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SYMPLECTIC FIELD THEORY OF THE GALILEAN COVARIANT SCALAR AND SPINOR REPRESENTATIONS

We explore the concept of the extended Galilei group, a representation for the symplectic quantum mechanics in the manifold $\mathcal{G}$, written in the light-cone of a five-dimensional de Sitter space-time in the phase space. The Hilbert space is constructed endowed with a symplectic structure. We study the unitary operators describing rotations and translations, whose generators satisfy the Lie algebra of $\mathcal{G}$. This representation gives rise to the Schrödinger (Klein–Gordon-like) equation for the wave function in the phase space such that the dependent variables have the position and linear momentum contents. The wave functions are associated to the Wigner function through the Moyal product such that the wave functions represent a quasiamplitude of probability. We construct the Pauli–Schrödinger (Dirac-like) equation in the phase space in its explicitly covariant form. Finally, we show the equivalence between the five-dimensional formalism of the phase space with the usual formalism, proposing a solution that recovers the non-covariant form of the Pauli–Schrödinger equation in the phase space.

Keywords: Galilean covariance, star-product, phase space, symplectic structure.

1. Introduction

In 1988, Takahashi *et. al.* [1] began a study of the Galilean covariance, where it was possible to develop an explicitly covariant non-relativistic field theory. With this formalism, the Schrödinger equation takes a similar form as the Klein–Gordon equation in the light-cone of a (4,1) de Sitter space [2, 3]. With the advent of the Galilean covariance, it was possible to derive the non-relativistic version of the Dirac theory, which is known in its usual form as the Pauli–Schrödinger equation. The goal in the present work is to derive a Wigner representation for such covariant theory.

The Wigner quasiprobability distribution (also called the Wigner function or the Wigner–Ville distribution in honor of Eugene Wigner and Jean–André Ville) was introduced by Eugene Wigner in 1932 [4] in order to study quantum corrections to classical statistical mechanics. The aim was to relate the wave function that appears in the Schrödinger equation to a probability distribution in the phase space. It is a generating function for all the spatial autocorrelation functions of a given quantum mechanical function $\psi(x)$. Thus, it maps the quantum density matrix onto the real phase space functions and operators introduced by Hermann Weyl in 1927 [5] in a context related to the theory of representations in mathematics (Weyl quantization in physics). Indeed, this is the Wigner–Weyl transformation of the density matrix; i.e., the realization of that operator in the phase space. It was later re-derived by Jean Ville in 1948 [6] as a quadratic representation (in sign) of the local time frequency energy of a signal, effectively a spectrogram. In 1949, José Enrique Moyal [7], who independently derived the Wigner function, as the functional generator of the quantum momentum, as a basis for an elegant codification of all expected values and, therefore, of quantum mechanics in the phase-space formulation (phase-space representation). This representation has been applied to a number of areas such as statistical mechanics, quantum chemistry, quantum optics, classical optics, signal analysis, electrical engineering, seismology, time-frequency analysis for music signals, spectrograms in biology and speech processing, and motor design. In order to derive a phase-space representation for the Galilean-covariant spin 1/2 particles, we use a symplectic representation for the Galilei group, which is associated with the Wigner approach [8–11].

This article is organized as follows. In Section 2, the construction of the Galilean covariance is presented. The Schrödinger (Klein–Gordon-like) equation and the Pauli–Schrödinger (Dirac-like) equation are derived showing the equivalence between our formalism and the usual non-relativistic formalism. In Section 4, a symplectic structure is constructed in the Galilean manifold. Using the commutation relations, the Schrödinger equation in five dimensions in the phase space is constructed. With a proposed solution, the Schrödinger equation in the phase space is restored to its non-covariant form in $(3+1)$ dimensions. The explicitly covariant Pauli–Schrödinger equation is derived in Section 5. We study a Galilean spin 1/2 particle in a external potential, and the solutions are proposed and discussed. In Section 6, the final concluding remarks are presented.

