GENERAL PHYSICS

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PROPAGATION OF THE MEASUREMENT ERROR AND THE MEASURED MEAN OF A PHYSICAL QUANTITY FOR ELEMENTARY FUNCTIONS a^x AND $\log_a x$

Rules have been obtained for the propagation of the error and the mean value for a measured physical quantity onto another one with a functional relation of the type a^x or $\log_a x$ between them. In essence, these rules are inherently based on the Gaussian weight scheme. Therefore, they should be valid in the framework of a real Gaussian weight scheme applied to discrete data of a real physical experiment (a sample). An analytical form that was used to present the rules concerned ("analytical propagation rules") and their character allow the processing and the analysis of experimental data to be simplified and accelerated.

K e y w o r d s: propagation of an error, propagation of an uncertainty.

1. Introduction

This work is devoted to an actual topic of estimating the errors for indirectly measured physical quantities. It makes a contribution to the general problem of error propagation. In more details, the problem of error propagation was described in works [1, 2]. It was also discussed in work [3].

There are two approaches to solve this problem. All modern theoretical and practical applications, methods, and developments concerning the propagation of errors are based exclusively on the expansion in a Taylor series ("differentiation") [4–13]. The best description of the problem of "analytical" error propagation is given in work [1]. In this paper, some efforts in this direction were made. Namely, two widely used elementary functions, a^x and $\log_a x$ (or, equivalently, e^x and $\ln x$) were considered. This work is a logical continuation of works [2, 3].

2. New Rules for the Calculation of Means and the Propagation of Errors in the Case of Elementary Functions a^x and $\log_a x$

To obtain analytical rules for two examined functions (a^x and $\log_a x$), the mean x_{av} and the "error" $k\sqrt{(\Delta x)^2_{av}}$ were related (formalized) to the basic notions of mathematical statistics:

$$x_{av} \approx E_x, \quad k^2(\Delta x)^2_{av} \approx D_x,$$

where E_x and D_x are, respectively, the mathematical expectation and the variance of the measured quantity x. At this formalization, it is assumed that, when measuring the physical quantity x, its specific values x_i are obtained in accordance with a certain function $f(x)$ describing the probability for any x-value to appear. Surely, this function has to depend on the measurement conditions: it implicitly depends on the measuring device, the chosen method, and so forth). As a rule, the function $f(x)$ is

normalized,

$$\int\limits_{-\infty}^{\infty} f(x)dx = 1. \tag{1}$$

In this case, it is called the function of the probability density for a physical quantity x with the continuous distribution. Accordingly, the true value μ of the physical quantity x – or its mathematical expectation E_x – can be calculated using the known function $f(x)$ by the formula

$$\mu = E_x = \int\limits_{-\infty}^{\infty} xf(x)dx. \tag{2}$$

Equation (2) is a definition of the mathematical expectation E_x [1]. The function $f(x)$ also determines the variance D_x of the physical quantity x [1] (the spread of its measured values owing to $f(x)$):

$$D_x = \int\limits_{-\infty}^{\infty} (x - E_x)^2 f(x)dx = \int\limits_{-\infty}^{\infty} (x - \mu)^2 f(x)dx, \tag{3}$$

where $\mu = E_x$. The so-called normal (Gaussian) probability distribution [1]:

$$f(x) = \frac{p}{\sqrt{\pi}} \exp[-p^2(x - \mu)^2], \tag{4}$$

where $p^2 = \frac{1}{2D_x}$ and $\mu = E_x$, is considered to be the most important one among other distributions.

In the case of a relationship by means of the function $y = h(x)$, the mathematical expectation E_h and the variance D_h for the function $h(x)$ equal [1], respectively,

$$\chi = E_h = \int\limits_{-\infty}^{\infty} h(x)f(x)dx, \tag{5}$$

$$D_h = \int\limits_{-\infty}^{\infty} [h(x) - E_h]^2 f(x)dx = \int\limits_{-\infty}^{\infty} [h(x) - \chi]^2 f(x)dx. \tag{6}$$

Expression (6) can be rewritten in a more convenient form,

$$D_h = \int\limits_{-\infty}^{\infty} [h^2(x) - 2h(x)E_h + E_h^2]f(x)dx =$$
$$= \int\limits_{-\infty}^{\infty} h^2(x) f(x)dx - E_h^2. \tag{7}$$

In Eqs. (4)–(7), the quantities $\mu = E_x$ and D_x enter to the function $f(x)$ as parameters. Therefore, strictly speaking, $f(x)$ can be written as $f(x, E_x, D_x)$. As a result,

$$E_h = \int\limits_{-\infty}^{\infty} h(x)f(x, E_x, D_x)dx, \tag{8}$$

$$D_h + E_h^2 = \int\limits_{-\infty}^{\infty} h^2(x) f(x, E_x, D_x)dx. \tag{9}$$

It is easy to see that Eqs. (8) and (9) are integral equations. By solving them, we would obtain a sought analytic relation between E_h and D_h (analogs of the means for the function $h(x)$), on the one hand, and E_x and D_x (analogs of the measured averages), on the other hand. It turned out that it is possible to choose tabulated integrals [4] similar to integrals (8) and (9), and, in such a way, to solve the problem for the elementary function a^x and, with its help, for the inverse function $\log_a x$ (see Appendix).

