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SPIN-ZERO COX'S PARTICLE WITH INTRINSIC STRUCTURE: GENERAL ANALYSIS IN EXTERNAL ELECTROMAGNETIC AND GRAVITATIONAL FIELDS

The relativistic theory of Cox's scalar non-point particle with intrinsic structure in the Proca approach in external uniform magnetic and electric fields in the Minkowski space is developed. A generalized Klein–Gordon–Fock equation is derived and is detailed in the presence of uniform magnetic and electric fields. The extension of this formalism to the arbitrary Riemannian space-time background is given. For a special class of curved metrics allowing for the existence of nonrelativistic wave equations, a generalized Schrödinger-type quantum mechanical equation for Cox's particle is derived. This generally covariant formalism is suitable in the presence of external magnetic and electric fields. It is shown that, in the most general form, the extended first-order Proca-like system of tensor equations contains non-minimal interaction terms through the electromagnetic tensor $F_{\alpha\beta}$ and the Ricci tensor $R_{\alpha\beta}$.

Keywords: zero spin, intrinsic structure, Cox's particle, generalized Schrödinger equation, magnetic field, electric field, Minkowski space, Riemannian space.

1. Cox's Wave Equation

In 1982, W. Cox [1] proposed a special wave equation for a scalar particle with a larger set of tensor components than the usual Proca approach includes: he used the set of a scalar, 4-vector, and antisymmetric and (irreducible) symmetric tensors, thus starting with the 20-component wave function. Such a more general theory for a scalar particle has attracted no significant attention till now. In the present paper, some further development of this theory will be presented, both in the Minkowski space and in arbitrary curved space models. In addition, we will elaborate some aspects of description of a Cox particle in external magnetic and electric fields.

First, let us consider the system of Cox's equations [1] in the Minkowski space. We will use a Proca-like generalized system obtained after the elimination of two second-rank tensors (applying the notation: $D_\alpha = i\hbar\partial_\alpha - \frac{e}{c}A_\alpha$ and $\mu = mc$; note the presence of an additional parameter λ which is associated with a non-trivial intrinsic structure of the scalar particle) from the initial system of Cox's equations

$$(\mu\delta_\alpha^\beta + \lambda F_\alpha{}^\beta)\Phi_\beta = D_\alpha\Phi, \quad D^\alpha\Phi_\alpha = \mu\Phi, \tag{1}$$

or, shorter,

$$\Lambda_\alpha{}^\beta\Phi_\beta = D_\alpha\Phi, \quad D^\alpha\Phi_\alpha = \mu\Phi. \tag{2}$$

The first equation in (2) can be multiplied by the inverse matrix $(\Lambda^{-1})_\rho{}^\alpha$. So, we obtain

$$(\Lambda^{-1})_\rho{}^\alpha D_\alpha\Phi = \Phi_\rho, \quad D^\rho\Phi_\rho = \mu\Phi. \tag{3}$$

From (3), one derives a generalized Klein–Gordon–Fock equation for the scalar function Φ:

$$\left[\mu D^\rho(\Lambda^{-1})_\rho{}^\alpha D_\alpha - \mu^2\right]\Phi = 0. \tag{4}$$

Equation (4) can be rewritten in the form

$$\{\mu(\Lambda^{-1})_{\rho}{}^{\alpha} D^{\rho} D_{\alpha} + \\ + \mu \left[i\hbar\partial^{\rho}(\Lambda^{-1})_{\rho}{}^{\alpha}\right] D_{\alpha} - \mu^2\} \Phi = 0. \quad (5)$$

The inverse matrix Λ^{-1} is of primary importance for the form of the generalized Klein–Gordon–Fock equation (5). We are to find an explicit form of the matrix Λ^{-1}. The calculation in this section is valid only in the Cartesian coordinates of a flat space; a generalization to the case of any curved space (or curvilinear coordinates in the flat space) will be given below. Let us introduce the notation

$$\Lambda = (\Lambda_{\alpha}{}^{\beta}) = \left| \begin{array}{cccc} \mu & -e_1 & -e_2 & -e_3 \\ -e_1 & \mu & -b_3 & b_2 \\ -e_2 & b_3 & \mu & -b_1 \\ -e_3 & -b_2 & b_1 & \mu \end{array} \right|, \quad (6)$$

$e_i = \lambda E_i \ , \quad b_i = \lambda B_i .$

The inverse matrix is defined by the formula

$$\Lambda^{-1} = \frac{1}{\det \Lambda} \left| \begin{array}{cccc} M_0{}^0 & -M_1{}^0 & +M_2{}^0 & -M_3{}^0 \\ -M_0{}^1 & +M_1{}^1 & -M_2{}^1 & +M_3{}^1 \\ +M_0{}^2 & -M_1{}^2 & +M_2{}^2 & -M_3{}^2 \\ -M_0{}^3 & +M_1{}^3 & -M_2{}^3 & +M_3{}^3 \end{array} \right| .$$

The determinant of the matrix Λ and the minors are

$$\det \Lambda = \mu^4 - \mu^2 \left(\mathbf{e}^2 - \mathbf{b}^2\right) - (\mathbf{eb})^2, \quad M_0{}^0 = \mu^3 + \mu \mathbf{b}^2,$$
$$M_1{}^1 = \mu^3 + \mu\left(b_1^2 - e_2^2 - e_3^2\right),$$
$$M_2{}^2 = \mu^3 + \mu\left(b_2^2 - e_1^2 - e_3^2\right),$$
$$M_3{}^3 = \mu^3 + \mu\left(b_3^2 - e_1^2 - e_2^2\right),$$
$$M_0{}^1 = -\mu^2 e_1 - \mu\left(e_2 b_3 - e_3 b_2\right) - b_1(\mathbf{eb}),$$
$$M_1{}^0 = -\mu^2 e_1 + \mu\left(e_2 b_3 - e_3 b_2\right) - b_1(\mathbf{eb}),$$
$$M_0{}^2 = \mu^2 e_2 + \mu\left(e_3 b_1 - e_1 b_3\right) + b_2(\mathbf{eb}),$$
$$M_2{}^0 = \mu^2 e_2 - \mu\left(e_3 b_1 - e_1 b_3\right) + b_2(\mathbf{eb}),$$
$$M_0{}^3 = -\mu^2 e_3 - \mu\left(e_1 b_2 - e_2 b_1\right) - b_3(\mathbf{eb}),$$
$$M_3{}^0 = -\mu^2 e_3 + \mu\left(e_1 b_2 - e_2 b_1\right) - b_3(\mathbf{eb}),$$
$$M_1{}^2 = \mu^2 b_3 - \mu\left(e_1 e_2 + b_1 b_2\right) - e_3(\mathbf{eb}),$$
$$M_2{}^1 = -\mu^2 b_3 - \mu\left(e_1 e_2 + b_1 b_2\right) + e_3(\mathbf{eb}),$$
$$M_1{}^3 = \mu^2 b_2 + \mu(e_1 e_3 + b_1 b_3) - e_2(\mathbf{eb}),$$
$$M_3{}^1 = -\mu^2 b_2 + \mu\left(e_1 e_3 + b_1 b_3\right) + e_2(\mathbf{eb}),$$
$$M_2{}^3 = \mu^2 b_1 - \mu\left(e_2 e_3 + b_2 b_3\right) - e_1(\mathbf{eb}),$$
$$M_3{}^2 = -\mu^2 b_1 - \mu\left(e_2 e_3 + b_2 b_3\right) + e_1(\mathbf{eb}).$$

