doi: 10.15407/ujpe60.06.0497

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PACS 11.10.Hi

ON DETERMINING THE RUNNING COUPLING FROM THE EFFECTIVE ACTION

The conformal anomaly has provided an expression for the effective action of gauge theories in the presence of a strong background field in terms of the running coupling constant. We exploit this result to find a novel expansion for the running coupling constant and to compare it with conventional expansions obtained by directly integrating the differential equation for the running coupling constant.

K e y w o r d s: running coupling, gauge theory.

1. Introduction

It has been long known that the introduction of a renormalization scale μ leads to a conformal anomaly. More explicitly, the trace of the energy-momentum tensor is no longer zero but rather is proportional to the renormalization group β-function [1]. From this result, one can show that the effective action for a gauge theory can be written in terms of the running gauge coupling when considered as a function of a strong background field [2]. At the same time, the effective action satisfies the renormalization group equation, which leads to the explicit summation of all its leading-log (LL), next-to-leading-log (NLL), *etc.* contributions [3]. In this paper, we exploit these two different expressions for the effective action to obtain a novel expression for the running gauge coupling in a gauge theory. It does not appear to be possible to obtain a similar expansion for running couplings that are not gauge couplings. We relate this new expansion to one previously derived by systematically solving the usual differential equation for the running coupling using techniques described below. This comparison shows that the conventional expansion for the running coupling is, for the first few terms of perturbation theory, identical to the novel expression that is derived below. We know of no way of obtaining our new expansion by directly solving the differential equation for the running coupling.

Of course, it is possible to obtain solutions for the running coupling with a truncated β function by using numerical integration techniques in conjunction with the usual differential equation for the running coupling.

2. The Running Coupling and the Effective Action

If the effective Lagrangian L is treated as a function of μ (the renormalization scale), $F_{\mu\nu}$ (the constant background field strength), and λ (the gauge coupling), then we have the renormalization group equation:

$$\mu\frac{dL}{d\mu} = \left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \gamma(\lambda)F_{\mu\nu}\frac{\partial}{\partial F_{\mu\nu}}\right)\times$$
$$\times L(\lambda, F_{\mu\nu}, \mu) = 0. \tag{1}$$

Since $\lambda F_{\mu\nu}$ is not renormalized [4], it follows that $\beta(\lambda) = -\lambda\gamma(\lambda)$ and Eq. (1) becomes

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\left(\frac{\partial}{\partial\lambda} - \frac{2}{\lambda}\Phi\frac{\partial}{\partial\Phi}\right)\right] L = 0, \tag{2}$$

where $\Phi = F_{\mu\nu}F^{\mu\nu}$.

For strong background fields (i.e., $\lambda\Phi \gg \mu^2$),

$$L = \sum_{n=0}^{\infty}\sum_{m=0}^{n} T_{n,m}\lambda^{2n}t^m\Phi = \sum_{n=0}^{\infty} S_n\left(\lambda^2 t\right)\lambda^{2n}\Phi, \tag{3}$$

where $t = \frac{1}{4}\ln\left(\frac{\lambda^2\Phi}{\mu^4}\right)$ [5] and

$$S_n\left(\lambda^2 t\right) = \sum_{m=0}^{\infty} T_{n+m,m}\left(\lambda^2 t\right)^m$$

($n = 0$ is LL, $n = 1$ is NLL *etc.*). Equation (2) leads to the nested equations ($n = 0, 1, 2, ...$)

$$-\frac{d}{d\xi}S_n(\xi)+2\sum_{\rho=0}^{n} b_{2\rho+3}\left[\xi\frac{d}{d\xi} + (n-\rho-1)\right]S_{n-\rho} = 0, \tag{4}$$

where $\beta(\lambda) = \sum_{\rho=0}^{\infty} b_{2\rho+3}\lambda^{2\rho+3}$ and $\xi = \lambda^2 t$. (We note that only b_3, b_5, b_7, b_9 are known explicitly in the MS scheme in QCD [12], though b_{11} is also known in QED [13].) The boundary condition for these equations is $S_n(\xi = 0) = T_{n,0}$. Solutions for $n = 0, 1, 2$ are respectively given by