Here, a comment should be made. If the variable x has a Gaussian distribution, the values of the nonlinear function $h(x)$ will be definitely not distributed according to the Gaussian law. Therefore, expressions (8) and (9) can be regarded as approximations to the expressions, in which the integration is carried out over the "correct" distribution:

$$E_h = \int\limits_{a}^{b} h \, Z(h, E_h, D_h)dh \quad D_h =$$
$$= \int\limits_{a}^{b} h^2 \, Z(h, E_h, D_h)dh - E_h^2,$$

where $Z(h, E_h, D_h)$ is a "pure" non-Gaussian distribution for h. The integration limits a and b in these integrals depend on the function $h(x)$. For example, for $h = \cos x$, we have $a = -1$ and $b = 1$.

However, the overwhelming majority of experimental works are based on the "classical" mixed summation scheme, Eqs. (8) and (9), which was expounded in work [1]. This is also true in the cases where Eqs. (8) and (9) are solved approximately by expanding $h(x)$ in the Taylor series [1]. Our expressions were also constructed in the framework of this ideology (see Appendix). Furthermore, all presented examples were also calculated, by using this scheme (see Section 3). In other words, both x_i and $h(x_i)$ were summed up with the Gaussian probabilities.

The means E_x, D_x, and Δ_x of the input set $\{x_i\}$ were calculated by the formulas

$$E_x = \sum x_i\, f_i; \quad D_h = \sum (x_i - E_x)^2\, f_i); \quad \Delta_x = \sqrt{D_x}.$$

The means E_h, D_h, and Δ_h of the output set $\{h_i = h(x_i)\}$ were calculated by the formulas

$$E_h = \sum h(x_i)\, f_i; \quad D_h = \sum (h(x_i) - E_h)^2\, f_i);$$
$$\Delta_x = \sqrt{D_x},$$

where $f_i = \frac{w_i}{\sum w_i}$ are Gaussian multipliers, $f_i = \frac{w_i}{\sum w_i}$; $w_i = \frac{p}{\sqrt{\pi}} \times \exp[-p^2(x_i - \mu)^2]$, $p^2 = \frac{1}{2D_x}$.

For the function a^x, the mentioned relations look like

$$E_h = E_{a^x} = a^{E_X}\, a^{D_X\,\ln a/2};$$
$$D_h = D_{a^x} = a^{2E_X}\, a^{D_X\,\ln a}(a^{D_X\,\ln a} - 1), \tag{10}$$

where E_X and D_X are the mean and the error, respectively, of the measured data, and E_{a^x} and D_{a^x} are the mean and the error, respectively, of the measurement results "propagated" with the use of the function a^x. For the function $\log_a x$, the analogous relations look like

$$E_h = E_{\log} = \frac{1}{2} \log_a\left(\frac{E_X^4}{D_X + E_X^2}\right);$$
$$D_h = D_{\log} = \frac{1}{\ln a} \log_a\left(\frac{D_X + E_X^2}{E_X^2}\right); \tag{11}$$

where $E_{\log}$ and $D_{\log}$ are the mean and the error, respectively, of the measurement results "propagated" with the use of the function $\log_a x$. For the "pure" exponential function $h = \exp x$ and the "pure" logarithmic function $h = \ln x$, those formulas can be written in a simpler form:

• for $h = \exp x$,

$$E_h = E_{\exp} = \exp E_X\, \exp\left(\frac{1}{2}\, D_X\right);$$
$$D_h = D_{\exp} = \exp 2E_X\, \exp D_X\, (\exp D_X - 1); \tag{12}$$

• for $h = \ln x$,

$$E_h = E_{\ln} = \frac{1}{2}\, \ln\left(\frac{E_x^4}{D_x + E_x^2}\right);$$
$$D_h = D_{\ln} = \ln\left(\frac{D_x + E_x^2}{E_x^2}\right); \tag{13}$$

where $E_{\exp}$ and $D_{\exp}$ are the mean and the error, respectively, of the measurement results "propagated"

with the use of the function $\exp x$, and $E_{\ln}$ and $D_{\ln}$ are their counterparts for the function $\ln x$.