2. Galilean Covariance

The Galilei transformations are given by

$$\mathbf{x}' = R\mathbf{x} + \mathbf{v}t + \mathbf{a}, \tag{1}$$

$$t' = t + b, \tag{2}$$

where R stands for the three-dimensional Euclidian rotations, v is the relative velocity defining the Galilean boosts, $\mathbf{a}$ and $\mathbf{b}$ stand for spatial and time translations, respectively. Consider a free particle with mass m; the mass shell relation is given by $\widehat{P}^2 - 2mE = 0$. Then we can define a 5-vector, $p^\mu = (p_x, p_y, p_z, m, E) = (p^i, m, E)$, with $i = 1, 2, 3$.

Thus, we can define a scalar product of the type

$$p_\mu p_\nu g^{\mu\nu} = p_i p_i - p_5 p_4 - p_4 p_5 = \widehat{P}^2 - 2mE = k, \tag{3}$$

where $g^{\mu\nu}$ is the metric of the space-time to be constructed, e $p_\nu g^{\mu\nu} = p^\mu$.

Let us define a set of canonical coordinates q^μ associated with p^μ, by writing a five-vector in M, $q^\mu = (\mathbf{q}, q^4, q^5)$, $\mathbf{q}$ is the canonical coordinate associated with $\widehat{P}$; q^4 is the canonical coordinate associated with E, and thus can be considered as the time coordinate; q^5 is the canonical coordinate associated with m explicitly given in terms of $\mathbf{q}$ and q^4, $q^\mu q_\mu = q^\mu q^\nu \eta_{\mu\nu} = \mathbf{q}^2 - q^4 q^5 = s^2 = 0$. From this $q^5 = \frac{\mathbf{q}^2}{2t}$, or infinitesimally, we obtain $\delta q^5 = \mathbf{v}\,\delta\frac{\mathbf{q}}{2}$. Therefore, the fifth component is basically defined by the velocity.

That can be seen as a special case of scalar product in G denoted as

$$(x|y) = g^{\mu\nu} x_\mu y_\nu = \sum_{i=1}^{3} x_i y_i - x_4 y_5 - x_5 y_4, \tag{4}$$

where $x^4 = y^4 = t$, $x^5 = \frac{x^2}{2t}$ e $y^5 = \frac{y^2}{2t}$. Hence, the following metric can be introduced:

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{5}$$

This is the metric of a Galilean manifold $\mathcal{G}$. In the sequence, this structure is explored in order to study unitary representations.

3. Hilbert Space and Sympletic Structure

Consider an analytical manifold $\mathcal{G}$, where each point is specified by the coordinates q_μ, with $\mu = 1, 2, 3, 4, 5$ and the metric specified by (5). The coordinates of every point in the cotangent-bundle $T^*\mathcal{G}$ will be denoted by (q_μ, p_μ). The space $T^*\mathcal{G}$ is equipped with a symplectic structure via the 2-form

$$\omega = dq^\mu \wedge dp_\mu \tag{6}$$

called the symplectic form (sum over repeated indices is assumed). We consider the following bidifferential operator on $C^\infty(T^*\mathcal{G})$ functions,

$$\Lambda = \frac{\overleftarrow{\partial}}{\partial q^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p^\mu}\frac{\overrightarrow{\partial}}{\partial q_\mu}, \tag{7}$$

such that, for C^∞ functions, $f(q,p)$ and $g(q,p)$, we have

$$\omega(f\Lambda, g\Lambda) = f\Lambda g = \{f, g\} \tag{8}$$

where

$$\{f, g\} = \frac{\partial f}{\partial q^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p^\mu}\frac{\partial g}{\partial q_\mu}. \tag{9}$$

It is the Poisson bracket, and $f\Lambda$ and $g\Lambda$ are two vector fields given by $h\Lambda = X_h = -\{h, \}$.