$$S_0 = -T_{0,0}w, \tag{5a}$$

$$S_1 = \frac{T_{0,0}b_5}{b_3}\ln|w| + T_{1,0}, \tag{5b}$$

$$S_2 = -\frac{T_{2,0}}{w} + \frac{b_7}{b_3}T_{0,0}\left(\frac{1+w}{w}\right) - \left(\frac{b_5}{b_3}\right)^2 T_{0,0}\left(\frac{\ln|w| + (1+w)}{w}\right), \tag{5c}$$

where $w = -1 + 2b_3\xi$. (Equation (5) corrects errors in ref. [3].) For the solutions of Eq. (4) for S_n ($n = 3 \ldots 6$), see Appendix.

An alternate expression for the effective action that follows from the conformal anomaly is [2] (again using MS renormalization)

$$L = -\frac{1}{4}\frac{\lambda_0^2}{\bar{\lambda}^2(t)}\Phi, \tag{6}$$

where the running coupling $\bar{\lambda}(t)$ satisfies

$$\frac{d\bar{\lambda}(t)}{dt} = \beta(\lambda(\bar{t})) \quad (\bar{\lambda}(t=0) = \lambda_0). \tag{7}$$

Equation (6) satisfies (1) provided $\mu = \mu_0$ is fixed. In ref. [3], it was shown that Eqs. (3) and (6) are consistent provided

$$T_{n,0} = -\frac{1}{4}\delta_{n,0}. \tag{8}$$

Furthermore, these two equations show that

$$\bar{\lambda}^2(t) = \frac{-\lambda_0^2}{4}\left[\sum_{n=0}^{\infty} S_n(\lambda_0^2 t)\lambda_0^{2n}\right]^{-1}. \tag{9}$$

More explicitly, from Eqs. (5), (8), (9), we obtain our principal result:

$$\bar{\lambda}^2(t) = \lambda_0^2\left[(1 - 2b_3\lambda_0^2 t) + \lambda_0^2\left(\frac{b_5}{b_3}\ln\left|-1 + 2b_3\lambda_0^2 t\right|\right) + \lambda_0^4\left(\frac{b_7}{b_3}\frac{2b_3\lambda_0^2 t}{-1+2b_3\lambda_0^2 t} - \left(\frac{b_5}{b_3}\right)^2 \times \frac{\ln\left|-1+2b_3\lambda_0^2 t\right| + 2b_3\lambda_0^2 t}{-1+2b_3\lambda_0^2 t}\right) + ...\right]^{-1}. \tag{10}$$

This rather unusual expression for $\bar{\lambda}^2(t)$ can be compared with what can be obtained directly from Eq. (7). (This is a non-trivial test for the correctness of using the conformal anomaly to obtain Eq. (10).) We make this comparison by perturbatively expanding Eq. (10) in powers of λ^2 and comparing this with what is obtained by systematically solving Eq. (7). For the lowest-order solution, from

$$\frac{d\bar{\lambda}^2(t)}{dt} = b_3\bar{\lambda}^3(t), \tag{11a}$$

we easily find

$$\bar{\lambda}^2(t) = \frac{\lambda_0^2}{1 - 2b_3\lambda_0^2 t}. \tag{11b}$$

While if we go the next order,

$$\frac{d\bar{\lambda}(t)}{dt} = b_3\bar{\lambda}^3(t) + b_5\bar{\lambda}^5(t), \tag{12a}$$

it follows that

$$We^w = e^{2b_3 t/\rho}W_0 e^{w_0}, \tag{12b}$$

where

$$W = (-1 - \rho\bar{\lambda}^2)/(\rho\bar{\lambda}^2),\ W_0 = (-1 - \rho\lambda_0^2)/(\rho\lambda_0^2). \tag{12c}$$

We thus encounter the Lambert W function [6].

Equation (11b) is identical to the lowest-order contribution to Eq. (10), while Eq. (10) yields no closed form expression, when b_3, b_5 are non-zero.