Thus, we obtained (see Appendix) the sought rules for the error propagation and the calculation of "shifted means" of the type $E_h = E_h(E_x, D_x)$ and $D_h = D_h(E_x, D_x)$ for the functions $h(x) = a^x$ and $h(x) = \log_a x$ (in the particular case, e^x and $\ln x$). Note that, in the limiting case $D_x = 0$, we have

$$
\begin{aligned}
E_h &= E_{a^x} = a^{E_x}; & D_h &= D_{a^x} = 0; \\
E_h &= E_{\log} = \log_a E_x; & D_h &= D_{\log} = 0; \\
E_h &= E_{\exp} = \exp E_x; & D_h &= D_{a^x} = 0; \\
E_h &= E_{\ln} = \ln E_x; & D_h &= D_{\ln} = 0.
\end{aligned}
$$

Therefore, the "standard" propagation rules can be applied:

$$E_{a^x} = a^{E_x}; \quad E_{\exp} = \exp E_x; \quad E_{\log} = \log_a E_x;$$
$$E_{\ln} = \ln E_x.$$

3. Application of the New Rules to Experimental Data

A set of experimental data is a collection of separate random values x_i obtained for a measured physical quantity x, i.e. the so-called "sample" $\{x_i\}$. The quantity x may have a continuous distribution [1]; in other words, values that were randomly "selected by a measuring device" from a continuous set of values are dealt with.

Let us consider how the relations obtained above are satisfied just in the case of discrete samples. For this purpose, let us calculate the mean values, by using the standard procedure (which we take as a reference) for four samples: two sets of experimental data $\{x_i\}$, one set of calculated values for the $\exp x$ function, and one set of calculated values for the $\ln x$ function. Then we compare them with the results obtained according to relations (12) and (13).

3.1. Two examples for the function $\exp x$

As an example for the function e^x, let us take an arbitrary sample $\{x_i\}$ of 20 "measurements" on the logarithmic scale:
$\{x_i\}$ = 8, 8.047, 8.094, 8.141, 8.188, 8.235, 8.282, 8.329, 8.376, 8.423, 8, 7.953, 7.906, 7.859, 7.812, 7.765, 7.718, 7.671, 7.624, and 7.577.
The arithmetic means calculated for this sample with a constant probability $w_i = 1/20$ give the following values: $E_n = 8$, $D_n = 0.06296$, and $\Delta_n =$
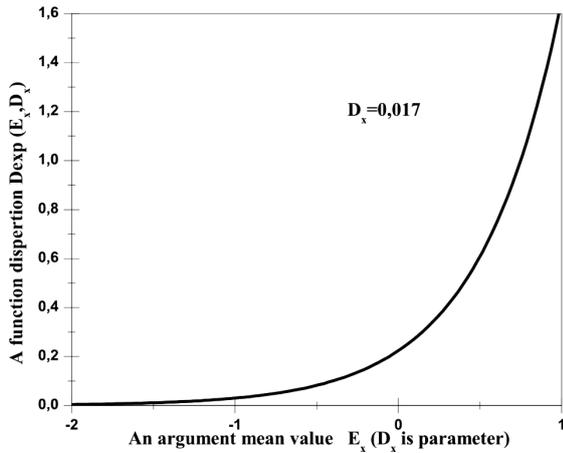
Fig. 1. Dependence of the variance $D_{\exp}$ of the function exp x on the mathematical expectation E_x of its argument for the argument variance $D_x = 0.017$

= 0.25092. Using them as the first approximation, we calculate the Gaussian means (2 to 3 iterations) according to the Gauss weighted scheme

$$E_x = \left(\sum x_i w_i\right) / \left(\sum w_i\right);$$
$$D_x = \left(\sum (x_i - E_x)^2 w_i\right) / \left(\sum w_i\right); \quad \Delta_x = \sqrt{D_x}, \quad (14)$$

where

$$w_i = \frac{p}{\sqrt{\pi}} \exp[-p^2(x_i - \mu)^2], \quad p^2 = \frac{1}{2D_x}. \quad (15)$$

As a result, we obtain $E_x = 8$, $D_x = 0.01726$, and $\Delta_x = 0.13138$. In other words, we have $E_x = 8 \pm 0.1$ for this sample.

In order to properly calculate the means for the function e^x, we should create a new statistical sample $\{\exp x_i\}$ and calculate its meaning values. The new sample looks like
$\{\exp x_i\}$ = 2980.95799, 3124.40767, 3274.76045, 3432.3485, 3597.52001, 3770.6399, 3952.09066, 4142.27321, 4341.60772, 4550.5346, 2980.95799, 2844.09445, 2713.51467, 2588.93015, 2470.06563, 2356.65849, 2248.45816, 2145.2256, 2046.73271, and 1952.76188.
Hereafter, we intentionally left more significant digits than it is required (two for D_x and one for Δ_x) in order to give the readers an opportunity to trace the calculations. Using the values of E_x, D_x, and Δ_x together with formulas (14) and (15), we obtain the sought means for the function e^x in the standard way: $E_{\exp} = 3006.8068$, $D_{\exp} = 157799.52$, and $\Delta_{\exp} = 397.23987$. The "propagation of errors"

according to the obtained relations (12) gives the following values: $E_{\exp} = 3006.7946$, $D_{\exp} = 157398.77$, and $\Delta_{\exp} = 396.735$. Here, we have an interesting example with large dispersions for the function e^x.