The space $T^*\mathcal{G}$ endowed with this symplectic structure is called the phase space and will be denoted by Γ. In order to associate the Hilbert space with the phase space Γ, we will consider the set of square-integrable complex functions, $\phi(q,p)$ in Γ such that

$$\int dp dq\, \phi^\dagger(q,p)\phi(q,p) < \infty \tag{10}$$

is a real bilinear form. In this case, $\phi(q,p) = \langle q,p|\phi\rangle$ is written with the aid of

$$\int dpdq|q,p\rangle\langle q,p| = 1, \tag{11}$$

where $\langle\phi|$ is the dual vector of $|\phi\rangle$. This symplectic Hilbert space is denoted by $H(\Gamma)$.

4. Symplectic Quantum Mechanics and the Galilei Group

In this section, we will study the Galilei group considering $H(\Gamma)$ as the space of representation. To do so, consider the unit transformations $U{:}\mathcal{H}(\Gamma) \to \mathcal{H}(\Gamma)$ such that $\langle\psi_1|\psi_2\rangle$ is invariant. Using the Λ operator, we define a mapping $e^{i\frac{\Lambda}{2}} = \star{:}\Gamma\times\Gamma \to \Gamma$ called a Moyal (or star) product and defined by

$$f\star g = f(q,p)\exp\left[\frac{i}{2}\left(\frac{\overleftarrow{\partial}}{\partial q^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p^\mu}\frac{\overrightarrow{\partial}}{\partial q_\mu}\right)\right]g(q,p),$$

it should be noted that we used $\hbar = 1$. The generators of U can be introduced by the following (Moyal–Weyl) star-operators:

$$\widehat{F} = f(q,p)\star = f\left(q^\mu + \frac{i}{2}\frac{\partial}{\partial p_\mu}, p^\mu - \frac{i}{2}\frac{\partial}{\partial q_\mu}\right).$$

To construct a representation of the Galilei algebra in $\mathcal{H}$, we define the operators

$$\widehat{P}^\mu \;=\; p^\mu\star = p^\mu - \frac{i}{2}\frac{\partial}{\partial q_\mu}, \tag{12a}$$

$$\widehat{Q}^\mu \;=\; q\star = q^\mu + \frac{i}{2}\frac{\partial}{\partial p_\mu}. \tag{12b}$$

and

$$\widehat{M}_{\nu\sigma} \;=\; M_{\nu\sigma}\star = \widehat{Q}_\nu\widehat{P}_\sigma - \widehat{Q}_\sigma\widehat{P}_\nu, \tag{12c}$$

where $\widehat{M}_{\nu\sigma}$ and $\widehat{P}_\mu$ are the generators of homogeneous and inhomogeneous transformations, respectively. From this set of unitary operators, we obtain, after some simple calculations, the following set of commutations relations:

$$\left[\widehat{P}_\mu, \widehat{M}_{\rho\sigma}\right] = -i(g_{\mu\rho}\widehat{P}^\sigma - g_{\mu\sigma}\widehat{P}^\rho),$$

$$\left[\widehat{P}_\mu, \widehat{P}_\sigma\right] = 0,$$

and

$$\left[\widehat{M}_{\mu\nu}, \widehat{M}_{\rho\sigma}\right] =$$

$$= -i(g_{\nu\rho}\widehat{M}_{\mu\sigma} - g_{\mu\rho}\widehat{M}_{\nu\sigma} + g_{\mu\sigma}\widehat{M}_{\nu\rho} - g_{\mu\sigma}\widehat{M}_{\nu\rho}).$$

