smallness of ε to exclude A from the right-hand side of (55) and iteratively express A in terms of $\mathcal{A}$ as follows.

Iteration 0 (order $O(\varepsilon^0)$):

$$A^{(0)}=\mathcal{A}. \tag{56}$$

Iteration 1 (order $O(\varepsilon^1)$):

$$A^{(1)}=\mathcal{A}-\mathtt{i}\varepsilon\,\alpha\,A_x. \tag{57}$$

Substituting the expression (56) for A in terms of $\mathcal{A}$ from the previous iteration in the right-hand side of Eq. (57), we get

$$A^{(1)}=\mathcal{A}-\mathtt{i}\varepsilon\,\alpha\,\mathcal{A}_x. \tag{58}$$

Iteration 2 (order $O(\varepsilon^2)$):

$$A^{(2)}=\mathcal{A}-\mathtt{i}\varepsilon\,\alpha\,A_x-\varepsilon^2\,(\beta\,A_{xx}+\gamma\,A^2\,\bar{A}). \tag{59}$$

Substituting the expression (57) for A in terms of $\mathcal{A}$ from the previous iteration in the right-hand side of Eq. (59), we get

$$A^{(2)}=\mathcal{A}-\mathtt{i}\,\varepsilon\,\alpha\,(\mathcal{A}-\mathtt{i}\varepsilon\,\alpha\,\mathcal{A}_x)_x-\varepsilon^2\,(\beta\,(\mathcal{A}-\mathtt{i}\varepsilon\,\alpha\,\mathcal{A}_x)_{xx}+$$
$$+\,\gamma\,(\mathcal{A}-\mathtt{i}\varepsilon\,\alpha\,\mathcal{A}_x)^2\,(\bar{\mathcal{A}}+\mathtt{i}\varepsilon\,\alpha\,\bar{\mathcal{A}}_x)).$$

Keeping only the terms of order up to $O(\varepsilon^2)$, we obtain

$$A^{(2)}=\mathcal{A}-\mathtt{i}\varepsilon\,\alpha\mathcal{A}_x-\varepsilon^2((\alpha^2+\beta)\mathcal{A}_{xx}+\gamma\,\mathcal{A}^2\,\bar{\mathcal{A}}). \tag{60}$$

Iteration 3 (order $O(\varepsilon^3)$ (similarly to the previous iteration):

$$A^{(3)}=\mathcal{A}-\mathtt{i}\varepsilon\,\alpha\mathcal{A}_x-\varepsilon^2((\alpha^2+\beta)\mathcal{A}_{xx}+\gamma\mathcal{A}^2\,\bar{\mathcal{A}})+$$
$$+\,\mathtt{i}\varepsilon^3((4\,\gamma\,\alpha-\xi)\mathcal{A}\,\bar{\mathcal{A}}\,\mathcal{A}_x-\zeta\,\mathcal{A}^2\bar{\mathcal{A}}_x+$$
$$+(\alpha^3+2\,\alpha\,\beta-\chi)\mathcal{A}_{xxx}). \tag{61}$$

Relation (61) gives the final expression for A in terms of the total amplitude $\mathcal{A}$, accurate to the fourth order of smallness with respect to ε. Then we calculate the derivative of $\mathcal{A}$ with respect to time τ:

$$\mathcal{A}_\tau=A_\tau+\mathtt{i}\varepsilon\alpha\,A_{x\tau}+\varepsilon^2\left(\beta\,A_{xx\tau}+\gamma(2A\bar{A}A_\tau+A^2\,\bar{A}_\tau)\right)+$$
$$+\,\mathtt{i}\,\varepsilon^3\left(\chi\,A_{xxx\tau}+\xi\,(A_\tau\,\bar{A}\,A_x+A\,\bar{A}_\tau A_x+A\,\bar{A}\,A_{x\tau})+\right.$$
$$\left.+\,\zeta(2\,AA_\tau\,\bar{A}_x+A^2\,\bar{A}_{x\tau})\right). \tag{62}$$

