B.I. LEV,[1] V.P. OSTROUKH,[2] V.B. TYMCHYSHYN,[1] A.G. ZAGORODNY[1]

[1] Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine
(14b, Metrolohichna Str., Kyiv 03680, Ukraine; e-mail: blev@bitp.kiev.ua)
[2] Taras Shevchenko National University of Kyiv
(4, Academician Glushkov Prosp., Kyiv 03187, Ukraine; e-mail: v.ostroukh@yandex.ua)

PACS 71.20.Nr, 72.20.Pa

ELECTRON DISTRIBUTION ON A DEFORMED LIQUID-HELIUM SURFACE

A simple quasiclassical statistical model for the description of electrons on a liquid-helium surface in an external electric field is proposed. The model involves the electrostatic interaction energy with due regards to the medium polarization and electron interactions associated with the surface deformation. The explanation of some phenomena observed in the experiments is given. In particular, a method for numerical calculation of inhomogeneities of the charge distribution in an external electric field is introduced. The obtained results can be used to explain the peculiarities of the electron distribution function measured in experiments.

K e y w o r d s: liquid helium, electrons, bubblon, two-dimensional electron systems.

Recently, much interest in the experimental and theoretical studies of low-dimensional Coulomb-like systems has been generated. Such systems are widely presented in experiments with emulsions, foams, polymers, colloidal suspensions, *etc.* Considerable attention is attracted to the special case of electrons on the surface of a dielectric substrate [1, 2]. Studies of these systems are not only of academic interest, but can also have some practical applications. For example, it is proposed to use the electrons on a dielectric surface for quantum computations [5].

A possibility of creating a two-dimensional system on the surface of a dielectric medium was predicted in [1, 2, 6]. First experiments were carried out one year later [7]. Two-dimensional electron systems are still extensively studied, and many interesting results have been obtained. Let us say, the first experimental realization of the Wigner solid, predicted in the well-known work [3], was made in an electron system on liquid helium [4]. Let us now discuss some properties of these systems. Electrons located on a dielectric surface have two degrees of freedom only [8, 9]. They can exist in the forms of a fluid or Wigner crystal [3, 4, 10]. Some interesting effects are caused by the deformation instability of the liquid helium surface that causes, for example, the phase transitions between triangle and square Wigner lattices [11]. Electrons on the surface of liquid helium become localized in macroscopic dimples, when the electric field perpendicular to the surface exceeds a critical value. These dimples form a two-dimensional hexagonal lattice [12]. The phase transition from a homogeneous two-dimensional charge distribution to the modulated charge density regime was investigated, and the hysteresis effect was observed, which means that the transition is discontinuous [13].

Modern research in the field of low-dimensional electron systems is based mostly on the quantum field theory [14] and the scaling theory [15]. For example, the electron transport properties in heterostructures and the electron structures on the liquid-helium surface can be studied using quantum field theory methods [16]. As regards the scaling theory, it was worked out in [17]. Nevertheless, these models are complicated for the analysis and require a lot of calcula-

tions. Therefore, it would be highly desirable to introduce simpler quasiclassical models of the type [18,19], which could be efficient for the description of properties of the low-dimensional electron systems. The main goal of the present contribution is to work out such model for the description of electrons on a helium surface.

It is well known that an electron is attracted to the surface of any dielectric media [20]. But, in addition to the attractive part of the electron-helium binding potential, there is the repulsive part. Electrons cannot just go through this surface, as being pushed out due to quantum effects [8, 21–23]. This means that the electrons are floating above the liquid helium surface. In the presence of an external field, the electrons can be pressed against the helium surface with a force that exceeds the gravitational one by many orders. Thus, as shown in [11], the pressing field results in distorting the surface at the position of each electron and, hence, in inducing an attractive force between the electrons. In [11], system's behavior is described in terms of the interplay between the repulsive Coulomb interaction and the attractive surface-induced interaction between individual electrons.

The system of electrons floating on the liquid helium surface is interesting as a representative of a wide class of systems with Coulomb-like interaction. These systems can significantly differ by physical properties, but their interparticle interaction resembles the Coulomb interaction. One of many possible examples can be dusty plasma. Electrons on the liquid helium surface and dust particles in plasma are much alike. Dust particles even can form some stable periodical structures similar to a Wigner crystal and undergo the melting or crystallization [18, 24–26].