Consider a vector $q^\mu \in G$ that obeys the set of linear transformations of the type

$$\bar{q}^\mu = G^\mu{}_\nu q^\nu + a^\mu. \tag{13}$$

A particular case of interest in these transformations is given by

$$\bar{q}^i = R^i_j q^j + v^i q^4 + a^i \tag{14}$$

$$\bar{q}^4 = q^4 + a^4 \tag{15}$$

$$\bar{q}^5 = q^5 - (R^i_j q^j)v_i + \frac{1}{2}\mathbf{v}^2 q^4. \tag{16}$$

In the matrix form, the homogeneous transformations are written as

$$G^\mu{}_\nu = \begin{pmatrix} R^1_{\ 1} & R^1_{\ 2} & R^1_{\ 3} & v^i & 0 \\ R^2_{\ 1} & R^2_{\ 2} & R^2_{\ 3} & v^2 & 0 \\ R^3_{\ 1} & R^3_{\ 2} & R^3_{\ 3} & v^3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ v_iR^i{}_j & v_iR^i{}_2 & v_iR^i{}_3 & \frac{\mathbf{v}^2}{2} & 1 \end{pmatrix}. \tag{17}$$

We can write the generators as

$$\begin{aligned} &\widehat{J}_i = \frac{1}{2}\epsilon_{ijk}\widehat{M}_{jk}, \quad \widehat{C}_i = \widehat{M}_{4i}, \\ &\widehat{K}_i = \widehat{M}_{5i}, \quad \widehat{D} = \widehat{M}_{54}. \end{aligned} \tag{18}$$

Hence, the non-vanishing commutation relations can be rewritten as

$$\begin{aligned} &\left[\widehat{J}_i, \widehat{J}_j\right] = i\epsilon_{ijk}\widehat{J}_k, && \left[\widehat{J}_i, \widehat{K}_j\right] = i\epsilon_{ijk}\widehat{K}_k, \\ &\left[\widehat{J}_i, \widehat{C}_j\right] = i\epsilon_{ijk}\widehat{C}_k, && \left[\widehat{K}_i, \widehat{C}_j\right] = i\delta_{ij}\widehat{D} + i\epsilon_{ijk}J_k, \\ &\left[\widehat{D}, \widehat{K}_i\right] = i\widehat{K}_i, && \left[\widehat{C}_i, \widehat{D}\right] = i\widehat{C}_i, \\ &\left[\widehat{P}_4, \widehat{D}\right] = i\widehat{P}_4, && \left[\widehat{J}_i, \widehat{P}_j\right] = i\epsilon_{ijk}\widehat{P}_k, \\ &\left[\widehat{P}_i, \widehat{K}_j\right] = i\delta_{ij}\widehat{P}_5, && \left[\widehat{P}_i, \widehat{C}_j\right] = i\delta_{ij}\widehat{P}_4, \\ &\left[\widehat{P}_4, \widehat{K}_i\right] = i\widehat{P}_i, && \left[\widehat{P}_5, \widehat{C}_i\right] = i\widehat{P}_i. \\ &\left[\widehat{D}, \widehat{P}_5\right] = i\widehat{P}_5, \end{aligned} \tag{19}$$

These relations have the Lie algebra of the Galilei group as a subalgebra in the case of $\mathcal{R}^3\times\mathcal{R}$, considering J_i the generators of rotations K_i of the pure Galilei transformations, P_μ the spatial and temporal translations. In fact, we can observe that Eqs. (14) and (15) are the Galilei transformations given by

Eq. (1) and (2) with $x^4 = t$. Equation (16) is the compatibility condition which represents the embedding

$$\mathcal{I} : \mathbf{A} \to A = \left(\mathbf{A}, A_4, \frac{\mathbf{A}^2}{2A_4}\right); \quad \mathbf{A} \in \mathcal{E}_3, A \in \mathcal{G}.$$