The derivative of A with respect to τ is given by Eq. (53), and the derivatives of A and A_τ with respect to x are obtained by differentiating Eq. (61) and Eq. (53) as many times as necessary. In doing so, Eq. (62) can be rewritten as

$$\mathtt{i}(\mathcal{A}_\tau+\omega'\mathcal{A}_{x_1})+\varepsilon\left(\frac{\omega''}{2}\mathcal{A}_{x_1x_1}+\omega k^2Q_3\mathcal{A}^2\bar{\mathcal{A}}\right)+$$
$$+\,\mathtt{i}\varepsilon^2\left(-\frac{\omega'''}{6}\mathcal{A}_{x_1x_1x_1}+\omega k\left(Q_{41}\mathcal{A}\bar{\mathcal{A}}\mathcal{A}_{x_1}+Q_{42}\mathcal{A}^2\bar{\mathcal{A}}_{x_1}\right)\right)+$$
$$+\,\varepsilon^3\left(-\frac{\omega''''}{24}\mathcal{A}_{x_1x_1x_1x_1}+\omega k^4Q_{51}\mathcal{A}^3\bar{\mathcal{A}}^2+\right.$$
$$+\,\varepsilon^3+\omega\big(Q_{52}\mathcal{A}\bar{\mathcal{A}}\mathcal{A}_{x_1x_1}+Q_{53}\mathcal{A}^2\bar{\mathcal{A}}_{x_1x_1}+$$
$$\left.+\,\varepsilon^3+Q_{54}\mathcal{A}\mathcal{A}_{x_1}\bar{\mathcal{A}}_{x_1}+Q_{55}\bar{\mathcal{A}}\mathcal{A}_{x_1}^2\big)\right)=0. \tag{63}$$

The coefficients of this equation are expressed as follows:

$$Q_3=q_3,\quad Q_{41}=q_{41},\quad Q_{42}=q_{42}+2\alpha\,q_3, \tag{64a}$$
$$Q_{51}=q_{51}-2\gamma\,q_3,\quad Q_{52}=q_{52}, \tag{64b}$$
$$Q_{53}=q_{53}-\omega''\gamma-2\alpha^2q_3-2\alpha\,q_{42}, \tag{64c}$$
$$Q_{54}=q_{54}-2\omega''\gamma+2(\alpha^2+2\beta)q_3-2\alpha\,q_{41}, \tag{64d}$$
$$Q_{55}=q_{55}-\omega''\gamma+(\alpha^2+2\beta)\,q_3. \tag{64e}$$

Thus, we have obtained Eq. (63) for the total amplitude $\mathcal{A}$ with the contributions of the first approximation (coefficient α), second approximation (coefficients β and γ), and third approximation (coefficients χ, ξ, and ς). It can be seen that the coefficients χ, ξ, and ς do not manifest themselves in the NLSE5 approximation.

Figure 3 demonstrates the coefficients Q_{41}, Q_{42} calculated by formulas (64) as compared to the coefficients q_{41}, q_{42}, coefficients $\hat{q}_{41}$, and $\hat{q}_{42}$, and coefficients $\tilde{q}_{41}$ and $\tilde{q}_{42}$. These are the results of Ref. [2] supplemented with the small corrections δ introduced

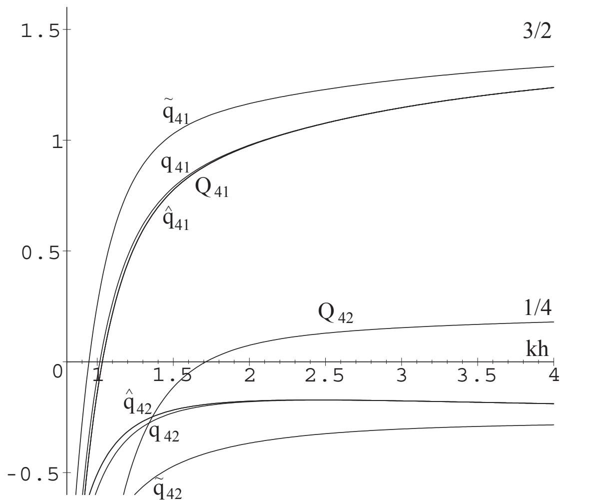

Fig. 3. Coefficients $\tilde{q}_{41}$, $\hat{q}_{41}$, q_{41}, Q_{41}, $\tilde{q}_{42}$, $\hat{q}_{42}$, q_{42}, and Q_{42} as functions of kh