In the tensor notation, the inverse matrix can be presented as follows:

$$\left(\Lambda^{-1}\right)_{\alpha}{}^{\beta} = \frac{1}{\mu^2\left(\mu^2 - \frac{\lambda^2}{2} F_{\rho}{}^{\sigma} F_{\sigma}{}^{\rho}\right) - \lambda^4 \left(\frac{1}{4} F_{\alpha}{}^{\beta} \overset{\times}{F}{}_{\beta}^{\alpha}\right)^2} \times$$
$$\times \left\{ \mu \left(\mu^2 - \frac{\lambda^2}{2} F_{\rho}{}^{\sigma} F_{\sigma}{}^{\rho}\right) \delta_{\alpha}^{\beta} - \lambda \left(\mu^2 - \frac{\lambda^2}{2} F_{\rho}{}^{\sigma} F_{\sigma}{}^{\rho}\right) \times \right.$$
$$\left. \times F_{\alpha}{}^{\beta} + \mu\lambda^2 F_{\alpha}{}^{\sigma} F_{\sigma}{}^{\beta} - \lambda^3 F_{\alpha}{}^{\sigma} F_{\sigma}{}^{\delta} F_{\delta}{}^{\beta} \right\}. \quad (7)$$

Let us consider two simple cases in detail. For the electric field $A_0 = -\mathbf{E}\,\mathbf{x}$, Eq. (4) yields

$$\Big[D_0^2 - \left(1 - \Gamma^2 \mathbf{E}^2\right) \mathbf{D}^2 - i\hbar \frac{e}{c} \Gamma \mathbf{E}^2 - \\ - \Gamma^2 (\mathbf{ED})^2 - \mu^2 \left(1 - \Gamma^2 \mathbf{E}^2\right) \Big] \Phi = 0. \quad (8)$$

For the magnetic field $\mathbf{A} = \frac{1}{2}\mathbf{x} \times \mathbf{B}$, we arrive at (for brevity, we use the parameter $\lambda/\mu = \Gamma$)

$$\Big[\left(1 + \Gamma^2 \mathbf{B}^2\right) D_0^2 - \mathbf{D}^2 + i\hbar \frac{e}{c} \Gamma \mathbf{B}^2 - \\ - \Gamma^2 (\mathbf{BD})^2 - \mu^2 \left(1 + \Gamma^2 \mathbf{B}^2\right) \Big] \Phi = 0. \quad (9)$$

2. Calculation of the Inverse Tensor $(\Lambda^{-1})_{\alpha}{}^{\beta}$ in the Case of the Riemannian Space

For simplicity, we will assume the metric tensor to be diagonal; however, the final results will be presented in the generally covariant form, so they are valid for any metric (including non-orthogonal) coordinate system of space–time).

In a diagonal metric, the following property of the electromagnetic tensor $F_{\alpha\beta}$ holds:

$$F_{00} = 0 \Longrightarrow F_0{}^0 = 0, \quad F_{11} = 0 \Longrightarrow F_1{}^1 = 0,$$
$$F_{22} = 0 \Longrightarrow F_2{}^2 = 0, \quad F_{33} = 0 \Longrightarrow F_3{}^3 = 0.$$

We start from the explicit form of the tensor $(F_{\alpha}{}^{\beta})$:

$$\left(F_{\alpha}{}^{\beta}\right) = \left| \begin{array}{cccc} 0 & F_0{}^1 & F_0{}^2 & F_0{}^3 \\ F_1{}^0 & 0 & F_1{}^2 & F_1{}^3 \\ F_2{}^0 & F_2{}^1 & 0 & F_2{}^3 \\ F_3{}^0 & F_3{}^1 & F_3{}^2 & 0 \end{array} \right| =$$
$$= \left| \begin{array}{cccc} 0 & g^{11} E_1 & g^{22} E_2 & g^{33} E_3 \\ -g^{00} E_1 & 0 & g^{22} B_3 & -g^{33} B_2 \\ -g^{00} E_2 & -g^{11} B_3 & 0 & g^{33} B_1 \\ -g^{00} E_3 & g^{11} B_2 & -g^{22} B_1 & 0 \end{array} \right| .$$

Below, we will use the notation

$$g^{11} E_1 = E^1, \quad g^{22} E_2 = E^2, \quad g^{33} E_3 = E^3, \quad g^{00} = h,$$
$$g^{22} g^{33} B_1 = B^1, \quad g^{33} g^{11} B_2 = B^2, \quad g^{11} g^{22} B_3 = B^3.$$

Let us compute the convolution of two tensors

$$(F_\alpha{}^\beta)(F_\beta{}^\rho) =$$

$$\begin{vmatrix} -hE^iE_i & -(E_2B^3 - E_3B^2) & -(E_3B^1 - E_1B^3) & -(E_1B^2 - E_2B^1) \\ -h(E^2B_3 - E^3B_2) & -hE_1E^1 - B_2B^2 - B_3B^3 & -hE_1E^2 + B^1B_2 & -hE_1E^3 + B^1B_3 \\ -h(E^3B_1 - E^1B_3) & -hE^1E_2 + B_1B^2 & -hE_2E^2 - B_1B^1 - B_3B^3 & -hE_2E^3 + B^2B_3 \\ -h(E^1B_2 - E^2B_1) & -hE^1E_3 + B_1B^3 & -hE^2E_3 + B_2B^3 & -hE_3E^3 - B_1B^1 - B_2B^2 \end{vmatrix}.$$

Next, we compute the convolution of three tensors

$$(F_\alpha{}^\beta)(F_\beta{}^\rho)(F_\rho{}^\sigma) = -\left(g^{00}E_iE^i + B_iB^i\right) \times$$
$$\times \begin{vmatrix} 0 & E^1 & E^2 & E^3 \\ -g^{00}E_1 & 0 & g^{22}B_3 & -g^{33}B_2 \\ -g^{00}E_2 & -g^{11}B_3 & 0 & g^{33}B_1 \\ -g^{00}E_3 & g^{11}B_2 & -g^{22}B_1 & 0 \end{vmatrix} +$$

$$+B_iE_i \begin{vmatrix} 0 & g^{11}B^1 & g^{22}B^2 & g^{33}B^3 \\ -g^{00}B^1 & 0 & g^{22}g^{00}E^3 & -g^{33}g^{00}E^2 \\ -g^{00}B^2 & -g^{11}g^{00}E^3 & 0 & g^{33}g^{00}E^1 \\ -g^{00}B^3 & g^{11}g^{00}E^2 & -g^{22}g^{00}E^1 & 0 \end{vmatrix}.$$

Then we compute the convolution of two tensors in two pairs of indices

$$\frac{1}{2}\left(F_\alpha{}^\beta F_\beta{}^\alpha\right) = -\left(g^{00}E_iE^i + B_iB^i\right) \equiv I(x).$$

Let us specify the dual electromagnetic tensor

$$(F^\times)^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\rho\sigma}(x)F_{\rho\sigma}, \quad \epsilon^{0123}(x) = \epsilon(x),$$
$$\epsilon^{\alpha\beta\rho\sigma}(x) = \epsilon^{[\alpha\beta\rho\sigma]}(x),$$

where

$$\epsilon(x) = \frac{1}{\sqrt{-\det g}} = \frac{1}{\sqrt{-g_{00}g_{11}g_{22}g_{33}}} =$$
$$= \sqrt{-g^{00}g^{11}g^{22}g^{33}}.$$