In this article, we will try to look at electrons on the liquid helium surface in a way more typical of Coulomb-like systems. On the other hand, we will develop the necessary formalism basing on modern research in the field of statistical equilibrium description of such a kind of systems in the mean-field approximation [18]. These methods will be used to develop some simple quasiclassical model that describes the electron behavior and can predict the parameters of structures they form depending on the external electric field intensity and the temperature. As a result, we will obtain the electron distribution function and conditions for the distribution to be changed from the uniform to the periodic one. The obtained results can be easily generalized to the case of three-dimensional systems using the appropriate expressions for interaction energy. This is an additional advantage of the proposed model, which provides a simpler description of the properties of the Coulomb-like systems than that given by a more sophisticated formalism [18, 19]. One more important difference between our studies and the results obtained earlier is that we consider simultaneously additional effects such as the polarization of a medium and the interaction due to a surface deformation.

Our theoretical description employs the free energy written in terms of the density distribution function in the mean-field approximation. Actually, the free energy consists of two parts, i.e.,

$$F = F_{\rm int} + F_s. \tag{1}$$

The first part of the free energy, which produces the interaction between electrons, can be written as

$$F_{\rm int} = \int\limits_{-L}^{+L}\int\limits_{-\infty}^{+\infty} \rho(\mathbf{r})\rho(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')d\mathbf{r}d\mathrm{r}', \tag{2}$$

where $V(\mathbf{r}-\mathbf{r}')$ is the energy of interaction between electrons on the surface. The entropy part F_s of the free energy can be written in the standard form [30]:

$$F_s = kT\int\big\{\rho(\mathbf{r})\ln\rho(\mathbf{r})+[1-\rho(\mathbf{r})]\ln[1-\rho(\mathbf{r})]\big\}d\mathbf{r}. \tag{3}$$

The interaction between electrons consists of two parts $V(\mathbf{r}-\mathbf{r}\,') = V_{e-e}(\mathbf{r}-\mathbf{r}') + V_{\rm def}(\mathbf{r}-\mathbf{r}')$, i.e., the direct electrostatic interaction and the interaction through a deformation of the surface [11]. The more detailed discussion of the electron-electron and electron-helium interactions, as far as the item V_{e-e}, can be found in [21]. As for the last term $V_{\rm def}$, we refer to [11].

All these items will be treated as being two-dimensional. Of course, there is some contribution caused by the third dimension. But it is known that this contribution can be neglected, and all electrons are supposed to be at the same distance above the helium surface [8]. That's why all calculations given in what follows have been done in the 2D-model. Any contribution to the action caused by the third dimension is supposed to be constant and having no influence on its minimization functional.

According to [11], the direct electron interaction energy can be presented as

$$V_{e-e}(r) = \frac{e^2}{4\pi\varepsilon_0 r} - \frac{4\Lambda_s}{\sqrt{r^2+(2d)^2}}, \qquad \Lambda_s = \frac{\varepsilon_s - \varepsilon_{\rm He}}{16\pi\varepsilon_0(\varepsilon_s+\varepsilon_{\rm He})}e^2, \tag{4}$$

where ε_s and $\varepsilon_{\rm He}$ are the dielectric constants of the substrate and liquid helium, respectively, r is the distance between electrons, d is the helium film thickness. The first part is caused by the ordinary Coulomb interaction, and the second one is induced by the liquid helium and substrate polarizations. Our direct interaction is different from the one in [11] by its second part that involves the helium polarization caused by electrons.

Electron localization on the helium surface is topological, in the sense that the surface of liquid helium is deformed, by exhibiting periodic troughs and peaks so that the electrons are more likely to be trapped inside the troughs. The lateral capillary interaction between two particles is given by [11]

$$V_{\rm def}(r) \propto \frac{f^2}{2\pi\sigma}{\rm K}_0(\lambda r) = c(E^2){\rm K}_0(\lambda r).$$

Here, $f = eE$ is the actual force that acts on each particle, σ is the surface tension, r is the distance between particles, K_0 is the modified Bessel function, and λ is the capillary length that depends on the fluid properties only. With no external field, the surface deformations are small, but the external electric field produces a greater deformation and changes the energy of interaction between electrons through a deformation of the helium surface. The coefficient of capillary interaction $c(E^2)$ will depend on the external electric field.