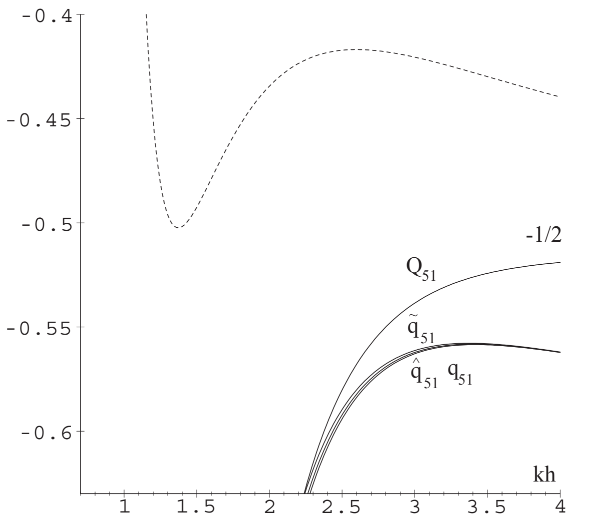

Fig. 4. Coefficients $\tilde{q}_{51}$, $\hat{q}_{51}$, q_{51}, and Q_{51} as functions of kh

in Ref. [3] for the coefficients $\hat{q}_{41}$ and $\hat{q}_{42}$, with the coefficients q_{41} and q_{42} used instead of $\hat{q}_{41}$ and $\hat{q}_{42}$. The curves demonstrating the coefficients $\hat{q}_{41}$ and $\hat{q}_{42}$ along with q_{41} and q_{42} in Fig. 3 stay close to one another for the depth $kh = 4.1$ at which the experiment of Ref. [1] was performed. This experiment proved NLSE4 to be an adequate model for the deterministic prediction of water waves. The authors of Ref. [1] used the expressions for q_{41} and q_{42} to make a comparison between the theory and the experiment.

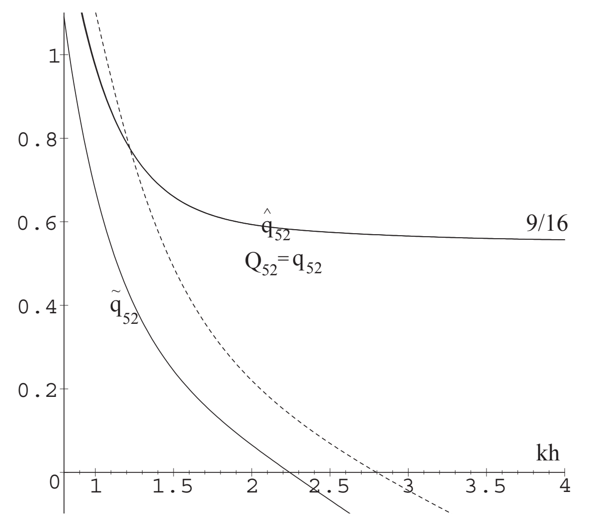

Fig. 5. Coefficients $\tilde{q}_{52}$, $\hat{q}_{52}$, q_{52}, and Q_{52} as functions of kh