Then we find the explicit expressions for components of the dual tensor:

$$(F^\times)^{01} = \epsilon^{0123}(x)F_{23} = \epsilon(x)F_{23} = \epsilon(x)B_1,$$

$$(F^\times)_0{}^1 = g_{00}\epsilon(x)B_1 = -\sqrt{-g}g^{11}B^1,$$

$$(F^\times)_1{}^0 = -g_{11}\epsilon(x)B_1 = +\sqrt{-g}\, g^{00}B^1,$$

$$(F^\times)^{02} = \epsilon^{0231}(x)F_{31} = \epsilon(x)F_{31} = \epsilon(x)B_2,$$

$$(F^\times)_0{}^2 = g_{00}\epsilon(x)B_2 = -\sqrt{-g}g^{22}B^2,$$

$$(F^\times)_2{}^0 = -g_{22}\epsilon(x)B_2 = +\sqrt{-g}g^{00}B^2,$$

$$(F^\times)^{03} = \epsilon^{0312}(x)F_{12} = \epsilon(x)F_{12} = \epsilon(x)B_3,$$

$$(F^\times)_0{}^3 = g_{00}\epsilon(x)B_3 = -\sqrt{-g}g^{33}B^3,$$

$$(F^\times)_3{}^0 = -g_{33}\epsilon(x)B_3 = +\sqrt{-g}g^{00}B^3,$$

$$(F^\times)^{23} = \epsilon^{2301}(x)F_{01} = \epsilon(x)F_{01} = \epsilon(x)E_1,$$

$$(F^\times)_2{}^3 = g_{22}\epsilon(x)E_1 = -\sqrt{-g}g^{00}g^{33}E^1,$$

$$(F^\times)_3{}^2 = -g_{33}\epsilon(x)E_1 = +\sqrt{-g}g^{00}g^{22}E^1,$$

$$(F^\times)^{31} = \epsilon^{3102}(x)F_{02} = \epsilon(x)F_{02} = \epsilon(x)E_2,$$

$$(F^\times)_3{}^1 = g_{33}\epsilon(x)E_2 = -\sqrt{-g}g^{00}g^{11}E^2,$$

$$(F^\times)_1{}^3 = -g_{11}\epsilon(x)E_2 = +\sqrt{-g}g^{00}g^{33}E^2,$$

$$(F^\times)^{12} = \epsilon^{1203}(x)F_{03} = \epsilon(x)F_{03} = \epsilon(x)E_3,$$

$$(F^\times)_1{}^2 = g_{11}\epsilon(x)E_3 = -\sqrt{-g}g^{00}g^{22}E^3,$$

$$(F^\times)_2{}^1 = -g_{22}\epsilon(x)E_3 = +\sqrt{-g}g^{00}g^{11}E^3.$$

So, we obtain

$$\frac{1}{4}(F^{\times\beta}_{\alpha})(F_\beta{}^\alpha) = \frac{1}{4}\mathrm{Sp}\left\{-\sqrt{-g} \times\right.$$
$$\times \begin{vmatrix} 0 & g^{11}B^1 & g^{22}B^2 & g^{33}B^3 \\ -g^{00}B^1 & 0 & g^{00}g^{22}E^3 & -g^{00}g^{33}E^2 \\ -g^{00}B^2 & -g^{00}g^{11}E^3 & 0 & g^{00}g^{33}E^1 \\ -g^{00}B^3 & g^{00}g^{11}E^2 & -g^{00}g^{22}E^1 & 0 \end{vmatrix} \times$$
$$\left.\times \begin{vmatrix} 0 & E^1 & E^2 & E^3 \\ -g^{00}E_1 & 0 & g^{22}B_3 & -g^{33}B_2 \\ -g^{00}E_2 & -g^{11}B_3 & 0 & g^{33}B_1 \\ -g^{00}E_3 & g^{11}B_2 & -g^{22}B_1 & 0 \end{vmatrix}\right\};$$

let us consider the expressions for diagonal elements of this product:

$$(00) = \sqrt{-g} g^{00} g^{11} g^{22} g^{33} (E_i B_i) = -\frac{1}{\sqrt{-g}} (E_i B_i),$$
$$(11) = -\frac{1}{\sqrt{-g}} (E_i B_i), \quad (22) = -\frac{1}{\sqrt{-g}} (E_i B_i),$$
$$(33) = -\frac{1}{\sqrt{-g}} (E_i B_i).$$

Thus, we arrive at the relation

$$\frac{1}{4} (F_\alpha^{\times\beta})(F_\beta{}^\rho) = -\frac{1}{\sqrt{-g}} (E_i B_i) \equiv J(x).$$

Now, we can obtain the following expansion for the convolution of three tensors:

$$F_\alpha{}^\beta F_\beta{}^\rho F_\rho{}^\sigma = I(x) F_\alpha{}^\sigma + J(x) F_\alpha^{\times\sigma}.$$

Here, the notation $I(x), J(x)$ for two invariants of the electromagnetic field is used.

Then we can easily find the explicit form of the convolution of four tensors

$$(F_\alpha{}^\beta F_\beta{}^\chi F_\chi{}^\sigma)(F_\sigma^\rho) = I(x) F_\alpha{}^\sigma F_\sigma{}^\rho -$$
$$- J(x) \frac{1}{\sqrt{-g}} (E_i B_i) \delta_\alpha{}^\rho = I(x) F_\alpha{}^\sigma F_\sigma{}^\rho + J^2(x) \delta_\alpha{}^\rho.$$

At the same time, the last relation defines the minimal polynomial of the matrix $(F_\alpha{}^\beta)$. Therefore, the tensor $(\Lambda^{-1})_\alpha{}^\beta$ inverse to $\Lambda_\sigma{}^\alpha = \mu \delta_\sigma{}^\alpha + \lambda F_\sigma{}^\alpha$ should be constructed in the form

$$(\Lambda^{-1})_\alpha{}^\beta = \lambda_1 \delta_\alpha{}^\beta + \lambda_2 F_\alpha{}^\beta + \lambda_3 F_\alpha{}^\rho F_\rho{}^\beta + \lambda_4 F_\alpha{}^\rho F_\rho{}^\sigma F_\sigma{}^\beta.$$

The identity

$$\Lambda_\sigma{}^\alpha (\Lambda^{-1})_\alpha{}^\beta = \{\mu \delta_\delta{}^\alpha + \lambda F_\sigma{}^\alpha\} \times$$
$$\times \{\lambda_1 \delta_\alpha{}^\beta + \lambda_2 F_\alpha{}^\beta + \lambda_3 F_\alpha{}^\rho F_\rho{}^\beta + \lambda_4 F_\alpha{}^\rho F_\rho{}^\chi F_\chi{}^\beta\} =$$
$$= \mu\lambda_1 \delta_\sigma{}^\beta + \mu\lambda_2 F_\sigma{}^\beta + \mu\lambda_3 F_\sigma{}^\rho F_\rho{}^\beta + \mu\lambda_4 F_\sigma{}^\rho F_\rho{}^\chi F_\chi{}^\beta +$$
$$+ \lambda\lambda_1 F_\sigma{}^\beta + \lambda\lambda_2 F_\sigma{}^\alpha F_\alpha{}^\beta + \lambda\lambda_3 F_\sigma{}^\alpha F_\alpha{}^\rho F_\rho{}^\beta +$$
$$+ \lambda\lambda_4 F_\sigma{}^\alpha F_\alpha{}^\rho F_\rho{}^\chi F_\chi{}^\beta = \delta_\sigma{}^\beta$$

yields the linear non-homogeneous system of equations for the parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$:

$$\mu\lambda_1 + \lambda\lambda_4 J^2 x = 1, \quad \lambda\lambda_1 + \mu\lambda_2 = 0,$$

$$\lambda\lambda_2 + \mu\lambda_3 + \lambda\lambda_4 I(x) = 0, \quad \lambda\lambda_3 + \mu\lambda_4 = 0.$$

Its solution is

$$\lambda_1 = \frac{\mu(\mu^2 - \lambda^2 I)}{\mu^2(\mu^2 - \lambda^2 I) - \lambda^4 J^2},$$
$$\lambda_2 = \frac{-\lambda(\mu^2 - \lambda^2 I)}{\mu^2(\mu^2 - \lambda^2 I) - \lambda^4 J^2},$$
$$\lambda_3 = \frac{\mu\lambda^2}{\mu^2(\mu^2 - \lambda^2 I) - \lambda^4 J^2},$$
$$\lambda_4 = -\frac{\lambda^3}{\mu^2(\mu^2 - \lambda^2 I) - \lambda^4 J^2}.$$