However, Eq. (10) can be related to what is obtained from a perturbative solution to Eq. (7), which is found in the following systematic way. We begin by letting $x = \bar{\lambda}^2$ and $2b_{2\rho+3} = \beta_\rho (\rho = 0, 1, 2...)$, so that Eq. (7) becomes [7]

$$\frac{dx}{dt} = x^2(\beta_0 + \beta_1 x + \beta_2 x^2 + ...). \quad (13)$$

If we now rescale $t \to t/\epsilon, x \to \epsilon x$ and then make the expansion $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...$ $(x_n(t = 0) = x\delta_{n,0})$, we find that, at successive orders in ϵ,

$$\frac{dx_0}{dt} = \beta_0 x_0^2, \quad (14a)$$

$$\frac{dx_1}{dt} = \beta_0 x_0^2 + 2\beta_1 x_0 x_1, \quad (14b)$$

$$\frac{dx_2}{dt} = \beta_0(x_1^2 + 2x_0 x_2) + 3\beta_1 x_1 x_0^2 + \beta_4 x_0^4. \quad (14c)$$

Solving these equations in turn leads to

$$x_0 = \frac{x}{1 - \beta_0 x t}, \quad (15a)$$

$$x_1 = -x^2 \frac{\beta_1}{\beta_0} \frac{\ln|1 - \beta_0 x t|}{(1 - \beta_0 x t)^2}, \quad (15b)$$

etc.

The solutions for $x_n (n = 2 \ldots 5)$ are given in Appendix.

An alternate approach is to systematically solving Eq. (7) is to write (in analogy with Eq. (3) [8])

$$x(\mu_0) = x(\mu) \sum_{n=0}^{\infty} \sum_{m=0}^{n} \tau_{n,m} x^n(\mu) \ln^m \left(\mu^2/\mu_0^2\right), \quad (16a)$$

$$\equiv \sum_{n=0}^{\infty} \sigma_n(\zeta) x^{n+1}(\mu) \quad (\sigma_n(0) = \delta_{n0}), \quad (16b)$$

where $\sigma_n(\zeta) = \sum_{m=0}^{\infty} \tau_{m+n,m} \zeta^m$ and $\zeta = x(\mu) \times \times \ln\left(\mu^2/\mu_0^2\right)$. If now $\beta(x) = x^2 \sum_{n=0}^{\infty} \beta_n x^n$ and

$$\mu^2 \frac{d}{d\mu^2} x(\mu_0) = 0, \quad (17a)$$

$$\mu^2 \frac{d}{d\mu^2} x(\mu) = \beta\left(x(\mu)\right), \quad (17b)$$

then we see that

$$(1 + \beta_0 \zeta)\sigma_0' = -\beta_0 \sigma_0, \quad (18a)$$

$$(1 + \beta_0 \zeta)\sigma_1' + 2\beta_0 \sigma_1 = (-\beta_1 \sigma_0 - \beta_1 \zeta \sigma_0'), \quad (18b)$$

$$(1 + \beta_0 \zeta)\sigma_2' + 3\beta_0 \sigma_2 = = (-\beta_2 \sigma_0 - \beta_2 \zeta \sigma_0') + (-2\beta_1 \sigma_1 - \beta_1 \zeta \sigma_1'). \quad (18c)$$

These equations have the solutions

$$\sigma_0 = (1 + \beta_0 \zeta)^{-1}, \quad (19a)$$

$$\sigma_1 = -\left(\frac{\beta_1}{\beta_0}\right) \frac{\ln|1 + \beta_0 \zeta|}{(1 + \beta_0 \zeta)^2}, \quad (19b)$$

$$\sigma_2 = \left(\left(\frac{\beta_1}{\beta_0}\right)^2 - \frac{\beta_2}{\beta_0}\right) \left(\frac{1}{(1 + \beta_0 \zeta)^2} - \frac{1}{(1 + \beta_0 \zeta)^3}\right) - - \left(\frac{\beta_1}{\beta_0}\right)^2 \frac{1}{(1 + \beta_0 \zeta)^3} \left(\ln|1 + \beta_0 \zeta| - \ln^2|1 + \beta_0 \zeta|\right) \quad (19c)$$

etc. These solutions to Eq. (18) are seen to be equivalent to those of Eq. (14).