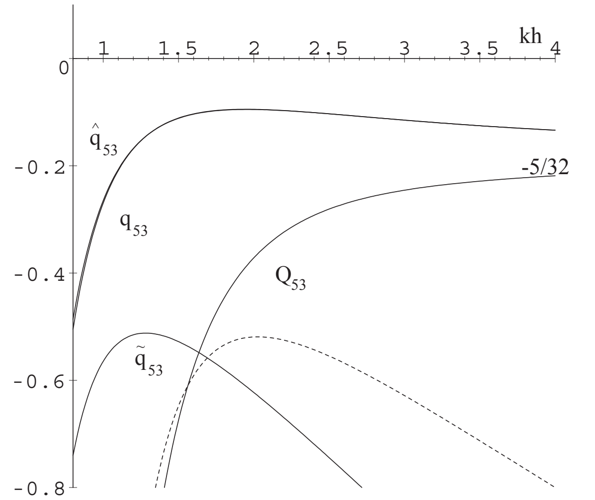

Fig. 6. Coefficients $\tilde{q}_{53}$, $\hat{q}_{53}$, q_{53}, and Q_{53} as functions of kh

Figures 4–8 demonstrate the coefficients Q_{51}, Q_{52}, Q_{53}, Q_{54}, and Q_{55} calculated by formulas (64) as compared to the coefficients q_{51}, q_{52}, q_{53}, q_{54}, and q_{55} of Eq. (51), coefficients $\hat{q}_{51}$, $\hat{q}_{52}$, $\hat{q}_{53}$, $\hat{q}_{54}$, and $\hat{q}_{55}$ of Eq. (45), and coefficients $\tilde{q}_{51}$, $\tilde{q}_{52}$, $\tilde{q}_{53}$, $\tilde{q}_{54}$, and $\tilde{q}_{55}$ of Eq. (37). The dotted curves demonstrate the corresponding plots of the coefficients Q_{nm} according to Ref. [3].

The limiting values of all the coefficients at $kh \to \infty$ are

$$\widetilde{q}_{51} = -\frac{5}{8},\quad \widehat{q}_{51} = -\frac{5}{8},\quad q_{51} = -\frac{5}{8},\quad Q_{51} = -\frac{1}{2},$$
$$\widetilde{q}_{52} = -\infty,\quad \widehat{q}_{52} = \frac{9}{16},\quad q_{52} = \frac{9}{16},\quad Q_{52} = \frac{9}{16},$$

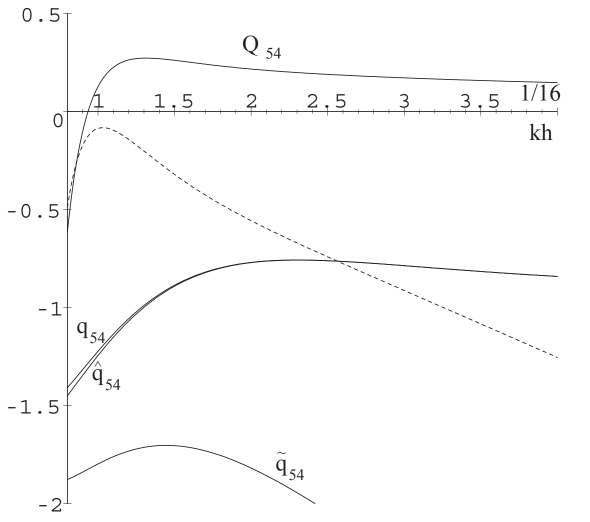

Fig. 7. Coefficients $\tilde{q}_{54}$, $\hat{q}_{54}$, q_{54}, and Q_{54} as functions of kh

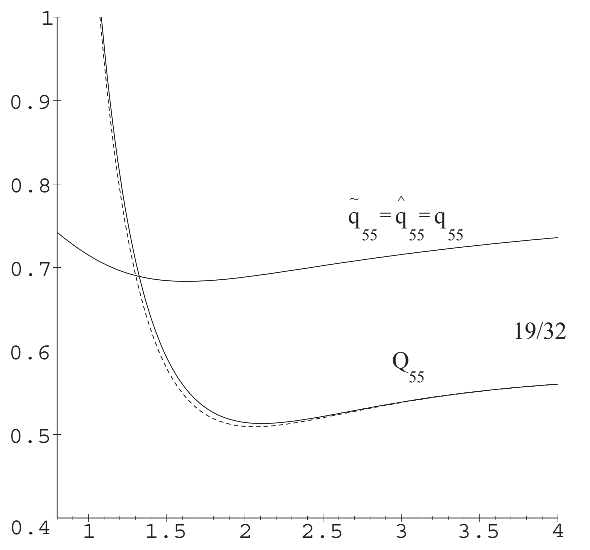

Fig. 8. Coefficients $\tilde{q}_{55}$, $\hat{q}_{55}$, q_{55}, and Q_{55} as functions of kh