Thus, we arrive at the following explicit representation of the inverse tensor:

$$(\Lambda^{-1})_\alpha{}^\beta = \frac{1}{\mu^2 (\mu^2 - \lambda^2 I) - \lambda^4 J^2} \times$$
$$\times \Big\{\mu \left(\mu^2 - \lambda^2 I\right) \delta_\alpha^\beta - \lambda \left(\mu^2 - \lambda^2 I\right) F_\alpha{}^\beta +$$
$$+ \mu\lambda^2 F_\alpha{}^\sigma F_\sigma{}^\beta - \lambda^3 F_\alpha{}^\sigma F_\sigma{}^\delta F_\delta{}^\beta\Big\}.$$

Using $F_\alpha{}^\sigma F_\sigma{}^\delta F_\delta{}^\beta = I F_\alpha{}^\beta + J F_\alpha^{\times\beta}$, we find a simpler representation (the symbol $\times$ stands for a dual tensor)

$$(\Lambda^{-1})_\alpha{}^\beta = \frac{1}{\mu^2 (\mu^2 - \lambda^2 I) - \lambda^4 J^2} \times$$
$$\times \Big\{\mu \left(\mu^2 - \lambda^2 I\right) \delta_\alpha^\beta - \lambda\mu^2 F_\alpha{}^\beta +$$
$$+ \mu\lambda^2 F_\alpha{}^\sigma F_\sigma{}^\beta - \lambda^3 J(x) F_\alpha^{\times\beta}\Big\}. \quad (10)$$

Substituting the above tensor $(\Lambda^{-1})_\alpha{}^\beta$ into Eq. (5) for a scalar function, we obtain (note that $D_\alpha = i\hbar\nabla_\alpha - \frac{e}{c} A_\alpha$)

$$\mu\Big[\mu \left(\mu^2 - \lambda^2 I\right) \delta_\alpha^\beta - \lambda\mu^2 F_\alpha{}^\beta +$$
$$+ \mu\lambda^2 F_\alpha{}^\sigma F_\sigma{}^\beta - \lambda^3 J F_\alpha^{\times\beta}\Big] D^\alpha D_\beta \Phi +$$
$$+ i\hbar\mu \det\Lambda \left[\nabla_\alpha \left(\frac{\mu \left(\mu^2 - \lambda^2 I\right)}{\det\Lambda} g^{\alpha\beta} - \frac{\lambda\mu^2}{\det\Lambda} F^{\alpha\beta} -\right.\right.$$
$$\left.\left. - \frac{\lambda^3 J}{\det\Lambda} F^{\times\alpha\beta} + \frac{\mu\lambda^2}{\det\Lambda} g_{\rho\sigma} F^{\alpha\rho} F^{\sigma\beta}\right)\right] \times$$
$$\times D_\beta \Phi - \mu^2 \det\Lambda \Phi = 0. \quad (11)$$

Let us specify the separate terms under the symbol of covariant derivative. The first term is reduced to the ordinary derivative of a scalar. The second and third terms are the 4-divergences of the antisymmetric tensor, and the fourth term is a 4-divergence of the symmetric tensor. They are calculated, by using the known formulas

$$\nabla_\alpha(A^{\alpha\beta}) = \left(\frac{1}{\sqrt{-g}}\partial_\alpha\sqrt{-g}A^{\alpha\beta}\right),$$

$$\nabla_\alpha(S^\alpha{}_\beta) = \left(\frac{1}{\sqrt{-g}}\partial_\alpha\sqrt{-g}S^\alpha{}_\beta\right) - \frac{1}{2}(\partial_\beta g_{\rho\sigma})S^{\rho\sigma}.$$

These results can be significantly simplified in the case of a purely magnetic or purely electric field.

Magnetic field:

$$(\Lambda^{-1})_\alpha{}^\beta = \frac{1}{\mu^2\left(\mu^2-\lambda^2 I\right)} \times$$
$$\times\Big\{\mu\left(\mu^2-\lambda^2 I\right)\delta^\beta_\alpha - \lambda\mu^2 F_\alpha{}^\beta + \mu\lambda^2 F_\alpha{}^\sigma F_\sigma{}^\beta\Big\},$$

$$I(x) = -(B_i B^i), \quad J(x) = 0,$$

$$\left(F_\alpha{}^\beta\right) = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & g^{22}B_3 & -g^{33}B_2 \\ 0 & -g^{11}B_3 & 0 & g^{33}B_1 \\ 0 & g^{11}B_2 & -g^{22}B_1 & 0 \end{vmatrix},$$

$$\left(F_\alpha{}^\sigma F_\sigma{}^\beta\right) =$$
$$= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -B_2B^2-B_3B^3 & B^1B_2 & B^1B_3 \\ 0 & B_1B^2 & -B_1B^1-B_3B^3 & B^2B_3 \\ 0 & B_1B^3 & B_2B^3 & -B_1B^1-B_2B^2 \end{vmatrix}.$$

Electric field:

$$(\Lambda^{-1})_\alpha{}^\beta = \frac{1}{\mu^2\left(\mu^2-\lambda^2 I\right)} \times$$
$$\times\left\{\mu(\mu^2-\lambda^2 I)\delta^\beta_\alpha - \lambda\mu^2 F_\alpha{}^\beta + \mu\lambda^2 F_\alpha{}^\sigma F_\sigma{}^\beta\right\},$$

$$I = -\left(g^{00}E_iE^i\right), \quad J = 0,$$

$$(F_\alpha{}^\beta) = \begin{vmatrix} 0 & E^1 & E^2 & E^3 \\ -g^{00}E_1 & 0 & 0 & 0 \\ -g^{00}E_2 & 0 & 0 & 0 \\ -g^{00}E_3 & 0 & 0 & 0 \end{vmatrix},$$

$$(F_\alpha{}^\sigma F_\sigma{}^\beta) =$$
$$= g^{00}\begin{vmatrix} -E^iE_i & 0 & 0 & 0 \\ 0 & -E_1E^1 & -E_1E^2 & -E_1E^3 \\ 0 & -E^1E_2 & -E_2E^2 & -E_2E^3 \\ 0 & -E^1E_3 & -E^2E_3 & -E_3E^3- \end{vmatrix}.$$

3. Non-Relativistic Approximation of the Generalized Wave Equation

The non-relativistic approximation of any wave equation can be substantial for its physical interpretation. Let us solve this task for Cox's particle (we use the general approach developed in [3]). We start from the system of equations in the form

$$K_\rho{}^\alpha\left(i\nabla_\alpha - \frac{e}{c\hbar}A_\alpha\right)\Phi = \frac{mc}{\hbar}\Phi_\rho,$$
$$\left(i\nabla_\alpha - \frac{e}{c\hbar}A_\alpha\right)\Phi^\alpha = \frac{mc}{\hbar}\Phi. \tag{12}$$

Equations (12) can be rewritten in a form more convenient for practical calculations:

$$K_\rho{}^\alpha\left(i\partial_\alpha - \frac{e}{c\hbar}A_\alpha\right)\Phi = \frac{mc}{\hbar}\Phi_\rho,$$
$$\left(\frac{i}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\sqrt{-g} - \frac{e}{c\hbar}A_\alpha\right)g^{\alpha\beta}\Phi_\beta = \frac{mc}{\hbar}\Phi. \tag{13}$$