Another example was created for a different domain of the argument of the function e^x (Fig. 1):
$\{x_i\}$ = 0, 0.047, 0.094, 0.141, 0.188, 0.235, 0.282, 0.329, 0.376, 0.423, 0, −0.047, −0.094, −0.141, −0.188, −0.235, −0.282, −0.329, −0.376, −0.423.
The arithmetic means calculated for this sample with a constant probability $w_i = 1/20$ give the following values: $E_n = 0$, $D_n = 0.06296$, and $\Delta_n = 0.25091$. Using them as the first approximation, we calculate the Gaussian means (here, four iterations are required) according to the Gaussian weight scheme (14) and (15). As a result, we obtain $E_x = 0$, $D_x = 0.02194$, and $\Delta_x = 0.14812$. In other words, we have $E_x = 0 \pm 0.1$ for this sample.

In order to correctly calculate the means for the function e^x, we have to construct a new statistical sample $\{\exp x_i\}$ and calculate the required meanings for it. The new sample looks like
$\{\exp x_i\}$ = 1, 1.04812, 1.09856, 1.15142, 1.20683, 1.26491, 1.32578, 1.38958, 1.45645, 1.52653, 1, 0.95409, 0.91028, 0.86849, 0.82861, 0.79057, 0.75427, 0.71964, 0.6866, 0.65508.
Using the values of E_x, D_x, and Δ_x together with formulas (14) and (15), we obtain the sought means for the function e^x in the standard way: $E_x = 1.01103$, $D_x = 0.02269$, and $\Delta_x = 0.15067$. The "propagation of errors" according to the obtained relations (12) gives the following values: $E_{\exp} = 1.01103$, $D_{\exp} = 0.02267$, and $\Delta_{\exp} = 0.15058$.

Taking the both examples into account, we may assert that the values of errors (the variance and the deviation) evaluated by relation (12) are the same as for our reference examples. In other words, in the case of the function e^x, the propagation of errors with the use of relations (12) is proper.

3.2. An example for the function ln x

Let us consider an example for the function $\ln x$ (Figs. 2 to 4). We use a sample of measurements of a quantity with the exponential dependence (say, the intensity measured, when determining the quantum yield of the activation energy):
$\{y_i\}$ = 2000, 2100, 2200, 2300, 2400, 2500, 2600, 2700, 2800, 2900, 2000, 1900, 1800, 1700, 1600, 1500, 1400, 1300, 1200, 1100.
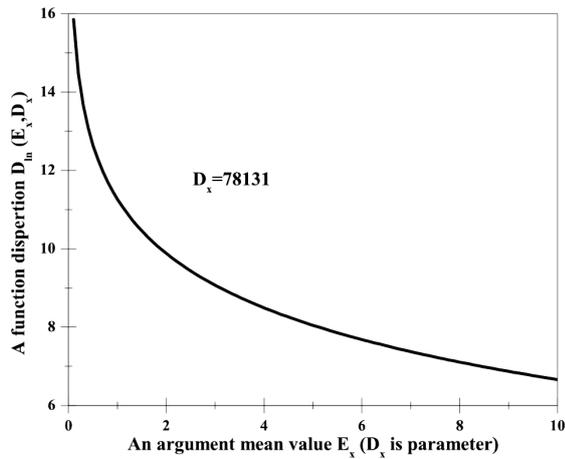
Fig. 2. Dependence of the variance $D_{\ln}$ of the function $\ln x$ on the mathematical expectation E_x of its argument for the argument variance $D_x = 78131.0$

Fig. 3. Dependence of the variance $D_{\ln}$ of the function $\ln x$ on the mathematical expectation E_x of its argument for the argument variance $D_x = 100.0$

Fig. 4. Dependence of the variance $D_{\ln}$ of the function $\ln x$ on the mathematical expectation E_x of its argument for the argument variance $D_x = 0.1$

Here, the variance of the argument is very large. Following the standard scheme for the calculation of means over the sample, we

(i) calculate the arithmetic means (with the probability $w_i = 1/20$): $E_n = 2000$, $D_n = 285000$, and $\Delta_n = 533.8539$;

(ii) calculate the Gaussian means (according to the weight scheme (14): $E_y = 2000$, $D_y = 78130.595$, and $\Delta_y = 279.5185$;

(iii) create a sample for the function $\ln y$:
$\{\ln y_i\} = 7.6009, 7.64969, 7.69621, 7.74066, 7.78322, 7.82405, 7.86327, 7.90101, 7.9373, 7.97247, 7.6009, 7.54961, 7.49554, 7.43838, 7.37776, 7.31322, 7.24423, 7.17012, 7.09008, 7.00307$;

(iv) statistically process this sample, by using the values obtained for E_y, D_y, and Δ_y, and get $E_{\ln} = 7.59081$, $D_{\ln} = 0.02068$, and $\Delta_{\ln} = 0.1438$.