$$\widetilde{q}_{53} = -\infty, \quad \widehat{q}_{53} = -\frac{3}{16}, \quad q_{53} = -\frac{3}{16}, \quad Q_{53} = -\frac{5}{32},$$

$$\widetilde{q}_{54} = -\infty, \quad \widehat{q}_{54} = -1, \quad q_{54} = -1, \quad Q_{54} = \frac{1}{16},$$

$$\widetilde{q}_{55} = \frac{13}{16}, \quad \widehat{q}_{55} = \frac{13}{16}, \quad q_{55} = \frac{13}{16}, \quad Q_{55} = \frac{19}{32}.$$

8. Conclusions

As compared to fiber optics, a peculiarity of deriving a NLSE for Stokes waves on the surface of a finite-depth fluid is the necessity of accounting for the low-frequency contribution of the zeroth harmonic to the motion of the high-frequency first harmonic. Using the boundedness condition (38) for the Ψ_3 component of the zeroth harmonic allowed us to obtain the NLSE5 for the total first-harmonic amplitude $\mathcal{A}$ of the envelope of surface profile with non-divergent coefficients in the infinite-depth limit. To the best of our knowledge, such a NLSE5 was derived here for the first time, and this fact represents the main result of this work.

We thank Dr. I.S. Gandzha for his valuable contribution to this work, in particular to the formulation of Section VII.

APPENDIX A

Fifth-order coefficients

$$q_{50}^{(a)} = \frac{1}{1024\sigma^8\nu^3}[(63\,\sigma^8 - 217\,\sigma^6 + 85\,\sigma^4 - 99\,\sigma^2 + 72)\times$$
$$\times(1-\sigma^2)^7\,kh^7 + \sigma\,(669\,\sigma^{10} - 444\,\sigma^8 - 554\,\sigma^6 + 4700\,\sigma^4 -$$
$$-1755\,\sigma^2 + 72)(\sigma^2-1)^5\,kh^6 - \sigma^2(2075\,\sigma^{12} - 5543\,\sigma^{10} + 478\,\sigma^8 +$$
$$+19578\,\sigma^6 + 5839\,\sigma^4 - 11619\,\sigma^2 - 1080)(\sigma^2-1)^3\,kh^5 +$$
$$+\sigma^3\,(\sigma^2-1)(2769\,\sigma^{14} - 11746\,\sigma^{12} + 26107\,\sigma^{10} -$$
$$-21956\,\sigma^8 - 2313\,\sigma^6 + 25726\,\sigma^4 - 24867\,\sigma^2 - 3960)\,kh^4 -$$
$$-\sigma^4(1501\,\sigma^{14} - 106\,\sigma^{12} + 5487\,\sigma^{10} - 57044\,\sigma^8 + 58283\,\sigma^6 -$$
$$-18890\,\sigma^4 + 21033\,\sigma^2 + 6120)\,kh^3 - \sigma^5(73\,\sigma^{12} - 13261\,\sigma^{10} +$$
$$+36538\,\sigma^8 + 4014\,\sigma^6 - 16427\,\sigma^4 - 2385\,\sigma^2 - 4968)\,kh^2 +$$
$$+\sigma^6\,(375\,\sigma^{10} - 9492\,\sigma^8 + 27058\,\sigma^6 - 5292\,\sigma^4 + 8127\,\sigma^2 -$$
$$-2088)\,kh - \sigma^7\,(101\,\sigma^8 - 2003\,\sigma^6 + 4983\,\sigma^4 + 3519\,\sigma^2 - 360)],$$