Considering the space-time models with the metric $dS^2 = c^2dt^2 + g_{kl}(x)\ dx^k dx^l$, let us perform the (3 + 1) splitting in (13):

$$K_0{}^0\Big(i\partial_0 - \frac{e}{c\hbar}A_0\Big)\Phi + K_0{}^k\Big(i\partial_k - \frac{e}{c\hbar}A_k\Big)\Phi = \frac{mc}{\hbar}\Phi_0,$$
$$K_j{}^0\Big(i\partial_0 - \frac{e}{c\hbar}A_0\Big)\Phi + K_j{}^k\Big(i\partial_k - \frac{e}{c\hbar}A_k\Big)\Phi = \frac{mc}{\hbar}\Phi_j,$$
$$\left(\frac{i}{\sqrt{-g}}\frac{\partial}{\partial x^0}\sqrt{-g} - \frac{e}{c\hbar}A_0\right)\Phi_0 +$$
$$+\left(\frac{i}{\sqrt{-g}}\frac{\partial}{\partial x^k}\sqrt{-g} - \frac{e}{c\hbar}A_k\right)g^{kl}\Phi_l = \frac{mc}{\hbar}\Phi. \tag{14}$$

Next, let us separate the rest energy by the substitutions

$$\Phi \Longrightarrow \exp\left(-i\frac{mc^2t}{\hbar}\right)\Phi, \quad \Phi_0 \Longrightarrow \exp\left(-i\frac{mc^2t}{\hbar}\right)\Phi_0,$$
$$\Phi_l \Longrightarrow \exp\left(-i\frac{mc^2t}{\hbar}\right)\Phi_l.$$

As a result, relation (14) yields

$$K_0{}^0\left(i\hbar\partial_t + mc^2 - eA_0\right)\Phi +$$
$$+K_0{}^k\left(ic\hbar\partial_k - eA_k\right)\Phi = mc^2\Phi_0, \tag{15}$$

$$K_j{}^0\left(i\hbar\partial_t + mc^2 - eA_0\right)\Phi +$$
$$+K_j{}^k\left(ic\hbar\partial_k - eA_k\right)\Phi = mc^2\Phi_j, \tag{16}$$

$$\left(i\hbar\partial_t + mc^2 + \frac{i\hbar}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial t} - eA_0\right)\Phi_0 + \\ + \left(\frac{ic\hbar}{\sqrt{-g}}\frac{\partial}{\partial x^k}\sqrt{-g} - eA_k\right) g^{kj}\Phi_j = mc^2\Phi. \quad (17)$$

Using (16), we now exclude the vector (non-dynamic) variable Φ_j:

$$K_0{}^0\left(i\hbar\partial_t + mc^2 - eA_0\right)\Phi + \\ + K_0{}^k\left(ic\hbar\partial_k - eA_k\right)\Phi = mc^2\,\Phi_0, \quad (18)$$

$$\left(i\hbar\partial_t + mc^2 + \frac{i\hbar}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial t} - eA_0\right)\times \\ \times\Phi_0 + \left(\frac{ic\hbar}{\sqrt{-g}}\frac{\partial}{\partial x^k}\sqrt{-g} - eA_k\right)\times \\ \times\frac{g^{kj}}{mc^2}\Big[K_j{}^0\left(i\hbar\partial_t + mc^2 - eA_0\right) + \\ + K_j{}^l(ic\hbar\partial_l - eA_l)\Big]\Phi = mc^2\Phi. \quad (19)$$

We introduce the notation

$$i\hbar\partial_t - eA_0 = D_t, \quad ic\hbar\partial_k - eA_k = cD_k, \\ \frac{ic\hbar}{\sqrt{-g}}\frac{\partial}{\partial x^k}\sqrt{-g} - eA_k = c\,\overset{\circ}{D}_k\,. \quad (20)$$

Then Eqs. (18) and (19) can be written as follows:

$$K_0{}^0\left(D_t + mc^2\right)\Phi + K_0{}^k cD_k\Phi = mc^2\Phi_0, \quad (21)$$

$$\left(D_t + mc^2 + \frac{i\hbar}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial t}\right)\Phi_0 + \overset{\circ}{D}_k\,\frac{g^{kj}}{m}\times \\ \times\left(\frac{1}{c}K_j{}^0\left(D_t + mc^2\right) + K_j{}^l D_l\right)\Phi = mc^2\Phi. \quad (22)$$

Following the method described in [3], we introduce the small component φ and the large component Ψ: $\Phi = (\Psi + \varphi)/2$, $\Phi_0 = (\Psi - \varphi)/2$. Substituting these relations in (21), (22), we obtain

$$K_0{}^0 D_t\frac{\Psi + \varphi}{2} + \left(K_0{}^0 - 1 + 1\right)mc^2\frac{\Psi + \varphi}{2} + \\ + K_0{}^k cD_k\frac{\Psi + \varphi}{2} = mc^2\frac{\Psi - \varphi}{2}, \quad (23)$$

$$\left(D_t + mc^2 + \frac{i\hbar}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial t}\right)\frac{\Psi - \varphi}{2} + \overset{\circ}{D}_k\,\frac{g^{kj}}{m}\times \\ \times\left(\frac{1}{c}K_j{}^0(D_t + mc^2) + K_j{}^l D_l\right)\frac{\Psi + \varphi}{2} = mc^2\frac{\Psi + \varphi}{2}. \quad (24)$$

From whence, neglecting the small component φ in comparison with the big one Ψ, we have

$$\Big(D_t + \left(K_0{}^0 - 1\right)\left(D_t + mc^2\right) + \\ + K_0{}^k cD_k\Big)\frac{\Psi}{2} = -mc^2\varphi, \quad (25)$$

$$\left(D_t + \frac{i\hbar}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial t}\right)\frac{\Psi}{2} + \overset{\circ}{D}_k\,\frac{g^{kj}}{m}\times \\ \times\left(\frac{1}{c}K_j{}^0\left(D_t + mc^2\right) + K_j{}^l D_l\right)\frac{\Psi}{2} = mc^2\varphi. \quad (26)$$

We assume that the energies of nonrelativistic particles are much smaller than the rest energy, i.e., we apply the approximation $(D_t + mc^2) \approx mc^2$. As a result, the first and second equations are simplified:

$$\left(D_t + \left(K_0{}^0 - 1\right)mc^2 + K_0{}^k cD_k\right)\frac{\Psi}{2} = -mc^2\varphi, \quad (27)$$

$$\left(D_t + \frac{i\hbar}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial t}\right)\frac{\Psi}{2} + \overset{\circ}{D}_k\,\frac{g^{kj}}{m}\times \\ \times\left(mcK_j{}^0 + K_j{}^l D_l\right)\frac{\Psi}{2} = mc^2\varphi. \quad (28)$$

With the help of (27), we can eliminate the small component φ from Eq. (28). So, we obtain

$$\left(D_t + \frac{i\hbar}{2\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial t} + \frac{1}{2}\left[(K_0{}^0 - 1)mc^2 + K_0{}^k cD_k\right]\right)\Psi = \\ = \overset{\circ}{D}_k\,\frac{(-g^{kj})}{2m}\left[K_j{}^l D_l + mcK_j{}^0\right]\Psi. \quad (29)$$

With the substitution $\Psi \Longrightarrow (-g)^{-1/4}\;\Psi$, Eq. (29) takes a simpler form

$$D_t\Psi = \frac{1}{2m}\,\overset{\circ}{D}_k\,\left(-g^{kj}\right)\left(K_j{}^l D_l + mcK_j{}^0\right)\Psi - \\ - \frac{1}{2}\left(\left(K_0{}^0 - 1\right)mc^2 + K_0{}^j cD_j\right)\Psi. \quad (30)$$

This is the non-relativistic Schrödinger equation for Cox's particle.