Due to the long-range nature of Coulomb-like forces, the boundary conditions may notably change the result, because the particles start to "feel" the size of the system. Moreover, they introduce restrictions to the properties of the density function, which simplifies not only analytical, but computer calculations as well. In [27–29], the exponentially decreasing potential was chosen, so the boundaries could be ignored. But, for the system under consideration, they should be taken into account.

We suppose that the whole system is placed in a square "metal box" with dimensions $-L \dots L$. This assumption is reasonable, because all experiments are performed for the system between grounded metal plates. Of course, we can expect electrons to leave the system through the grounded walls. But the electron leakage through the boundaries is prevented by the "guard ring" and the "guard potential" [21]. We assume that such "guard field" acts only on the electrons in a vicinity of the walls and can be neglected anywhere else [22]. The existence of these rings means in our model only that the charge in this system can be treated as a constant, but it does not violate the boundary conditions that should be in agree with the principle of images. This makes our calculations inexact near the boundary, but the density disturbance we are interested in appears far from the "walls" as well.

At this moment, we should explain some features of the obtained results, because they may seem non-intuitive. It may seem, that electrons should be smeared all around the walls. But this is not true. Electrons are attracted by the walls, but the more electrons are placed near the walls, the less others will be attracted there. If we remember that electrons cannot come closer to the wall, then the guard ring allows us to conclude that the minimum of the potential energy, even taking only the Coulomb interaction into account, will be not achieved when all electrons are on the border near walls, because distance between them will be "close to zero," and the distances between them and their images – not. So some of them should be far from the walls. This means that we have to expect some complicated charge distribution in this system.

Now, let us consider some mathematical model we will work with. Due to classical electrodynamics, it can be shown that we should consider imaginary charges along with real ones (principle of images) when calculating the total potential energy. To make our calculations simpler, we will analytically extend the density distribution function to $(-\infty;\infty)$. Moreover, we claim it to be symmetric (due to system's symmetry) and anti-periodic (due to metal walls):

$$\rho(x;y) = \sum_{i,j=0}^{\infty} C_{i,j}\cos(x\alpha_i)\cos(y\alpha_j), \tag{5}$$

where $\alpha_i = (\pi/2L)[2i+1]$. We leave to the reader to prove the fact that the charge density $e\rho(x;y)$ in the form (5) automatically holds the principle of symmetry ($e\rho(x;y) = e\rho(-x;y) = e\rho(x;-y) = e\rho(y;-x)$). Moreover, when $|x| > L$ or $|y| > L$,

$e\rho(x;y)$ can be treated as the value of imaginary charge, for example $e\rho(L-x;y) = -e\rho(L+x;y)$. Every function that satisfies these conditions can be represented in the form of the mentioned series.

Changing the variables $x' \to \chi = x - x'$ and $y' \to \gamma = y - y'$ in (2) and then taking into account that V depends on $|r - r'|$ in our case, we obtain

$$F_{\text{int}} \approx \int\limits_{-\infty}^{+\infty}\int\limits_{-L}^{+L} \rho(x;y)\rho(x-\chi;y-\gamma)dxdy\,V(\chi;\gamma)d\chi d\gamma.$$

The inner integral has limits from $-L$ to $+L$, because we are integrating all over the system. On the other hand, the outer integral limits are set from $-\infty$ to $+\infty$ because of image charges. Since every electron has the infinite number of images, calculating its energy makes us take them all into account. The quality in this equation is approximate because of the summand in V that is not caused by electrostatical forces (4). But we may notice that this summand is very "short-range", so we can consider the integration with infinite limits as well.