$$q_{50}^{(b)} = \frac{1}{1536\sigma^4\nu^2}[-(165\,\sigma^4 - 50\,\sigma^2 - 3)\,(\sigma^2-1)^6\,kh^7 +$$
$$+\sigma\,(939\,\sigma^6 - 1193\,\sigma^4 - 1823\,\sigma^2 - 99)(\sigma^2-1)^4\,kh^6 -$$
$$-3\,\sigma^2\,(771\,\sigma^8 - 600\,\sigma^6 + 2806\,\sigma^4 + 1744\,\sigma^2 + 143)(\sigma^2-1)^2\,kh^5 +$$
$$+\sigma^3\,(3255\,\sigma^{10} + 985\,\sigma^8 + 6054\,\sigma^6 - 18030\,\sigma^4 + 7411\,\sigma^2 + 2373)\,kh^4 -$$
$$-\sigma^4\,(2895\,\sigma^8 + 7348\,\sigma^6 - 7638\,\sigma^4 - 3884\,\sigma^2 + 3327)\,kh^3 +$$
$$+3\,\sigma^5\,(555\,\sigma^6 + 1423\,\sigma^4 - 1823\,\sigma^2 + 485)\,kh^2 -$$
$$-3\,\sigma^6\,(193\,\sigma^4 + 50\,\sigma^2 - 99)\,kh + 3\,\sigma^7\,(31\,\sigma^2 - 91)],$$

$$q_{50}^{(c)} = -\frac{1}{768\sigma^4\nu^2}[\,(33\,\sigma^4 + 134\,\sigma^2 + 9)\,(\sigma^2-1)^6\,kh^7 -$$
$$-\sigma\,(63\,\sigma^6 - 1093\,\sigma^4 - 2659\,\sigma^2 - 279)\,(\sigma^2-1)^4 kh^6 -$$
$$-3\,\sigma^2\,(25\,\sigma^8 + 584\,\sigma^6 - 1710\,\sigma^4 - 672\,\sigma^2 - 19)\,(\sigma^2-1)^2 kh^5 +$$
$$+\sigma^3\,(165\,\sigma^{10} - 3317\,\sigma^8 - 4110\,\sigma^6 + 16566\,\sigma^4 - 4775\,\sigma^2 -$$
$$-2481)\,kh^4 + \sigma^4\,(195\,\sigma^8 + 6356\,\sigma^6 - 7662\,\sigma^4 - 4396\,\sigma^2 +$$
$$+3459)kh^3 - 3\,\sigma^5\,(191\,\sigma^6 + 659\,\sigma^4 - 2403\,\sigma^2 + 145)kh^2 +$$
$$+3\,\sigma^6\,(141\,\sigma^4 - 486\,\sigma^2 - 535)\,kh - 3\,\sigma^7\,(35\,\sigma^2 - 239)],$$

$$\omega'''' = -\frac{\omega}{384k^4\,\sigma^4}[(\sigma^2-1)\,(105\,\sigma^6 - 51\,\sigma^4 - 5\,\sigma^2 + 15)\,kh^4 -$$