Let us specify the case where only a magnetic field is present. Then the Schrödinger equation (30) becomes much simpler:

$$D_t\Psi = \frac{1}{2m}\,\overset{\circ}{D}_k\,\left(-g^{kj}\right)K_j{}^l D_l\Psi. \quad (31)$$

Let us detail the operator $K_j{}^l D_l$:

$$K_1{}^l D_l = K_1{}^1 D_1 + K_1{}^2 D_2 +$$

$$+\,{K_1}^3 D_3 = \frac{1}{\mu^2+\lambda^2 B_i B^i}\times$$
$$\times\Big[\mu^2 D_1 + \mu\lambda\left(B_2 D^3 - B_3 D^2\right) + \lambda^2 B^1 (B_i D_i)\Big],$$
$${K_2}^l D_l = {K_2}^1 D_1 + {K_2}^2 D_2 +$$
$$+\,{K_2}^3 D_3 = \frac{1}{\mu^2+\lambda^2 B_i B^i}\times$$
$$\times\Big[\mu^2 D_2 + \mu\lambda\left(B_3 D^1 - B_1 D^3\right) + \lambda^2 B^2 (\mathbf{BD})\Big],$$
$${K_3}^l D_l = {K_3}^1 D_1 + {K_3}^2 D_2 +$$
$$+\,{K_3}^3 D_3 = \frac{1}{\mu^2+\lambda^2 B_i B^i}\times$$
$$\times\Big[\mu^2 D_3 + \mu\lambda\left(B_1 D^2 - B_2 D^1\right) + \lambda^2 B^3 \,(B_i D_i)\Big].$$

Thus, we have (we use the notation $\Gamma = \lambda/\mu$)

$$\overset{*}{D}_1 = {K_1}^l D_l = \frac{1}{1+\Gamma^2 B_i B^i}\times$$
$$\times\Big[D_1 + \Gamma\left(B_2 D^3 - B_3 D^2\right) + \Gamma^2 B^1 (B_i D_i)\Big],$$
$$\overset{*}{D}_2 = {K_2}^l D_l = \frac{1}{1+\Gamma^2 B_i B^i}\times$$
$$\times\Big[D_2 + \Gamma\left(B_3 D^1 - B_1 D^3\right) + \Gamma^2 B^2 (\mathbf{BD})\Big], \tag{32}$$
$$\overset{*}{D}_3 = {K_3}^l D_l = \frac{1}{1+\Gamma^2 B_i B^i}\times$$
$$\times\Big[D_3 + \Gamma\left(B_1 D^2 - B_2 D^1\right) + \Gamma^2 B^3 (B_i D_i)\Big].$$

Therefore, Eq. (31) can be written in a compact form as follows:

$$D_t \Psi = -\frac{1}{2m}\,\overset{\circ}{D}_k\, g^{kj}(x)\,\overset{*}{D}_j\,\Psi. \tag{33}$$

In the case of the Cartesian coordinates of the flat space, the metric tensor is trivial, and the equation is simplified to the form

$$D_t \Psi = \frac{1}{2m}\mathbf{D}\left(\frac{\mathbf{D} - \Gamma \mathbf{B}\times\mathbf{D} + \Gamma^2 \mathbf{B}(\mathbf{BD})}{1+\Gamma^2 \mathbf{B}^2}\right)\Psi. \tag{34}$$

If the magnetic field in the flat space is uniform, Eq. (34) can be simplified still more

$$D_t \Psi = \frac{1}{2m}\frac{1}{1+\Gamma^2 \mathbf{B}^2}\times$$
$$\times\left(\mathbf{D}^2 - \Gamma \mathbf{D}(\mathbf{B}\times\mathbf{D}) + \Gamma^2 (\mathbf{BD})^2\right)\Psi. \tag{35}$$

Due to the identity $\mathbf{D}(\mathbf{B}\times\mathbf{D}) = +i\frac{e\hbar}{c}\mathbf{B}^2$, Eq. (35) can be represented as follows:

$$D_t \Psi = \frac{1}{2m}\frac{1}{1+\Gamma^2 \mathbf{B}^2}\times$$
$$\times\left(\mathbf{D}^2 - i\frac{e\hbar}{c}\Gamma \mathbf{B}^2 + \Gamma^2 (\mathbf{BD})^2\right)\Psi. \tag{36}$$

Note that the presence of the term $i(e\hbar/c)\Gamma\mathbf{B}^2$ means that the parameter Γ must be purely imaginary. The explicit form of (36) also implies that there is a steady shift of all levels by the value determined by the amplitude of the magnetic field and the parameter $i\Gamma$.

Now, we consider the case of a uniform electric field. The operator ${K_j}^l D_l + \mu {K_j}^0$ is

$${K_1}^l D_l + \mu {K_1}^0 = \frac{1}{1+\Gamma^2 E_i E^i}\times$$
$$\times\Big[D_1 + \Gamma^2 (E_i E^i) D_1 + \Gamma^2 E_1 (E^i D_i) + \mu\Gamma E_1\Big],$$
$${K_2}^l D_l + \mu {K_2}^0 = \frac{1}{1+\Gamma^2 E_i E^i}\times$$
$$\times\Big[D_2 + \Gamma^2 (E_i E^i) D_2 + \Gamma^2 E_2 (E^i D_i) + \mu\Gamma E_2\Big],$$
$${K_3}^l D_l + \mu {K_3}^0 = \frac{1}{1+\Gamma^2 E_i E^i}\times$$
$$\times\Big[D_3 + \Gamma^2 (E_i E^i) D_3 + \Gamma^2 E_3 (E^i D_i) + \mu\Gamma E_3\Big].$$

Thus,

$$({K_j}^l D_l + \mu {K_j}^0) = \frac{1}{1+\Gamma^2 E_i E^i}\times$$
$$\times\Big[D_j + \Gamma^2 (E_i E^i) D_j + \Gamma^2 E_j (E^i D_i) + \mu\Gamma E_j\Big]. \tag{37}$$

So, we have the representation

$$\left({K_0}^0 - 1\right) mc^2 + {K_0}^j c D_j =$$
$$= -c\frac{\Gamma^2 E_i E^i \mu + \Gamma E^j D_j}{1+\Gamma^2 E_i E^i}. \tag{38}$$

Thus, we obtain

$$\left(D_t - c\frac{\Gamma^2 E_i E^i \mu + \Gamma E^j D_j}{2\left(1+\Gamma^2 E_i E^i\right)}\right)\Psi = \frac{1}{2m}\,\overset{\circ}{D}_k\,\left(-g^{kj}\right)\times$$
$$\times\left[D_j + \frac{\Gamma^2 E_j (E^i D_i) + \mu\Gamma E_j}{1+\Gamma^2 E_i E^i}\right]\Psi; \tag{39}$$

this is the Schrödinger equation for Cox's particle in the electric field.

4. Non-Minimal Coupling to the Curvature of Space–Time in the System of Cox's Equations

We turn to the initial complete Cox's system of equations [1], which includes the symmetric and antisymmetric tensors:

$$\lambda_1 D^\beta \Phi_\beta - \mu\Phi = 0,$$
$$\lambda_1^* D_\beta \Phi + \lambda_2 D^\alpha \Phi_{[\alpha\beta]} - \lambda_3 D^\alpha \Phi_{(\alpha\beta)} - \mu\Phi_\beta = 0,$$
$$\lambda_2^* (D_\alpha \Phi_\beta - D_\beta \Phi_\alpha) - \mu \Phi_{[\alpha\beta]} = 0, \tag{40}$$
$$\lambda_3^* \left(D_\alpha \Phi_\beta + D_\beta \Phi_\alpha - \frac{1}{2} g_{\alpha\beta} D^\rho \Phi_\rho \right) - \mu \Phi_{(\alpha\beta)} = 0,$$

where the auxiliary numerical parameters λ_1, λ_2, and λ_3 subject to the additional constraints

$$\lambda_2\lambda_2^* - \lambda_3\lambda_3^* = 0, \quad \lambda_1\lambda_1^* - \frac{3}{2}\lambda_3\lambda_3^* = 1; \tag{41}$$

symbol D_α denotes the derivative, which takes the presence of external electromagnetic and gravitational fields into account:

$$D_\alpha = i\hbar\nabla_\alpha - \frac{e}{c} A_\alpha, \quad \mu = Mc.$$

With the help of the third and fourth equations in (40), let us exclude the tensor components

$$\mu^{-1} \left(\lambda_2^* (D_\alpha \Phi_\beta - D_\beta \Phi_\alpha) \right) = \Phi_{[\alpha\beta]},$$
$$\mu^{-1} \lambda_3^* \left(D_\alpha \Phi_\beta + D_\beta \Phi_\alpha - \frac{1}{2} g_{\alpha\beta} D^\rho \Phi_\rho \right) = \Phi_{(\alpha\beta)}. \tag{42}$$