Substituting (5) into the previous equation, expanding by the known trigonometric formulas $\cos\big([x-\chi]\alpha_i\big)$ and $\cos\big([y-\gamma]\alpha_j\big)$, and taking the relations

$$\int\limits_{-L}^{+L} \cos(x\alpha_i)\sin(x\alpha_j)dx = 0,$$

$$\int\limits_{-L}^{+L} \cos(x\alpha_i)\cos(x\alpha_j)dx = \delta_{i;j}$$

into account, it can be shown that (2) can be written as

$$F_{e-e} = \sum_{i,j=0}^{\infty} C_{i;j}^2 k_{i;j}, \tag{6}$$

where $k_{i;j}$ depends on ε_{He}, ε_s, L, and d only:

$$k_{i,j} = \int\limits_{-\infty}^{+\infty}\int \cos(\chi\alpha_i)\cos(\gamma\alpha_j)V(\chi;\gamma)d\chi d\gamma.$$

Now, we recall to the reader that the charge in such a system is constant. So it should be $Q_{\text{total}} = \iint_{-L}^{+L} \rho(\mathbf{r})d\mathbf{r}$. With $\rho(\mathbf{r})$ being taken into account, it is possible to show that the previous relation can be reduced to $Q_{\text{total}} = \sum_{i,j=0}^{\infty} C_{i;j}q_{i;j}$, where $q_{i;j}$ depends only on L.

At this point, it might be preferable to employ the numerical methods. With the previous result being taken into account, the free energy F can be minimized by means of the gradient descent method. This method was chosen as most suitable due to its simplicity and a low calculation complexity that makes possible to work with a big amount of variables $C_{i;j}$. Since we will not get into technical details for those readers unfamiliar with the gradient descent method, we recommend to refer to the appropriate literature (e.g., [31]). But it should be mentioned that there are some features, when using the gradient descent method in this paper.

The method can find the values of variables, when some function reaches its minimum. But here, we should find when not a function, but rather a functional is minimal. Decomposing $\rho(\mathbf{r})$ in a Fourier series and substituting into the functional, we reduce the problem to finding the coefficients of the Fourier series. Of course, there is the infinite number of these coefficients, so we should limit ourselves with first N ones. The more we take them, the more precise the result will be after the minimization procedure (in this paper, 10^4 coefficients were taken), until we still have a significant number of electrons in one period of the lowest term of the expansion to use the continuous approximation. The physical parameters were taken as follows [22]: $L = 1$ cm; $\varepsilon_s = 11$ (silicium); $d = 0.1$ cm. The second feature is that ρ should be nonnegative so the sign of $\rho(x;y)$ is checked on every step of the gradient descent method. Due to this calculation, we will obtain some values of the coefficients $C_{i;j}$ that minimize the free energy F.

The result is that, without an external field, the electrons form some 2D periodic structure if the temperature is sufficiently low (Fig. 1). The lighter areas correspond to higher electron concentrations (higher probability of finding some electrons there). The obtained results mean that the ground state of the electron system on the liquid helium surface is not a uniform Wigner crystal. The density of electrons and the Wigner lattice period are periodically modulated. This form of the surface can be treated as a Wigner crystal formed by electron dimples, if we think of them as charged quasiparticles. The temperature growth makes it possible to observe some smear-

ing of the electron density function, which confirms our intuitive speculations (Fig. 2). Since the boundaries are grounded, the particle distribution function is always equal to zero there, so we cannot expect that the distribution function is uniform. However, it tends to be so as the temperature is growing up.

We have already mentioned that some inhomogeneities of the electron distribution on the liquid-helium surface exist even without an external field. Switching-on an external field can only deepen the difference between the local maximum and the minimum of the electron distribution function. The computer simulation results confirm again the intuitive conclusions and show some sharpening of the density function (Fig. 3). This means that the electrons are collected together due to some additional effective attraction that is caused by helium-film deformations. The external field presses the electrons into liquid helium. But, due to some quantum effects, they are pushed out. So, the electrons make the helium film to "feel" the external field and to be deformed according to it. Of course, the zones with higher particle concentrations undergo stronger deformations than those with lower particle concentrations. Helium film deformations cause the presence of some additional terms in the free energy of the system. So, deformations of the liquid-helium film with an external field are not only some "sharpened" version of themselves without an external field, the system rather should find some form of the helium surface that minimizes the total free energy and can be qualitatively different from that without any field. Figure 3 shows that, besides being sharpened, both local maximum and minimum are shifted. When the electron density is sufficiently large, they can be pushed inside helium and form some kind of a bubble filled with electrons (bubblon). Such behavior cannot be taken into account in terms of our model, but some appropriate estimates can be done.