With the solution to Eq. (7) given by Eq. (15) (or alternatively Eq. (19)), we find that this is equivalent to the expression for the running coupling given by Eq. (9), where the running coupling appearing in Eq. (9) is expanded in powers of λ_0^2. (Recall that $x = \bar{\lambda}^2$ and $\beta_p = 2b_{2p+3}$.) This holds true to the order that we have computed (λ_0^{12}), and we anticipate that it would be true to all orders in λ_0^2. The novel expansion of (9) is distinct from all previous expansions that have been derived in that the dependence of $\bar{\lambda}^2(t)$ on t is exclusively in the denominator.

The sums $\sum_{n=0}^{\infty} S_n(\lambda^2 t)\lambda^{2n}\Phi$ and $\sum_{n=0}^{\infty} \sigma_n(\zeta) \times \times x^{n+1}$ in Eqs. (3) and (16a), (16b) represent leading-log (LL) contributions (for $n = 0$), next-to-leading-log (NLL) contributions (for $n = 1$), and, in general, $N^p LL$ contribution (for $n = p$) for L and $\bar{\lambda}^2$, respectively. It proves possible to use the renormalization group equation to perform parts of these sums, as was done in ref. [9] when considering the effective potential.

We illustrate this by first considering $\sigma_n(\zeta)$. From Eqs. (16b) and (17a), (17b), we find that

$$\left[(1 + \beta_0 \zeta)\frac{d}{d\zeta} + (n + 1)\beta_0\right] \sigma_n + + \sum_{\rho=1}^{n} \beta_\rho \left[\zeta \frac{d}{d\zeta} + (n + 1 - \rho)\right] \sigma_{n-\rho} = 0. \quad (20)$$

(This generalizes Eqs. (18a), (18b), (18c).) The general form of $\sigma_n(\zeta)$ that follows from Eq. (20) is

$$\sigma_n = \sum_{i=0}^{n} \sum_{j=0}^{i} \sigma_{i,j}^n \frac{L^j}{U^{i+1}}, \quad (21)$$

where $U = 1 + \beta_0\zeta$ and $L = \ln U$. Substitution of Eq. (21) into Eq. (20) leads to the recursion relation

$$\beta_0\left[(j+1)\sigma^n_{i,j+1} + (n-i)\sigma^n_{i,j}\right] + \\ + \sum_{\rho=1}^{n}\Bigg[(j+1)\sigma^{n-\rho}_{i,j+1} + i\sigma^{n-\rho}_{i-1,j} - \\ - (j+1)\sigma^{n-\rho}_{i-1,j+1} + (n-i+\rho)\sigma^{n-\rho}_{i,j}\Bigg] = 0. \quad (22)$$

If we set $i = n+1$ in Eq. (22), then

$$\sigma^n_{n,j+1} = \rho_1\left[\frac{n}{j+1}\sigma^{n-1}_{n-1,j} - \sigma^{n-1}_{n-1,j+1}\right], \quad (23)$$

where $\rho_n = -\beta_n/\beta_0$. If we set $j = n-1$ in Eq. (23), then

$$\sigma^n_{n,n} = \rho_1\sigma^{n-1}_{n-1,n-1} = (\rho_1)^n, \quad (24)$$

as, by Eq. (19a), $\sigma^0_{0,0} = 1$. Restricting σ^n_{ij} in Eq. (21) to $\sigma^n_{n,n}$, we find from Eq. (16b) that

$$x(\mu_0) = \sum_{n=0}^{\infty}\rho_1^n\frac{L^n}{U^{n+1}}x^{n+1}(\mu) = \frac{x(\mu)}{U - \rho_1 Lx(\mu)} \quad (25)$$

or, more explicitly (reversing the roles of μ and μ_0),

$$x(\mu) = \\ = \frac{x(\mu_0)}{1 - \beta_0 x(\mu_0)\ln\left(\frac{\mu^2}{\mu_0^2}\right) + \frac{\beta_1}{\beta_0}\ln\left(1 - \beta_0 x(\mu_0)\ln\left(\frac{\mu^2}{\mu_0^2}\right)\right)x(\mu_0)}, \quad (26)$$

which is consistent with Eq. (10).