In two other ones, we have

$$\lambda_1 D^\beta \Phi_\beta - \mu\Phi = 0, \tag{43}$$
$$\lambda_1^* D_\beta \Phi + \lambda_2 D^\alpha \mu^{-1} \Big[\lambda_2^* (D_\alpha \Phi_\beta - D_\beta \Phi_\alpha) \Big] -$$
$$- \lambda_3 D^\alpha \mu^{-1} \lambda_3^* \left(D_\alpha \Phi_\beta + D_\beta \Phi_\alpha - \frac{1}{2} g_{\alpha\beta} D^\rho \Phi_\rho \right) -$$
$$- \mu \Phi_\beta = 0. \tag{44}$$

Equation (44) can be presented as

$$\lambda_1^* D_\beta \Phi - \mu^{-1} (\lambda_2\lambda_2^* + \lambda_3\lambda_3^*) D^\alpha D_\beta \Phi_\alpha +$$
$$+ \frac{1}{2}\mu^{-1}\lambda_3\lambda_3^* D_\beta D^\rho \Phi_\rho - \mu\Phi_\beta = 0. \tag{45}$$

In view of (41), one can use the identity

$$(\lambda_2\lambda_2^* + \lambda_3\lambda_3^*) = 2\lambda_3\lambda_3^*.$$

Hence,

$$\lambda_1^* D_\beta \Phi - \mu^{-1} 2\lambda_3\lambda_3^* D_\alpha D_\beta \Phi^\alpha +$$
$$+ \frac{1}{2}\mu^{-1}\lambda_3\lambda_3^* D_\beta D_\alpha \Phi^\alpha - \mu\Phi_\beta = 0. \tag{46}$$

We use the identity

$$D_\alpha D_\beta \Phi^\alpha = D_\beta D_\alpha \Phi^\alpha + (D_\alpha D_\beta - D_\beta D_\alpha) \Phi^\alpha =$$
$$= D_\beta D_\alpha \Phi^\alpha + \hbar^2 \left(-i\frac{e}{\hbar c} F_{\alpha\beta} - R_{\alpha\beta} \right) \Phi^{a\alpha}. \tag{47}$$

Then Eq. (46) can be converted to the following one:

$$\lambda_1^* D_\beta \Phi - \mu^{-1} 2\lambda_3\lambda_3^* \Big[D_\beta D_\alpha \Phi^\alpha +$$
$$+ \hbar^2 \left(-i\frac{e}{\hbar c} F_{\alpha\beta} - R_{\alpha\beta} \right) \Phi^\alpha \Big] +$$
$$+ \frac{1}{2}\mu^{-1}\lambda_3\lambda_3^* D_\beta D_\alpha \Phi^\alpha - \mu\Phi_\beta = 0$$

or

$$\lambda_1^* D_\beta \Phi + \mu^{-1} 2\lambda_3\lambda_3^* \hbar^2 \left(i\frac{e}{c} F_{\alpha\beta} + R_{\alpha\beta} \right) \Phi^\alpha -$$
$$- \frac{3}{2}\mu^{-1}\lambda_3\lambda_3^* D_\beta (D_\alpha \Phi^\alpha) - \mu\Phi_\beta = 0. \tag{48}$$

With regard for Eq. (43),

$$D_\alpha \Phi^\alpha = \frac{\mu}{\lambda_1}\Phi,$$

we obtain

$$\lambda_1\lambda_1^* D_\beta \Phi + \mu^{-1} 2\lambda_3\lambda_3^* \hbar^2 \left(i\frac{e}{\hbar c} F_{\alpha\beta} + R_{\alpha\beta} \right) \lambda_1 \Phi^\alpha -$$
$$- \frac{3}{2}\lambda_3\lambda_3^* D_\beta \Phi - \mu\lambda_1\Phi_\beta = 0. \tag{49}$$

With the use of the second condition in (41)

$$\lambda_1\lambda_1^* - \frac{3}{2}\lambda_3\lambda_3^* = 1,$$

Eq. (49) can be simplified to the form

$$D_\beta \Phi + \mu^{-1} 2\lambda_3\lambda_3^* \hbar^2 \times$$
$$\times \left(i\frac{e}{\hbar c} F_{\alpha\beta} + R_{\alpha\beta} \right) \lambda_1 \Phi^\alpha - \mu\lambda_1\Phi_\beta = 0. \tag{50}$$

One should remember the additional equation (43)

$$\lambda_1 D^\beta \Phi_\beta - \mu\Phi = 0. \tag{51}$$

The parameter λ_1 can be included in the designation of the vector components $\lambda_1 \Phi_\beta \longrightarrow \Phi_\beta$. So, we arrive at the extended Proca equations

$$D^\beta \Phi_\beta - \mu\Phi = 0,$$
$$D_\beta \Phi - \mu\Phi_\beta - i\frac{\hbar^2}{Mc}(2\lambda_3\lambda_3^*)\left(\frac{e}{\hbar c}F_{\beta\alpha} + iR_{\beta\alpha}\right)\Phi^\alpha = 0. \qquad (52)$$

These equations should be compared with those, from which we started in Section **1**:

$$D^\beta \Phi_\beta - \mu\Phi = 0, \quad D_\beta \Phi - \mu\Phi_\beta - \lambda F_{\beta\alpha}\Phi^\alpha = 0; \qquad (53)$$

they partly correlate if (note that λ is an imaginary number)

$$\lambda = \frac{\hbar^2}{Mc}\frac{e}{\hbar c}(2i\lambda_3\lambda_3^*). \qquad (54)$$

Obviously, system (52) is more general than (53), it involves the non-minimal interaction of Cox's scalar particle with the external geometric background through the Ricci tensor.

Equations (52) can be rewritten as

$$D^\beta \Phi_\beta - \mu\Phi = 0,$$
$$D_\beta \Phi - \lambda\left(F_{\beta\alpha} + i\frac{\hbar c}{e}R_{\beta\alpha}\right)\Phi^\alpha - \mu\Phi_\beta = 0. \qquad (55)$$

In the absence of an electromagnetic field, Eqs. (55) are simplified (parameter $i\lambda$ is a real-valued one)

$$D^\beta \Phi_\beta = \mu\Phi,$$
$$D_\beta \Phi = \left(i\lambda\frac{\hbar c}{e}R_{\beta\alpha}(x) + \mu g_{\beta\alpha}(x)\right)\Phi^\alpha. \qquad (56)$$

This is a purely geometric modification of the theory of a scalar particle in Cox's approach.