The commutation of K_i and P_i is naturally nonzero in this context, so P_5 will be related to the mass, which is the extension parameter of the Galilei group or an invariant of the extended Galilei–Lie algebra. So, the invariants of this algebra in the light cone of the de Sitter space-time are

$$I_1 = \widehat{P}_\mu \widehat{P}^\mu \tag{20}$$

$$I_2 = \widehat{P}_5 = -mI \tag{21}$$

$$I_3 = \widehat{W}_{5\mu} \widehat{W}_5^\mu, \tag{22}$$

where I is the identity operator, m is the mass, $W_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\alpha\beta\rho\nu} P^\alpha M^{\beta\rho}$ is the 5-dimensional Pauli–Lubanski tensor, and $\epsilon_{\mu\nu\alpha\beta\rho}$ is the totally antisymmetric tensor in five dimensions. In the scalar represantation, we can defined $I_3 = 0$. Using the Casimir invariants I_1 and I_2 and applying them to Ψ, we have

$$\widehat{P}_\mu \widehat{P}^\mu \Psi = k^2 \Psi,$$
$$\widehat{P}_5 \Psi = -m\Psi.$$

We obtain

$$\left(p^2 - i\mathrm{p}\,\nabla - \frac{1}{4}\nabla^2 - k^2\right)\Psi =$$
$$= 2\left(p_4 - \frac{i}{2}\partial_t\right)\left(p_5 - \frac{i}{2}\partial_5\right)\Psi,$$

and a solution of this equation is

$$\Psi = e^{-i2p_5 q^5} \rho(q^5) e^{-2ip_4 t} \chi(t) \Phi(\mathbf{q}, \mathbf{p}). \tag{23}$$

Thus,

$$\left(p^2\Phi - i\mathrm{p}\,\nabla\Phi - \frac{1}{4}\nabla^2\Phi - k^2\right)\frac{1}{\Phi} =$$
$$= \frac{1}{2}\left(i\partial_t\chi\right)\left(i\partial_5\rho\right)\frac{1}{\chi\rho},$$

which yields

$$i\partial_t\chi = \alpha\chi, \quad \text{and} \quad i\partial_5\rho = \beta\rho.$$

Thus, our solution for χ and ρ is

$$\chi = e^{-i\alpha t}, \quad \rho = e^{-i\beta q^5}. \tag{24}$$

Using the fact that

$$\widehat{P}_4\Psi = \left(p_4 - \frac{i}{2}\partial_t\right) e^{-i(2p_4+\alpha)t} = -E\, e^{-i(2p_4+\alpha)t}$$

and

$$\widehat{P}_5\Psi = \left(p_5 - \frac{i}{2}\partial_5\right) e^{-i(2p_5+\beta)q^5} = -m\, e^{-i(2p_5+\beta)q^5},$$

we can conclude that

$$\alpha = 2E, \quad \beta = 2m. \tag{25}$$

So, we have

$$\frac{1}{2m}\left(p^2 - i\boldsymbol{p}\,\nabla - \frac{1}{4}\nabla^2\right)\Phi = \left(E + \frac{k^2}{2m}\right)\Phi,$$

which is the usual form of the Schrödinger equation in the phase space for a free particle with mass m and with an additional kinetic energy of $\frac{k^2}{2m}$, that we can always set as the zero of energy.

This equation and its complex conjugate can also be obtained by using the Lagrangian density in the phase space (we use $d^\mu = d/dq_\mu$)

$$\mathcal{L} = \partial^\mu\Psi(q,p)\partial\Psi^*(q,p) + \frac{i}{2}p^\mu[\Psi(q,p)\partial^\mu\Psi^*(q,p) -$$
$$- \Psi^*(q,p)\partial^\mu\Psi(q,p)] + \left[\frac{p^\mu p_\mu}{4} - k^2\right]\Psi.$$