If $j = n-2$ in Eq. (23), an explicit expression for $\sigma^n_{n,n-1}$ can be found following the approach of ref. [5]; this further modifies the expression for $x(\mu)$ in Eq. (26).

In a similar fashion, one can use Eq. (4) to see that

$$S_n(\xi) = \sum_{i=0}^{n}\sum_{j=0}^{i} S^n_{ij}\frac{L^j}{w^{i-1}}; \quad (27)$$

in analogy with Eq. (22), we find that

$$(j+1)S^n_{i,j+1} + (n-i)S^n_{ij} + \\ + \sum_{\rho=1}^{n-1}\chi_{2\rho+3}\Bigg[(j+1)S^{n-\rho}_{i-1,j+1} - (i-2)S^{n-\rho}_{i-1,j} + \\ + (j+1)S^{n-\rho}_{i,j+1} + (n-\rho-i)S^{n-\rho}_{ij}\Bigg] = 0, \quad (28)$$

where $\chi_{2\rho+3} = b_{2\rho+3}/b_3 (\rho = 1, 2...)$. For $i = n$ and $j = n-1$, Eq. (28) reduces to

$$S^n_{n,n} - \chi_5\frac{(n-2)}{n}S^{n-1}_{n-1,n-1} = 0. \quad (29)$$

As $S^0_{0,0} = \frac{1}{4}$ (by Eqs. (5a), (8)), we see by Eq. (29) that $S^1_{1,1} = -\chi_5/4, S^n_{n,n} = 0 (n \geq 2)$. If we only consider the contributions to S_n coming from $S^n_{n,n}$, it follows from Eq. (9) that

$$\bar{\lambda}^2(t) = -\frac{\lambda_0^2}{4}\left[\frac{1}{4}w - \frac{\chi_5}{4}(\ln w)\lambda_0^2\right]^{-1}, \quad (30)$$

which is identical to Eq. (26).

Further results that follow from Eq. (28) are, in turn,

$$S^2_{2,0} = -\frac{1}{4}(\chi_7 - \chi_5^2) \text{ (from Eq. (5c))}, \quad (31a)$$

$$S^3_{3,1} = -\frac{\chi_5\chi_7}{4} \text{ (from Eq. (5c) and Eq. (28) with } n = i = 3 \text{ and } j = 0), \quad (31b)$$

$$S^1_{1,0} = 0 \text{ (from Eq. (5b) and Eq. (28))}, \quad (31c)$$

$$S^n_{n,n-1} = \frac{1}{4}\frac{\chi_5^n}{n-1} \; (n \geq 2) \text{ (from Eq. (28) with } i = n, j = n-2), \quad (31d)$$

$$S^n_{n,n-2} = -\frac{\chi_5^{n-2}\chi_7}{4} - \\ - \frac{\chi_5^n}{4}\left(\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-2}\right) \; (n \geq 4) \\ \text{(from Eq. (28) with } i = n, j = n-3), \quad (31e)$$

$$S^n_{n-1,n-1} = 0 \; (n \geq 1) \\ \text{(from Eq. (28) with } i = j = n-1), \quad (32)$$

$$S^2_{1,0} = \frac{\chi_5^2}{4} - \frac{\chi_7}{4} \\ \text{(from Eq. (28) with } n = 2, i = 1, j = 0), \quad (33a)$$

$$S^3_{2,1} = 0 \\ \text{(from Eq. (28) with } n = 3, i = 2, j = 1), \quad (33b)$$

$$S^n_{n-1,n-2} = 0 \; (n \geq 3), \quad (33c)$$

$$S^n_{n-1,n-3} = \frac{1}{4}\left(\chi_7\chi_5^{n-2} - \chi_5^n\right) \; (n \geq 3) \\ \text{(from Eq. (28) with} i = n, j = n-3). \quad (34)$$