Calculations by relations (13) give us the following values: $E_{\ln} = 7.59123$, $D_{\ln} = 0.019344$, and $\Delta_{\ln} = 0.13908$. It is evident that the agreement is good in this case. In other words, the propagation of errors with the use of formulas (13) is proper for the function $\ln x$.

4. Some General Features of the Obtained Relations

The analytical form obtained for the rules of error propagation makes it easy to distinguish the specific features of corresponding relations and even to plot graphical dependences, which is very useful while planning a physical experiment and analyzing its results.
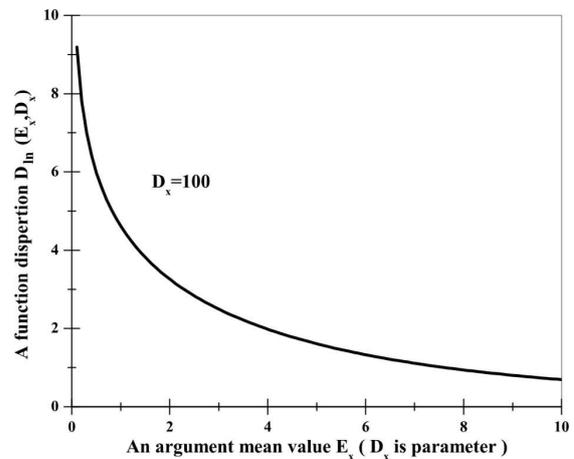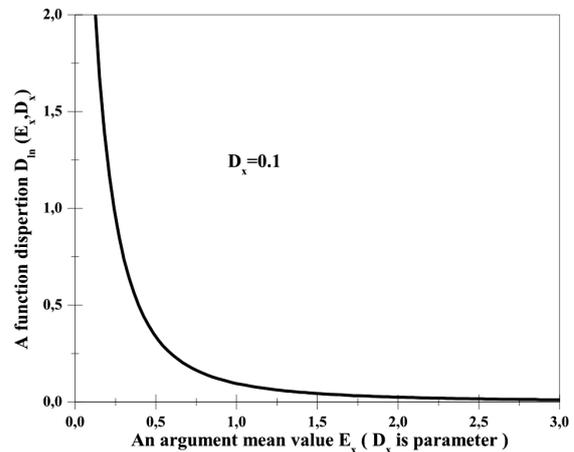
It should be emphasized that the quantities E_h, D_h, E_x, and D_x are mutually related. Furthermore, the quantities E_h and D_h are functions of two arguments rather than one:

$$E_h = E_h(E_x, D_x), \quad D_h = D_h(E_x, D_x).$$

It is sometimes difficult to get used to this fact, as well as to the fact that the errors of the functions $h(x)$, e.g., $\Delta_{\exp}$ or $\Delta_{\ln}$, depend on the measured mean value x. All those things are clearly illustrated in Figs. 1 to

4, where the dependences of the function variances on the values of the measured "means" of the corresponding argument E_x (the variance D_x is regarded as a parameter) are depicted. The very opportunity to have a graphical representation for the obtained relations makes it possible to discuss the character of future measurements and plan them.

5. Conclusions

Similarly to what was done in work [6], relations (11) and (13) are considered to be semiempirical ones because of their construction on the basis of inverse functions. However, all inverse functions constructed in this way [2, 3], as well as relations (10)–(13) in this work, provide proper results for samples and can be widely used to shorten and to substantially simplify the calculation procedures for the functions a^x and $\log_a x$. Since the variances D_h and the errors σ for both considered functions practically coincide with their real values, the propagation of errors can be performed in a chain of functions such as $\log_a b^{\log(\cdots)}$, $a^{\log_b()\cdots}$, or another combination of the indicated functions together with the functions x^2, $\sqrt{x}$, $\cos x$, and $\arccos x$ considered in works [2,3].

Therefore, on the basis of the obtained analytical relations, it is possible to construct two simple general algorithms for the calculation of pairs of separate values (E_{a^x}, D_{a^x}) and $(E_{\log}, D_{\log})$ for the variable x with the Gaussian distribution. They can be built-in as separate program modules (subroutines) into any program procedure. At the same time, this algorithm remains transparent for the understanding. This is impossible in principle for other error propagation methods, because they required the expansion in a series or the differentiation of the relevant superposition of functions as a whole entity. Moreover, every problem demands to construct a separate procedure.