The association of this representation with the Wigner formalism is given by

$$f_w(q,p) = \Psi(q,p) \star \Psi^\dagger(q,p),$$

where $f_w(q,p)$ is the Wigner function. To prove this, we recall that Eq. (23) can be written as

$$\widehat{P}_\mu \widehat{P}^\mu \Psi = p^2 \star \Psi(q,p).$$

Multiplying the right-hand side of the above equation by $\Psi^\dagger$, we obtain

$$(p^2 \star \Psi) \star \Psi^\dagger = k^2 \Psi \star \Psi^\dagger. \tag{26}$$

But $\Psi^\dagger \star p^2 = k^2 \Psi^\dagger$. Thus,

$$\Psi \star (\Psi^\dagger \star p^2) = k^2 \Psi \star \Psi^\dagger. \tag{27}$$

Subtracting (27) from (26), we have

$$p^2 \star f_w(q,p) - p^2 \star f_w(q,p) = 0, \tag{28}$$

which is the Moyal brackets $\{p^2, f_w\}_M$. In view of Eq. (12a), Eq. (28) becomes

$$p_\mu \partial_{q_\mu} f_w(q,p) = 0, \tag{29}$$

where the Wigner function in the Galilean manifold is a solution of this equation.

5. Spin 1/2 Symplectic Representaion

In order to study the representations of spin-1/2 particles, we introduce $\gamma^\mu \widehat{P}_\mu$, where $\widehat{P}_\mu = p_\mu - \frac{i}{2}\partial_\mu$ in such a way that, acting on the 5-spinor in the phase space $\Psi(q,p)$, we have

$$\gamma^\mu \left(p_\mu - \frac{i}{2}\partial_\mu\right)\Psi(p,q) = k\Psi(p,q), \tag{30}$$

which is the Galilean-covariant Pauli–Schrödinger equation. Consequently, the mass shell condition is obtained by the usual steps:

$$(\gamma^\mu \widehat{P}_\mu)(\gamma^\nu \widehat{P}_\nu)\Psi(q,p) = k^2\Psi(q,p). \tag{31}$$

Therefore,

$$\gamma^\mu\gamma^\nu(\widehat{P}_\mu \widehat{P}_\nu) = k^2 = \widehat{P}^\mu \widehat{P}_\nu. \tag{32}$$

Since $\widehat{P}_\mu \widehat{P}_\nu = \widehat{P}_\nu \widehat{P}_\mu$, we have

$$\frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\widehat{P}_\mu \widehat{P}_\nu = \widehat{P}^\mu \widehat{P}_\nu, \tag{33}$$

so

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{34}$$

Equation (30) can be derived from the Lagrangian density for spin-1/2 particles in the phase space, which is given by

$$\mathcal{L} = -\frac{i}{4}\Big((\partial_\mu\bar{\Psi})\gamma^\mu\Psi - \bar{\Psi}(\gamma^\mu\partial_\mu\Psi)\Big) - (k - \gamma^\mu p_\mu)\Psi\bar{\Psi},$$

where $\bar{\Psi} = \zeta\Psi^\dagger$, with$\zeta = -\frac{i}{\sqrt{2}}\{\gamma^4 + \gamma^5\} = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}$. In the Galilean-covariant Pauli–Schrödinger equation case, the association to the Wigner function is given by $f_w = \Psi \star \bar{\Psi}$, with each component satisfying Eq. (29).

Let us now examine the gauge symmetries in the phase space demanding the invariance of the Lagrangian under a local gauge transformation given by $e^{\Lambda(q,p)}\Psi$. This leads to the minimum coupling,

$$\widehat{P}_\mu\Psi \to (\widehat{P}_\mu - eA_\mu)\Psi = \left(p_\mu - \frac{i}{2}\partial_\mu - eA_\mu\right)\Psi.$$

This describes an electron in an external field with the Pauli–Schrödinger equation given by

$$\left[\gamma^\mu\left(p_\mu - \frac{i}{2}\partial_\mu - eA_\mu\right) - k\right]\Psi = 0. \tag{35}$$