These contributions to L in Eq. (3) can now be easily summed. (For the contribution of Eq. (31), see Appendix). The final result for L/Φ coming from Eqs. (31)–(34) is the following (with $\Lambda = 1 - \frac{\chi_5 L\lambda^2}{w}$):

$$\frac{4L}{\Phi} = w - \chi_5 L\lambda^2 - \chi_5 \ln\Lambda\lambda^2 + \\ + \left(\chi_5^2 - \chi_7\right)\frac{\lambda^4}{w} - \frac{\chi_5\chi_7}{w^2}L\lambda^6 + (\chi_5 - \chi_7^2)\lambda^4 - \\ - \frac{\chi_7\chi_5^2 L^2}{w^3\Lambda}\lambda^8 - \frac{\chi_5^2}{w\lambda}(1 - \Lambda + \ln\Lambda)\lambda^4 + \\ + \left(\frac{\chi_5\chi_7 - \chi_5^3 w}{w^2\Lambda}\right)\lambda^6, \quad (35)$$

where there are contributions from all terms of order N^pLL. From (6), it follows that $\overline{\lambda}^2(t) = -(\lambda_0^2\Phi)/ /(4L)$ with L given by Eq. (35).

3. Discussion

By exploiting the conformal anomaly, the effective action for a constant external gauge field can be expressed in terms of the running coupling. We have used this result to find an alternative expression for the running coupling that is perturbatively equivalent to the usual solutions to Eq. (7).

We have also shown how portions of all N^pLL contributions to L coming from Eqs. (29), (31)–(34) can be summed to give Eq. (35). This leads, in turn, to an expansion of $\overline{\lambda}^2(t)$ that incorporates the portions of the N^pLL contributions for all p. Having contributions to $\lambda(t)$ coming from all order of perturbation theory is not possible if one were to systematically integrate Eq. (7) directly.

In [10], a different approach was used to integrate Eq. (7). In this reference, one takes

$$t = \int \frac{d\lambda}{b_3\lambda^3 + b_5\lambda^5 + ...} = \\ = -\frac{1}{2b_3}\left[\frac{1}{\lambda^2} + \frac{b_5}{b_3}\ell n\lambda^2 + \left(\frac{b_7}{b_3} - \frac{b_5^2}{b_7^2}\right)\lambda^2 + ...\right], \quad (36)$$

which is obtained by expanding the denominator of the integral. This is now solved iteratively to yield

$$\lambda^2 = -\frac{1}{2b_3 t} + \frac{b_5}{4b_3^3}\frac{1}{t^2}\ell n\left(-\frac{1}{2b_3 t}\right) + \dots . \quad (37)$$

A systematic approach to using (36) to expand λ^2 in powers of t^{-1} and $\ell n\, t$ is given in [11]; the techniques used resemble those that lead to (35) above. However, the renormalization group equation is not employed directly in ref. [11] as it is here; one is systematically solving (7) directly rather than using (2).

In addition to having all-orders contributions to $\lambda(t)$ coming from Eq. (9), we also have an unusual analytic dependence on t, as all dependence on t occurs in the denominator. Thus, we can gain additional insights into the asymptotic behaviour of $\lambda(t)$; this is currently being considered.

I thank A. Patrushev who assisted in these calculations and Roger Macleod for a helpful suggestion.

APPENDIX

The solutions for $x_n (n = 2 \dots 5)$ in Eq. (14) are as follows:

$$x_2 = \frac{1}{\beta_0^2 w^3}\Big[x^3\left(\beta_1^2\left(w - \ln^2 w + \right.\right. \\ \left.\left. + \ln(w) + 1\right) - \beta_0\beta_2(w+1)\right)\Big], \quad (38a)$$

$$x_3 = -\frac{1}{2\beta_0^3 w^4}x^4\Big[\beta_0^2\beta_3\left(w^2 - 1\right) + \beta_1^3\left((w+1)^2 + \right. \\ \left. + 2\ln^3 w - 5\ln^2 w - 4(w+1)\ln w\right) - \\ - 2\beta_0\beta_2\beta_1(w(w+1) - (2w+3)\ln(w))\Big], \quad (38b)$$