Suppose that the electron concentration is sufficiently small and the external field is getting more and more stronger. The numerical calculation shows that if the external field is strong enough, it becomes possible that almost all electrons are collected in the center of the helium film (Fig. 4). We can see that such distribution function is convex similarly to the case of a high temperature and no field. The main difference between the two distribution functions is

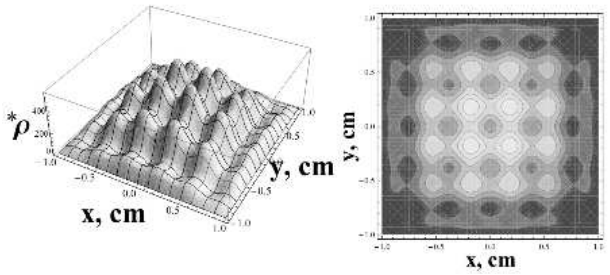

Fig. 1. Plot of the electron distribution function: $T = 0.02$ K, ${}^*\rho = \rho L^2/Q_{\text{total}} \times 100\%$, no external field

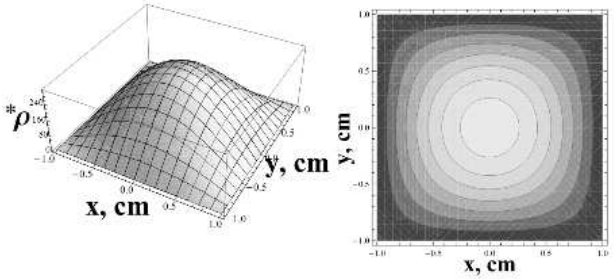

Fig. 2. Plot of the electron distribution function: $T = 0.4$ K, ${}^*\rho = \rho L^2/Q_{\text{total}} \times 100\%$, no external field

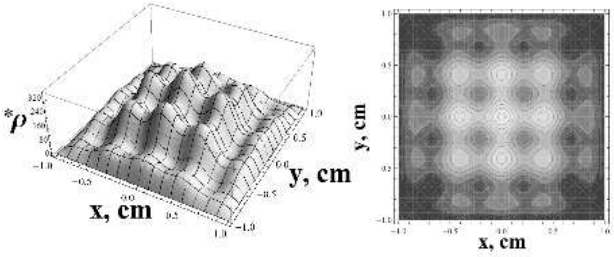

Fig. 3. Plot of the electron distribution function: $T = 0.02$ K, $E = 1.4 \times 10^5$ V/m, ${}^*\rho = \rho L^2/Q_{\text{total}} \times 100\%$

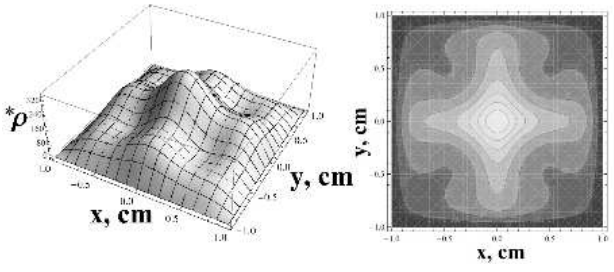

Fig. 4. Plot of the electron distribution function: $T = 0.02$ K, $E = 2 \times 10^5$ V/m, ${}^*\rho = \rho L^2/Q_{\text{total}} \times 100\%$

that the distribution function is very sharp in the first case and tends to become constant in the second case (Fig. 2).

We can conclude that a simple quasiclassical model can be built, by using an effective electron potential on the liquid-helium surface as some empirical data. This model predicts some effects that can be observed for the electrons on the liquid helium surface. On the other hand, it is rather simple and thus may be considered with no use of clusters or other powerful computer tools. Moreover, after some additional simplifications, it can be even treated analytically. We expect that the electron distribution function on the liquid-helium surface could be measured by means of modern experimental methods, and it would be possible to compare our results to the experimental data.

We are grateful for the joint grant of the Russian Foundation for basic researches and the National Academy of Science of Ukraine.