The proposed method makes it possible to predict the error value of the function and to plot its profile in the intended measurement area of physical quantities.

Of interest is a possibility to obtain the shifted mean values for E_{a^x} and $E_{\log}$. In the examples given above, this shift does not affect the values of those means and does not play any role, but it does exist, and its meaning can be used in some applications.

Since the analytical expressionfor the means (E_{a^x}, D_{a^x}) and $(E_{\log}, D_{\log})$ are connected with the Gaussian distribution, then the calculated values make it possible to compare them with the values of the same quantities, but calculated for other distributions. The best distribution will correspond to the minimum of D_{a^x} or $D_{\log}$.

APPENDIX

In this appendix, a mathematical proof is given that the relations obtained for two functions, a^x and $\log_a x$, by reducing the integral equations (8) and (9) to tabulated integrals and transforming the obtained formulas to the convenient forms (10)–(13), are proper.

Mathematical expectation
E_h for the function $h(x) = a^x$

By applying sequentially the substitutions $k = \ln a$, $h(x) = a^x = \exp(x \ln a) = \exp(kx)$, $y = x - \mu$, and $x = y + \mu$ to Eq. (8) and taking the Gaussian distribution (4) for $h(x)$ into account, we can make the following chain of transformations:

$$\chi = E_h = E_{a^x} = \frac{p}{\sqrt{\pi}} \int_{-\infty}^{\infty} a^x \exp[-p^2(x-\mu)^2]dx =$$

$$= \frac{p}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp[k(y+\mu)] \exp[-p^2 y^2]dy =$$

$$= \frac{p}{\sqrt{\pi}} \exp[k\mu] \int_{-\infty}^{\infty} \exp[-p^2 y^2 + ky]dy = \frac{p}{\sqrt{\pi}} \exp[k\mu] J =$$

$$= \frac{p}{\sqrt{\pi}} \exp[\ln a\mu] J = \frac{p}{\sqrt{\pi}} a^{\mu} J. \tag{16}$$

The integral

$$J = \int\limits_{-\infty}^{\infty} \exp[-p^2 y^2 + ky] dy$$

is similar to the tabulated integral [4, integral (3.923.2)]

$$T_2 = \int\limits_{-\infty}^{\infty} \exp[-a_1 y^2 - 2by - c] \cos[p^2 y^2 + qy + r] dy.$$

The latter, at $c = p = q = r = 0$, equals

$$T_2 = \int\limits_{-\infty}^{\infty} \exp[-a_1 y^2 - 2by] dy = \sqrt{\frac{\pi}{a_1}} \, \exp\frac{b^2}{a_1}. \qquad (17)$$

Hence, the integrals J and T_2 are identical with an accuracy to the substitutions $a_1 = p^2$, $-2b = k$, and $b = -\frac{k}{2}$. Therefore, taking into account that $k = \ln a$ and $1/p^2 = 2D_x$ [see Eq. (4)], we obtain

$$J = \sqrt{\frac{\pi}{p^2}} \, \exp\frac{k^2}{4p^2} = \frac{\sqrt{\pi}}{p} \, \exp[(\ln a)^2 \, 2D/4] =$$

$$= \frac{\sqrt{\pi}}{p} \, a^{D \, \ln a/2}.$$

Substituting this expression into expression (16), we find a final relation of the integral $E_h = E_{a^x}$ to the integrals E_x and D_x. Taking into account that $\mu = E_x$ [see Eq. (2)] and $p^2 = 1/(2D_x)$ [see Eq. (4)], this relation looks like

$$E_h = E_a x = \frac{p}{\sqrt{\pi}} a^{\mu} \frac{\sqrt{\pi}}{p} \, a^{D_X \, \ln a/2} = a^{\mu} \, a^{D_X \, \ln a/2} =$$

$$= a^{E_X} \, a^{D_X \, \ln a/2}. \qquad (18)$$

If $a = e$, then

$$E_h = E_{\exp} = e^{E_x} e^{D_x/2}. \qquad (19)$$

This is an "intuitively" expected result. As $D_x \approx 0$, there is a small shift associated with the multiplier $e^{D_x/2} \approx 1$, and Eqs. (18) and (19) acquire a "natural" form

$$E_h = E_{a^x} \approx a^{E_x}, \quad E_h = E_{\exp} \approx \exp E_x.$$

Under certain conditions, this multiplier can be ignored, but Eqs. (18) and (19) are exact working formulas for $h(x) = a^x$ and $h(x) = \exp x$.