$$x_4 = \frac{1}{6\beta_0^4 w^5}x^5\Big[-2\beta_0^2\left(\beta_0\beta_4\left(w^3+1\right) - \right. \\ \left. - \beta_2^2(w-5)(w+1)^2\right) - 6\beta_0\beta_2\beta_1^2\left(-\left(2w^2+5w+3\right)\times\right. \\ \left. \times \ln w + (w-3)(w+1)^2 + 3(w+2)\ln^2 w\right) + \\ + \beta_1^4\left(-6\left(w^2+5w+4\right)\ln w + (w+1)^2(2w-7) - \right. \\ \left. - 6\ln^4 w + 26\ln^3 w + 9(2w+1)\ln^2 w\right) + \beta_0^2\beta_3\beta_1 \times \\ \times \left(4w^3 + 3w^2 - 6\left(w^2-2\right)\ln w + 1\right)\Big], \quad (38c)$$

$$x_5 = -\frac{1}{12\beta_0^5 w^6}x^6\Big[\beta_1^5\big(6\left(3w^2+26w+23\right)\ln^2 w + \\ + (w+1)^3(3w-17) + 12\ln^5 w - 77\ln^4 w + \\ + (22-48w)\ln^3 w - 2(w+1)^2(4w-11)\ln w\big) + \\ + 3\beta_0^3\left(\beta_0\beta_5\left(w^4-1\right) - 2\beta_2\beta_3\left(-w^2+w+2\right)^2\right) + \\ + \beta_0^2\beta_3\beta_1^2\big(\left(9w^2-22w+23\right)(w+1)^2 + \\ + 6\left(3w^2-10\right)\ln^2 w - 2\left(8w^3+15w^2-7\right)\ln w\big) - \\ - 6\beta_0\beta_2\beta_1^3\big((w+1)^2\left(2w^2-8w-3\right) + \left(6w^2+26w+27\right)\times \\ \times \ln^2 w + \left(-4w^3+2w^2+30w+24\right)\ln w - \\ - 4(2w+5)\ln^3 w\big) + \beta_0^2\beta_1\big(2\beta_0\beta_4\left(-3w^4-2w^3 + \right. \\ \left. + 2\left(2w^3+5\right)\ln w + 1\right) + \beta_2^2(w+1)\left(9w^3 - 29w^2 + \right. \\ \left. + \left(-8w^2+44w+100\right)\ln w - 37w + 1\right)\big)\Big]. \quad (38d)$$

The solutions for $S_n (n = 3 \dots 6)$ in (4) are as follows:

$$S_3 = -\frac{1}{8w^2}\Big[\chi_9\left(w^2-1\right) - 2\chi_7\chi_5\left(w^2+w-\ln(w)\right) + \\ + \chi_5^3\left((w+1)^2 - \ln^2 w\right)\Big], \quad (39a)$$

$$\begin{aligned}
S_4 &= \frac{1}{24w^3}\Big[-2\chi_{11}\left(w^3+1\right)+\chi_9\chi_5\times\\
&\times\left(4w^3+3w^2+6\ln(w)+1\right)+2\chi_7^2(w-2)\times\\
&\times(w+1)^2-6\chi_7\chi_5^2\left((w-1)(w+1)^2+\ln^2 w-\right.\\
&\left.-(w+1)\ln(w)\right)+\chi_5^4\left((w+1)^2(2w-1)+\right.\\
&\left.+2\ln^3 w-3\ln^2 w-6(w+1)\ln(w)\right)\Big], \qquad (39b)
\end{aligned}$$

$$\begin{aligned}
S_5 &= -\frac{1}{48w^4}\Big[\chi_{13}\big(-6\chi_7\chi_5^3\big((w+1)^2\times\\
&\times\left(2w^2-2w-1\right)+\left(-2w^2+2w+4\right)\ln(w)-\\
&-2\ln^3 w+(2w+5)\ln^2 w\big)-3\left(2\chi_7\chi_9\times\right.\\
&\left.\times\left(w^4-w^2+2w+2\right)-\chi_{13}\left(w^4-1\right)\right)+\\
&+\chi_9\chi_5^2\left(9w^4+8w^3-6\left(w^2-1\right)\ln(w)+\right.\\
&\left.+12w-18\ln^2 w+11\right)+\chi_5\left(2\chi_{11}\left(-3w^4-\right.\right.\\
&\left.-2w^3+6\ln(w)+1\right)+\chi_7^2(w+1)\times\\
&\left.\times\left(9w^3-5w^2-13w+24\ln(w)+1\right)\right)+\\
&+\chi_5^5\left((w+1)^3(3w-5)-3\ln^4 w+10\ln^3 w+\right.\\
&\left.+12(w+1)\ln^2(w)-6(w+1)^2\ln(w)\right)\big)\Big], \qquad (39c)
\end{aligned}$$