$$-4\,\sigma\,(\sigma^2-1)\,(15\,\sigma^4-2\,\sigma^2+3)\,kh^3-6\,\sigma^2\,(\sigma^2-1)\,(3\,\sigma^2+1)\,kh^2-$$
$$-12\,\sigma^3\,(\sigma^2-1)\,kh-15\,\sigma^4)],$$
$$\widetilde{q}_{51}=\frac{1}{1024\sigma^{10}\nu^2}[-(72\,\sigma^{12}-1023\,\sigma^{10}+2589\,\sigma^8-4382\,\sigma^6+$$
$$+3978\,\sigma^4-675\,\sigma^2+81)(\sigma^2-1)^4\;kh^4+$$
$$+4\sigma(72\,\sigma^{14}-807\,\sigma^{12}+1054\,\sigma^{10}+31\,\sigma^8-2580\,\sigma^6+4023\,\sigma^4-$$
$$-594\,\sigma^2+81)(\sigma^2-1)^2\,kh^3-$$
$$-2\sigma^2(248\,\sigma^{16}-1597\,\sigma^{14}+2081\,\sigma^{12}-2237\,\sigma^{10}+5689\,\sigma^8-$$
$$-10431\,\sigma^6+12987\,\sigma^4-1863\,\sigma^2+243)kh^2+$$
$$+4\sigma^3(104\,\sigma^{14}-535\,\sigma^{12}+766\,\sigma^{10}+735\,\sigma^8-1556\,\sigma^6+3303\,\sigma^4-$$
$$-594\,\sigma^2+81)\,kh-\sigma^4\,(136\,\sigma^{12}-1055\,\sigma^{10}+3837\,\sigma^8-3198\,\sigma^6+$$
$$+2538\,\sigma^4-675\,\sigma^2+81)],$$
$$\widetilde{q}_{52}=\frac{1}{128\sigma^8\nu^2}[-(20\,\sigma^{10}-119\,\sigma^8+40\,\sigma^6-138\,\sigma^4+252\,\sigma^2+9)\times$$
$$\times(\sigma^2-1)^5\,kh^6+2\,\sigma\,(52\,\sigma^{12}-175\,\sigma^{10}-145\,\sigma^8+2\,\sigma^6-174\,\sigma^4+$$
$$+669\,\sigma^2+27)(\sigma^2-1)^3\;kh^5-\sigma^2\;(\sigma^2-1)(220\,\sigma^{14}-481\,\sigma^{12}-$$
$$-402\,\sigma^{10}+1081\,\sigma^8-224\,\sigma^6-2143\,\sigma^4+2838\,\sigma^2+135)\,kh^4+$$
$$+4\,\sigma^3\,(60\,\sigma^{14}-85\,\sigma^{12}-134\,\sigma^{10}+53\,\sigma^8+88\,\sigma^6+733\,\sigma^4-$$
$$-798\,\sigma^2-45)\,kh^3-\sigma^4\,(140\,\sigma^{12}-149\,\sigma^{10}-591\,\sigma^8+194\,\sigma^6+$$
$$+1146\,\sigma^4-2013\,\sigma^2-135)\,kh^2+$$
$$+2\,\sigma^5\,(20\,\sigma^{10}-43\,\sigma^8-124\,\sigma^6+134\,\sigma^4-312\,\sigma^2-27)\,kh-$$
$$-\sigma^6\,(4\,\sigma^8-19\,\sigma^6-35\,\sigma^4-69\,\sigma^2-9)],$$
$$\widetilde{q}_{53}=\frac{1}{64\sigma^6\nu^2}[-(10\,\sigma^6-9\,\sigma^4+4\,\sigma^2-9)(\sigma^2-1)^6\;kh^6+$$
$$+2\,\sigma\,(26\,\sigma^8-27\,\sigma^6+21\,\sigma^4+3\,\sigma^2-15)(\sigma^2-1)^4\,kh^5-$$
$$-\sigma^2(110\,\sigma^{10}-207\,\sigma^8-32\,\sigma^6-186\,\sigma^4-118\,\sigma^2-15)(\sigma^2-1)^2kh^4+$$
$$+4\,\sigma^3\,(30\,\sigma^{12}-105\,\sigma^{10}+15\,\sigma^8+42\,\sigma^6+28\,\sigma^4-89\,\sigma^2+15)\,kh^3-$$
$$-\sigma^4\,(70\,\sigma^{10}-283\,\sigma^8+32\,\sigma^6+290\,\sigma^4-150\,\sigma^2+105)\,kh^2+$$
$$+2\,\sigma^5\,(10\,\sigma^8-59\,\sigma^6+93\,\sigma^4+67\,\sigma^2+33)\,kh-$$
$$-\sigma^6\,(2\,\sigma^6-21\,\sigma^4+84\,\sigma^2+15)],$$