In order to illustrate such result, let us consider a electron in an external field given by $A_\mu(\mathbf{A}, A_4, A_5)$, with $A_4 = -\phi$ and $A_5 = 0$. Considering the representation of the γ^μ matrices

$$\gamma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix},\ \gamma^4 = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0\end{pmatrix},\ \gamma^5 = \begin{pmatrix} 0 & -\sqrt{2} \\ 0 & 0\end{pmatrix}.$$

where σ^i are the Pauli matrices, and $\sqrt{2}$ is the identity 2×2 matrix multiplied by $\sqrt{2}$. We can rewrite the object Ψ, as $\Psi = \begin{pmatrix}\varphi \\ \chi\end{pmatrix}$, where φ and χ are 2-spinors dependent on $x^\mu; \mu = 1, ..., 5$. Thus, in the representation where $k = 0$, the Eq. (35) becomes

$$\begin{aligned} &\sigma\left(\mathbf{p} - \frac{i}{2}\partial_q - e\mathbf{A}\right)\varphi - \sqrt{2}\left(p_5 - \frac{i}{2}\partial_5\right)\chi = 0, \\ &\sqrt{2}\left(p_4 - \frac{i}{2}\partial_t - e\phi\right)\varphi - \boldsymbol{\sigma}\left(\mathbf{p} - \frac{i}{2}\partial_q - e\mathbf{A}\right)\chi = 0. \end{aligned} \tag{36}$$

Solving the coupled equations, we get an equation for φ and χ. Replacing the eigenvalues of $\widehat{P}_4$ and $\widehat{P}_5$, we have

$$\left[\frac{1}{2m}\left(\boldsymbol{\sigma}\left(\mathbf{p} - \frac{i}{2}\partial_q - e\mathbf{A}\right)\right)^2 + e\phi\right]\varphi = E\varphi,$$

$$\left[\frac{1}{2m}\left(\boldsymbol{\sigma}\left(\mathbf{p} - \frac{i}{2}\partial_q - e\mathbf{A}\right)\right)^2 + e\phi\right]\chi = E\chi.$$

These are the non-covariant form of the Pauli–Schrödinger equation in the phase space independent of the time with

$$f_w = \Psi \star \bar{\Psi} = i\varphi \star \chi^\dagger - i\chi \star \varphi^\dagger.$$

This leads to

$$E_n = \frac{eB}{m}\left(n + \frac{1}{2} - \frac{s}{2}\right) - \frac{k^2}{2m},$$

where $s = \pm 1$. It should be noted that the above expression represents the Landau levels which show the spin-splitting feature.

The above Figures 1 and 2 show the Wigner functions for the ground and first excited states, respec-

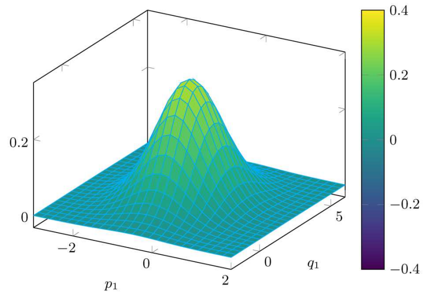

Fig. 1. Wigner Function (cut in q_1,p_1),Ground State

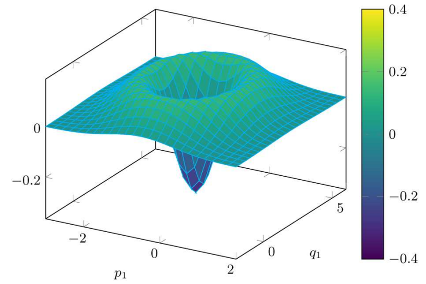

Fig. 2. Wigner Function (cut in q_1,p_1), First Excited State

tively, in the cut (q_1, p_1). These are the same solutions known in the literature using the usual Wigner method.

6. Concluding Remarks

We study the spin-1/2 particle equation, the Pauli–Schrödinger equation, in the context of the Galilean covariance, considering a symplectic Hilbert space. We begin with a presentation on the Galilean manifold which is used to review the construction of the Galilean covariance and the representations of quantum mechanics in this formalism, namely, the spin-1/2 and scalar representations and the Schrödinger (Klein–Gordon-like) and Pauli–Schrödinger (Dirac-like) equations, respectively.