$$\begin{aligned}
S_6 &= \frac{1}{240w^5}\Big[-10\chi_7\chi_5^4\big((w+1)^3\times\\
&\times\left(6w^2-12w+7\right)+\left(6w^2-3w-9\right)\ln^2 w+\\
&+6\ln^4 w-2(3w+13)\ln^3 w-6(w-4)\times\\
&\times(w+1)^2\ln(w)\big)+\chi_5^6\big(3(w+1)^3\left(4w^2-7w-1\right)+\\
&+30\left(w^2+5w+4\right)\ln^2 w+12\ln^5 w-65\ln^4 w-\\
&-30(2w+1)\ln^3 w-10(w+1)^2(2w-7)\ln(w)\big)+\\
&+\chi_9\chi_5^3\big(30\left(w^2-5\right)\ln^2 w+3(w+1)^2\left(16w^3-\right.\\
&\left.-17w^2+8w+1\right)-10\left(4w^3+3w^2+18w+19\right)\times\\
&\times\ln(w)+120\ln^3 w\big)-2\big(6\chi_{15}\left(w^5+1\right)+2\chi_7^3\times\\
&\times(w+1)^3\left(3w^2-9w+13\right)+2\chi_{11}\chi_7\left(-6w^5+\right.\\
&\left.+5w^3+15w+14\right)-3\chi_9^2\left(2w^5+5w^2-3\right)\big)+\\
&+\chi_5^2\Big(\chi_7^2\big(3\left(24w^3-33w^2+2w+39\right)(w+1)^2-\\
&-20\left(w^3-9w^2-15w-5\right)\ln(w)-\\
&-60(3w+4)\ln^2 w\big)+2\chi_{11}\big(10\left(w^3+1\right)\ln(w)+\\
&+3\left(-6w^5-5w^4+10w+9\right)-60\ln^2(w)\big)\Big)+\\
&+\chi_5\big(3\chi_{13}\left(8w^5+5w^4+20\ln(w)+3\right)-\\
&-2\chi_7\chi_9(w+1)\left(36w^4-21w^3-14w^2+29w+\right.\\
&\left.+30(w-4)\ln(w)-14\right)\big)\Big]. \qquad (39d)
\end{aligned}$$

We also employ, in evaluating the contributions of Eq. (31e) to L, the result

$$\begin{aligned}
&\sum_{n=4}^{\infty} x^n\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-2}\right)=\\
&=\frac{1}{2}(x^4+x^5+x^6+\ldots)+\frac{1}{3}(x^5+x^6+\ldots)=\\
&=\frac{1}{2}\frac{x^4}{1-x}+\frac{1}{3}\frac{x^5}{1-x}+\ldots=-\frac{x^2}{1-x}\left(x+\ln(1-x)\right). \qquad (40)
\end{aligned}$$

From this, we see that if $n \geq 4$

$$\begin{aligned}
&\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-2}\right)=\\
&=\frac{-1}{n!}\frac{d^n}{dx^n}\bigg|_{x=0}\frac{x^2}{1-x}(x+\ln(1-x)). \qquad (41)
\end{aligned}$$

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Received 18.12.14

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ПРО ВИЗНАЧЕННЯ
БІЖУЧОЇ КОНСТАНТИ ЗВ'ЯЗКУ
ІЗ ЕФЕКТИВНОЇ ДІЇ

Р е з ю м е

За наявності конформной аномалії отримана формула для ефективної дії градієнтних теорій в присутності сильного фонового поля через біжучу константу зв'язку. Цей результат дозволив знайти нове розкладання для біжучої константи зв'язку і порівняти його з відомими розкладаннями, отриманими прямим інтегруванням диференціального рівняння для цієї константи.