1. M.W. Cole, Phys. Rev. B **2**, 4239 (1970).
2. M.W. Cole and M.H. Cohen, Phys. Rev. Lett. **23**, 1238 (1969).
3. E. Wigner, Phys. Rev. **46**, 1002 (1934).
4. C.C. Grimes, and G. Adams, Phys. Rev. Lett. **42**, 795 (1979).
5. P.M. Platzman and M.I. Dykman, Science **284**, 1967 (1999).
6. V.B. Shikin, Sov. Phys. JETP **31**, 936 (1970).
7. F.I.B. Williams, R.S. Crandall, and A.H. Willis, Phys. Rev. Lett. **26**, 7 (1971).
8. V.S. Edel'man, Usp. Fiz. Nauk **23**, 227 (1980).
9. T. Ando, A. Fowler, and F. Stern, Rev. Mod. Phys. **54**, 437 (1982).
10. P.M. Platzman and H. Fukuyama, Phys. Rev. B **10**, 3150 (1974).
11. M. Haque, I. Paul, and S. Pankov, Phys. Rev. B **68**, 045427 (2003).
12. P. Leiderer and M. Wanner, Phys. Lett. A **73**, 189 (1979).
13. W. Ebner and P. Leiderer, Phys. Lett. A **80**, 277 (1980).
14. J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, 1996).
15. D.J. Thouless, Phys. Rep. **13**, 93 (1974).
16. S. Datta, *Electronic Transport in Mesoscopic Systems* (Cambridge Univ. Press, Cambridge, 1995).
17. Yu.P. Monarkha and K. Kono, *Two-Dimensional Coulomb Liquids and Solids* (Springer, Berlin, 2004).
18. B.I. Lev and A.G. Zagorodny, Phys. Rev. E **84**, 061115 (2011).
19. B.I. Lev and A.Y. Zhugaevych, Phys. Rev. E **57**, 6460 (1998).
20. L.D. Landau and E.M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon Press, New York, 1984).
21. I. Skachko, *Phase Diagram of a 2-Dimensional Electron System on the Surface of liquid helium*, Ph.D. thesis, Rutgers, The State Univ. of New Jersey (2006).
22. S. Koutsoumpos, *Surface State Electron Dynamics on Deformed Liquid Helium Films*, Ph.D. thesis, Konstanz (2010).
23. D.K. Lambert, *Electrons on the Surface of Liquid Helium*, Ph.D. thesis, Lawrence Berkeley Laboratory (1979).
24. B.I. Lev and A.G. Zagorodny, Phys. Lett. A, **373**, 1101 (2009).
25. B.I. Lev, V.B. Tymchyshyn, and A.G. Zagorodny, Cond. Matter Phys. **12**, 1 (2009).
26. B.I. Lev, V.B. Tymchyshyn, and A.G. Zagorodny, Phys. Lett. A **375**, 593 (2011).
27. A.J. Archer, Phys. Rev. E **78**, 031402 (2008).
28. A. Imperio and L. Reatto, J. Chem. Phys. **124**, 164712 (2006).
29. A. Imperio and L. Reatto, Phys. Rev. E **76**, 040402 (2007).
30. B.I. Lev, V.P. Ostroukh, V.B. Tymchyshyn, and A.G. Zagorodny, arXiv:1308.3852 (2013).
31. Yu. Nesterov, *Introductory Lectures on Convex Optimization. A Basic Course* (Kluwer, Dordrecht, 2004).

Received 11.10.13

Б.І. Лев, В.П. Остроух,
В.Б. Тимчишин, А.Г. Загородній

РОЗПОДІЛ ЕЛЕКТРОНІВ НА ДЕФОРМОВАНІЙ ПОВЕРХНІ РІДКОГО ГЕЛІЮ

Р е з ю м е

Запропоновано просту квазікласичну модель для опису електронів на поверхні рідкого гелію в зовнішньому електричному полі. В межах моделі приймається до уваги енергія електростатичної взаємодії з урахуванням поляризації середовища та міжелектронної взаємодії, пов'язаної з деформацією поверхні. Дається пояснення деяких ефектів, що спостерігаються експериментально. В тому числі приводиться спосіб чисельного розрахунку неоднорідностей розподілу заряду у зовнішньому полі. Отримані результати можуть бути використані для пояснення особливостей функції розподілу електронів, що вимірюється експериментально.