The quantum mechanics formalism in the phase space is derived in this context of the Galilean covariance giving rise to the representations of spin-0 and spin-1/2 equations. For the spin-1/2 equation (the Dirac-like equation), we study the electron in an external field. Solving it, we were able to recover the non-covariant Pauli–Schrödinger equation in phase space and to analyze, in this context, the Landau levels.

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1. Y. Takahashi. Towards the many-body theory with the Galilei invariance as a guide: Part I. *Fortschr. Phys.* **36**, 63 (1988).
2. M. Omote, S. Kamefuchi, Y. Takahashi, Y. Ohnuki. Galilean covariance and the Schrödinger equation. *Fortschr. Phys.* **37**, 933 (1989).
3. A.E. Santana, F.C. Khanna, Y. Takahashi. Galilei covariance and (4,1) de Sitter space. *Prog. Theor. Phys.* **99**, 327 (1998).
4. E. Wigner. Über das überschreiten von potentialschwellen bei chemischen reaktionen. *Z. Phys. Chem.* **19**, 203 (1932).
5. H. Weyl. Quantenmechanik und gruppentheorie. *Z. Phys.* **46**, 1 (1927).
6. J. Ville. Theorie et application de la notion de signal analytique. *Cables et Transmissions* **2**, 61 (1948).
7. J.E. Moyal. Quantum mechanics as a statistical theory. *Math. Proc. Cambr. Phil. Soc.* **45**, 99 (1949).
8. M.D. Oliveira, M.C.B. Fernandes, F.C. Khanna, A.E. Santana, J.D.M. Vianna. Symplectic quantum mechanics. *Ann. Phys.* **312**, 492 (2004).
9. R.G.G. Amorim, F.C. Khanna, A.P.C. Malbouisson, J.M.C. Malbouisson, A.E. Santana. Quantum field theory in phase space. *Int. J. Mod. Phys. A* 1950037 (2019).
10. R.A.S. Paiva, R.G.G. Amorim. Quantum physics in phase space: An analysis of simple pendulum. *Adv. Theor. Comp. Phys.* **1**, 1 (2018).
11. H. Dessano, R.A.S. Paiva, R.G.G. Amorim, S.C. Ulhoa, A.E. Santana. Wigner function and non-classicality for oscillator systems. *Brazilian J. Phys.* 1 (2019).

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СИМПЛЕКТИЧНА ТЕОРІЯ ПОЛЯ
ГАЛІЛЕЄВО-КОВАРІАНТНИХ СКАЛЯРНОГО
І СПІНОРНОГО ПРЕДСТАВЛЕНЬ

Р е з ю м е

Ми досліджуємо концепцію розширеної групи Галілея, деякого представлення для симплектичної квантової механіки на многовиді $\mathcal{G}$, заданого на світловому конусі п'ятивимірного простору-часу де Сіттера у фазовому просторі. Побудувано Гільбертів простір, наділений симплектичною структурою. Ми вивчаємо унітарні оператори, що описують повороти і трансляції, генератори яких утворюють алгебру Лі в $\mathcal{G}$. Це представлення породжує рівняння Шредінгера (типу Кляйна–Гордона) для хвильової функції у фазовому просторі, так що змінні мають зміст положення і лінійного імпульсу. Хвильові функції пов'язані з функцією Вігнера через добуток Мойала, так що хвильові функції репрезентують квазіамплітуду ймовірності. Ми будуємо рівняння Паулі–Шредінгера (типу рівняння Дірака) у фазовому просторі в явно коваріантній формі. На завершення ми показуємо еквівалентність між п'ятивимірним формалізмом фазового простору і звичайним формалізмом, пропонуючи розв'язок, що відновлює нековаріантну форму рівняння Паулі–Шредінгера у фазовому просторі.