$$\widetilde{q}_{54}=\frac{1}{256\nu^2\sigma^6}[(43\,\sigma^8-165\,\sigma^6-11\,\sigma^4+329\,\sigma^2-324)\times$$
$$\times(\sigma^2-1)^5\,kh^6-2\,\sigma\,(111\,\sigma^{10}-654\,\sigma^8+76\,\sigma^6+258\,\sigma^4+$$
$$+357\,\sigma^2-660)\,(\sigma^2-1)^3\,kh^5+\sigma^2\,(\sigma^2-1)(465\,\sigma^{12}-3557\,\sigma^{10}+$$
$$+4286\,\sigma^8-890\,\sigma^6-1683\,\sigma^4+1071\,\sigma^2-1740)\,kh^4-$$
$$-4\,\sigma^3\,(125\,\sigma^{12}-983\,\sigma^{10}+1514\,\sigma^8-1106\,\sigma^6+525\,\sigma^4+377\,\sigma^2+$$
$$+60)\,kh^3+3\,\sigma^4\,(95\,\sigma^{10}-568\,\sigma^8+138\,\sigma^6-756\,\sigma^4+527\,\sigma^2-$$
$$-460)\,kh^2-2\,\sigma^5\,(39\,\sigma^8-55\,\sigma^6-1107\,\sigma^4-465\,\sigma^2-588)\,kh+$$
$$+\sigma^6\,(7\,\sigma^6+90\,\sigma^4-917\,\sigma^2-300)],$$
$$\widetilde{q}_{55}=\frac{1}{256\sigma^8\nu^2}[-(27\,\sigma^{10}+63\,\sigma^8-85\,\sigma^6-337\,\sigma^4+414\,\sigma^2-18)\times$$
$$\times(\sigma^2-1)^5\,kh^6+2\sigma(63\,\sigma^{12}-138\,\sigma^{10}-98\,\sigma^8+58\,\sigma^6-727\,\sigma^4+$$
$$+1152\,\sigma^2-54)\,(\sigma^2-1)^3kh^5-\sigma^2\,(\sigma^2-1)(225\,\sigma^{14}-1297\,\sigma^{12}+$$
$$+1792\,\sigma^{10}+440\,\sigma^8+161\,\sigma^6-5481\,\sigma^4+5454\,\sigma^2-270)kh^4+$$
$$+4\sigma^3(\sigma^2-1)(45\,\sigma^{12}-246\,\sigma^{10}+422\,\sigma^8-322\,\sigma^6-369\,\sigma^4+$$
$$+1584\,\sigma^2-90)\,kh^3-\sigma^4\,(45\,\sigma^{12}+180\,\sigma^{10}-740\,\sigma^8-2524\,\sigma^6+$$
$$+3905\,\sigma^4-4464\,\sigma^2+270)\,kh^2-2\,\sigma^5\,(9\,\sigma^{10}-301\,\sigma^8+$$
$$+1101\,\sigma^6-93\,\sigma^4+810\,\sigma^2-54)\,kh+\sigma^6\,(9\,\sigma^8-198\,\sigma^6+$$
$$+739\,\sigma^4+252\,\sigma^2-18)].$$

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НЕЛІНІЙНЕ РІВНЯННЯ
ШРЕДІНҐЕРА П'ЯТОГО ПОРЯДКУ
ДЛЯ ХВИЛЬ НА ПОВЕРХНІ ШАРУ РІДИНИ

В продовження попередньої роботи автора (ЖЕТФ, **97**, 2003) виведено наступного, п'ятого порядку нелінійне рівняння Шредінґера для обвідної осцилюючих хвиль на поверхні шару безвихорної, нестискаємої, нев'язкої рідини у випадку плоского дна. Це рівняння враховує четвертого порядку лінійну дисперсію, третього і п'ятого порядку нелінійність і кубічні по дисперсії нелінійно-дисперсійні ефекти. Коефіцієнти цього рівняння даються як функції безрозмірного параметра kh, де k – хвильове число несучої хвилі, h – глибина незбуреної рідини і лишаються обмеженими в граничному випадку нескінченної глибини.

Ключові слова: теорія нелінійних хвиль, слабка нелінійність, нелінійне рівняння Шредінґера, солітони обвідної, детерміністичний опис хвиль.