Variance D_h for the function $h(x) = a^x$

From Eq. (9) with the use of Eq. (4), we obtain the following expression for the error propagation:

$$D_h = D_{a^x} = \frac{p}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} a^{2x} \exp[-p^2 (x-\mu)^2] dx - E_h^2 =$$

$$= \frac{p}{\sqrt{\pi}} J_0 - E_h^2. \qquad (20)$$

Now, we transform

$$J_0 = \int\limits_{-\infty}^{\infty} a^{2x} \exp[-p^2 (x-\mu)^2] dx$$

into a tabulated integral, by using the sequential substitutions $y = x - \mu$, $x = y + \mu$, and $k = \ln a$ [see Eq. (8)]:

$$J_0 = \int\limits_{-\infty}^{\infty} a^{2x} \exp[-p^2 (x-\mu)^2] dx =$$

$$= \int\limits_{-\infty}^{\infty} a^{2y+2\mu} \exp[-p^2 y^2] dy =$$

$$= a^{2\mu} \int\limits_{-\infty}^{\infty} a^{2y} \exp[-p^2 y^2] dy =$$

$$= a^{2\mu} \int\limits_{-\infty}^{\infty} \exp[\ln a \, 2y] \exp[-p^2 y^2] dy =$$

$$= a^{2\mu} \int\limits_{-\infty}^{\infty} \exp[-p^2 y^2 + 2ky] = a^{2\mu} J_{01}. \qquad (21)$$

It is evident that

$$J_{01} = \int\limits_{-\infty}^{\infty} \exp[-p^2 y^2 + 2ky] dy$$

is also integral (17) with $b = -k = -\ln a$, $a_1 = p^2$, and $1/p^2 = 2D_x$. Therefore,

$$J_{01} = \frac{\sqrt{\pi}}{p} a^{2D_x \, \ln a}.$$

Using this result and Eq. (18) for E_h, from Eqs. (20) and (21), we obtain the variance D_h for the function $h(x) = a^x$ in the form

$$D_h = D_{a^x} = a^{2E_x} a^{D_x \, \ln a} (a^{D_x \, \ln a} - 1). \qquad (22)$$

In the case $a = e$, we have a simplified form

$$D_h = D_{\exp} = e^{2E_x} e^{D_x} (e^{D_x} - 1). \qquad (23)$$

This formula is the error propagation rule for the function $h(x) = a^x$.

The mean E_h and the variance D_h for the function $h(x) = \log_a x$

The straightforward calculation of the quantities $E_{\log_a x}$ and $D_{\log_a x}$ with the help of tabulated integrals is a rather problematic task. The sought relations can be obtained by considering the function $\log_a x$ as an inverse function to the function a^x and by using Eqs. (18) and (22). Really, these equations give us an explicit relation between four integrals (numbers)–E_{a^x}, D_{a^x}, E_x, and D_x:

$$E_{a^x} = E_{a^x}(E_x, D_x), \quad D_{a^x} = D_{a^x}(E_x, D_x). \qquad (24)$$

Equations (18) and (22) can also be used to determine the functions

$$E_x = E_x(E_{a^x}, D_{a^x}), \quad D_x = D_x(E_{a^x}, D_{a^x}), \qquad (25)$$

which are inverse to functions (24) and also have to properly describe the mathematical relations among E_{a^x}, D_{a^x}, E_x, and D_x. Furthermore, if E_{a^x} and D_{a^x} are obtained in a different way (e.g., they can be measured experimentally) and have the same numerical values as were calculated from formulas (24), then they will again satisfy Eqs. (8) and (9) that describe the relations among four integrals if and only if the quantities E_x and D_x have the same values that are given in Eq. (24).

In other words, if we have the relation $y = a^x$ and the inverse one $x = \log y$, then relations (25) inverse to Eqs. (18) and (22) will give us the proper values of integral expressions for the mathematical expectation E_x and the variance D_x, determined by Eqs. (8) and (9), for the function of the random variable y that is connected with the variable x by the law $y = a^x$ (or $x = \log_a y$). Hence, by solving Eqs. (18) and (22) with respect to x, we can obtain, by using a simple calculation procedure, the values of E_x and D_x from the values of E_y and D_y, which are the means for the measured random variable y that is related to x by the formula $x = \log_a y$.

In order to solve Eqs. (18) and (22), let us rewrite them in the form

$$a^{D_x \ln a} = \frac{E_y^2}{a^{2E_x}} \qquad (26)$$

and

$$D_y = D_{a^x} = a^{2E_x} a^{D_x \ln a} (a^{D_x \ln a} - 1), \qquad (27)$$

respectively. Bearing in mind that the integrals E_y and D_y are associated with the function $y = a^x$, whereas the integrals E_x and D_x with the function $y = \log_a x$, let us solve these equations with respect to the integrals E_x and D_x, i.e. let us obtain the equations inverse to Eqs. (18) and (22), and to Eqs. (26) and (27). Substituting Eq. (26) into Eq. (27), we obtain

$$D_y = E_y^2 \left(\frac{E_y^2}{a^{2 E_x}} - 1 \right),$$

whence

$$\frac{D_y}{E_y^2} + 1 = \frac{D_y + E_y^2}{E_y^2} = \frac{E_y^2}{a^{2E_x}},$$

so that

$$a^{2E_x} = \frac{E_y^4}{D_y + E_y^2} \qquad (28)$$

and, finally,

$$E_x = \frac{1}{2} \log_a \left(\frac{E_y^4}{D_y + E_y^2} \right). \qquad (29)$$

Substituting expression (28) into Eq. (26), after a series of elementary transformations, we obtain the final expression for D_x:

$$D_x = \frac{1}{\ln a} \log_a \left(\frac{D_y + E_y^2}{E_y^2} \right). \qquad (30)$$

Let us rewrite the obtained relations (18), (23), (29), and (30) in a symbolic form, where x is a measured physical quantity (argument), and h is the corresponding function (a^x,

$\log_a x$, e^x, or $\ln x$):

$$E_h = E_{a^X} = a^{E_X} a^{D_X \ln a/2};$$
$$D_h = D_{a^X} = a^{2E_X} a^{D_X \ln a} (a^{D_X \ln a} - 1);$$
$$E_h = E_{\log} = \frac{1}{2} \log_a \left(\frac{E_X^4}{D_X + E_X^2} \right); \qquad (31)$$
$$D_h = D_{\log} = \frac{1}{\ln a} \log_a \left(\frac{D_X + E_X^2}{E_X^2} \right).$$

For the "pure" exponential function ($h(x) = e^x$ or $\ln x$), these expressions look like

$$E_h = E_{\exp} = \exp E_X \exp \left(\frac{D_X}{2} \right);$$
$$D_h = D_{\exp} = \exp 2E_X \exp D_X (\exp D_X - 1);$$
$$E_h = E_{\ln} = \frac{1}{2} \ln \left(\frac{E_X^4}{D_X + E_X^2} \right); \qquad (32)$$
$$D_h = D_{\ln} = \ln \left(\frac{D_X + E_X^2}{E_X^2} \right).$$

With regard for the results of works [2, 3], it is useful to reproduce the "analytic" rules of error propagation for two other pairs of functions:

for $\cos x$ and $\arccos x$,

$$E_h = E_{\cos} = \exp \left(-\frac{D_x}{2} \right) \cos E_x;$$
$$D_h = D_{\cos} = \frac{1}{2} \left[1 - \exp(-D_x) \right] \left[1 - \exp(-D_x) \cos 2E_x \right];$$
$$E_h = E_{\arccos} = \arccos \frac{E_x}{\pm \sqrt{E_x^2 + \sqrt{(1 - E_x^2)^2 - 2D_x}}};$$
$$D_h = D_{\arccos} = \ln \left(\frac{1}{E_x^2 + \sqrt{(1 - E_x^2)^2 - 2D_x}} \right). \qquad (33)$$

and for x^2 and $\sqrt{x}$,

$$E_{x^2} = E_x^2 + D_x,$$
$$D_{x^2} = 2D_x^2 + 4E_x^2 D_x,$$
$$E_{\sqrt{x}}^4 = E_x^2 - \frac{1}{2} D_x, \qquad (34)$$
$$D_{\sqrt{x}} = E_x - \sqrt{E_x^2 - \frac{1}{2} D_x}.$$

In view of the formalization

$$x_{av} \approx E_x, \quad k^2 (\Delta x)_{av}^2 \approx D_x,$$

the obtained relations give us the sought "propagation rules" for the means and the errors of the functions a^x, $\log_a x$, e^x, $\ln x$, x^2, $\sqrt{x}$, $\cos x$, and $\arccos x$:

$$X_{av} \to H_{av}, \quad |\Delta X|_{av} \to |\Delta H|_{av}.$$

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Г.Г. Роде

ПЕРЕНОС ПОХИБОК
ТА СЕРЕДНІХ ВИМІРІВ ФІЗИЧНОЇ ВЕЛИЧИНИ
ДЛЯ ЕЛЕМЕНТАРНИХ ФУНКЦІЙ a^x ТА $\log_a x$

Р е з ю м е

Отримані "правила переносу похибки та середнього" однієї вимірюваної фізичної величини на іншу, пов'язану з нею зв'язком a^x або $\log_a x$. В ці правила по природі закладена вагова схема Гауса. Тому вони мають добре працювати в рамках реальної вагової схеми Гауса з дискретними даними реального фізичного дослідження (з "вибірками"). Аналітична форма, в якій представлені згадані правила ("аналітичні правила переносу"), а також їх характер дозволяють спростити і прискорити процедуру обробки й аналізу експериментальних даних.