5. Calculation of the Tensor $(\Lambda^{-1})_\alpha{}^\beta$ with the Ricci Tensor Included

We write Eq. (56) in the form ($\lambda^* = -\lambda$; temporarily, the coefficient $\frac{\hbar c}{e}$ will be a part of the designation of the Ricci tensor)

$$D^\beta \Phi_\beta = \mu\Phi,$$
$$\left[\mu\delta_\alpha{}^\beta + \lambda\left(F_\alpha{}^\beta + iR_\alpha{}^\beta\right)\right]\Phi_\beta = D_\alpha\Phi. \qquad (57)$$

With the use of the notation

$$\Lambda_\alpha{}^\beta = \mu\delta_\alpha{}^\beta + \lambda\left(F_\alpha{}^\beta + iR_\alpha{}^\beta\right), \qquad (58)$$

Eq. (57) can be written as

$$\Phi_\rho = (\Lambda^{-1})_\rho{}^\alpha D_\alpha\Phi, \quad D^\rho\Phi_\rho = \mu\Phi. \qquad (59)$$

This yields a generalized scalar equation

$$\left[D^\rho\left(\Lambda^{-1}\right)_\rho{}^\alpha(x)D_\alpha - \mu\right]\Phi(x) = 0. \qquad (60)$$

Since the characteristic equation

$$G^4 = g_0 + g_1 G + g_2 G^2 + g_3 G^3 \qquad (61)$$

for the matrix $F_\alpha{}^\beta + iR_\alpha{}^\beta = G_\alpha{}^\beta$ allows us to express the fourth power of the matrix G through I, G, G^2, and G^3, we can look for the inverse matrix in the form

$$\left(\Lambda^{-1}\right)_\rho{}^\alpha = \lambda_0 + \lambda_1 G + \lambda_2 G^2 + \lambda_3 G^3. \qquad (62)$$

From the equation $\Lambda\Lambda^{-1} = I$, we obtain

$$I = (\mu + \lambda G)\left(\lambda_0 + \lambda_1 G + \lambda_2 G^2 + \lambda_3 G^3\right) =$$
$$= \mu\lambda_0 + \mu\lambda_1 G + \mu\lambda_2 G^2 + \mu\lambda_3 G^3 +$$
$$+ \lambda\lambda_0 G + \lambda\lambda_1 G^2 + \lambda\lambda_2 G^3 +$$
$$+ \lambda\lambda_3\left(g_0 + g_1 G + g_2 G^2 + g_3 G^3\right).$$

So, we have a linear system of equations

$$I\!: \ \mu\lambda_0 + \lambda\lambda_3 g_0 = 1,$$
$$G\!: \ \mu\lambda_1 + \lambda\lambda_0 + \lambda\lambda_3 g_1 = 0,$$
$$G^2\!: \ \mu\lambda_2 + \lambda\lambda_1 + \lambda\lambda_3 g_2 = 0, \qquad (63)$$
$$G^3\!: \ \mu\lambda_3 + \lambda\lambda_2 + \lambda\lambda_3 g_3 = 0$$

or, in the matrix form,

$$\begin{pmatrix} \mu & 0 & 0 & \lambda g_0 \\ \lambda & \mu & 0 & \lambda g_1 \\ 0 & \lambda & \mu & \lambda g_2 \\ 0 & 0 & \lambda & \mu + \lambda g_3 \end{pmatrix} \begin{vmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}. \qquad (64)$$

Its solution is

$$\lambda_0 = \frac{-\left(\mu^3 + \mu^2\lambda g_3 - \mu\lambda^2 g_2 + \lambda^3 g_1\right)}{-\mu^4 - \mu^3\lambda g_3 + \mu^2\lambda^2 g_2 - \mu\lambda^3 g_1 + \lambda^4 g_0},$$
$$\lambda_1 = \frac{-\left(-\mu^2\lambda - \mu\lambda^2 g_3 + \lambda^3 g_2\right)}{-\mu^4 - \mu^3\lambda g_3 + \mu^2\lambda^2 g_2 - \mu\lambda^3 g_1 + \lambda^4 g_0},$$

$$\lambda_2 = \frac{-\left(\mu\lambda^2 + \lambda^3 g_3\right)}{-\mu^4 - \mu^3\lambda g_3 + \mu^2\lambda^2 g_2 - \mu\lambda^3 g_1 + \lambda^4 g_0},$$
$$\lambda_3 = \frac{\lambda^3}{-\mu^4 - \mu^3\lambda g_3 + \mu^2\lambda^2 g_2 - \mu\lambda^3 g_1 + \lambda^4 g_0}.$$

We introduce the new notation

$$g_0 = p_4, \quad g_1 = p_3, \quad g_2 = p_2, \quad g_3 = p_1,$$
$$G^4 = p_1 G^3 + p_2 G^2 + p_3 G + p_4. \tag{65}$$

Then

$$\lambda_0 = \frac{\mu^3 + \mu^2\lambda p_1 - \mu\lambda^2 p_2 + \lambda^3 p_3}{\mu^4 + \mu^3\lambda p_1 - \mu^2\lambda^2 p_2 + \mu\lambda^3 p_3 - \lambda^4 p_4},$$
$$\lambda_1 = \frac{-\mu^2\lambda - \mu, \lambda^2 p_1 + \lambda^3 p_2}{\mu^4 + \mu^3\lambda p_1 - \mu^2\lambda^2 p_2 + \mu\lambda^3 p_3 - \lambda^4 p_4},$$
$$\lambda_2 = \frac{\mu\lambda^2 + \lambda^3 p_1}{\mu^4 + \mu^3\lambda p_1 - \mu^2\lambda^2 p_2 + \mu\lambda^3 p_3 - \lambda^4 p_4}, \tag{66}$$
$$\lambda_3 = \frac{-\lambda^3}{\mu^4 + \mu^3\lambda p_1 - \mu^2\lambda^2 p_2 + \mu\lambda^3 p_3 - \lambda^4 p_4}.$$

We recall that

$$(\Lambda^{-1})^{\alpha}_{\rho} = \lambda_0 + \lambda_1 G + \lambda_2 G^2 + \lambda_3 G^3.$$

Degrees of the matrix G can be associated with the following invariants (see Chap. IV in [5]):

$$\mathrm{Sp}(G) = g_1 + g_2 + g_3 + g_4 = s_1,$$
$$s_1 = G_{\alpha}{}^{\alpha}(x),$$
$$\mathrm{Sp}(G^2) = g_1^2 + g_2^2 + g_3^2 + g_4^2 = s_2,$$
$$s_2 = G_{\alpha}{}^{\rho}(x) G_{\rho}{}^{\alpha}(x),$$
$$\mathrm{Sp}(G^3) = g_1^3 + g_2^3 + g_3^3 + g_4^3 = s_3, \tag{67}$$
$$s_3 = G_{\alpha}{}^{\rho}(x) G_{\rho}{}^{\sigma}(x) G_{\sigma}{}^{\alpha}(x),$$
$$\mathrm{Sp}(G^4) = g_1^4 + g_2^4 + g_3^4 + g_4^4 = s_4,$$
$$s_4 = G_{\alpha}{}^{\rho}(x) G_{\rho}{}^{\delta}(x) G_{\delta}{}^{\sigma}(x) G_{\sigma}{}^{\alpha}(x).$$

Here, the quantities $g_1, ..., g_4$ stand for four eigenvalues of the matrix G.

The invariants s_i and p_i obey the Newton recurrence formulas (see [5]):

$$p_1 = s_1 = \mathrm{Sp}(G),$$
$$p_2 = \frac{1}{2}(s_2 - p_1 s_1) = \frac{1}{2}\Big[\mathrm{Sp}\left(G^2\right) - p_1 \mathrm{Sp}(G)\Big],$$
$$p_3 = \frac{1}{3}(s_3 - p_1 s_2 - p_2 s_1) =$$
$$= \frac{1}{3}\Big[\mathrm{Sp}\left(G^3\right) - p_1 \mathrm{Sp}\left(G^2\right) - p_2 \mathrm{Sp}(G)\Big],$$
$$p_4 = \frac{1}{4}(s_4 - p_1 s_3 - p_2 s_2 - p_3 s_1) =$$
$$= \frac{1}{4}\Big[\mathrm{Sp}\left(G^4\right) - p_1 \mathrm{Sp}\left(G^3\right) - p_2 \mathrm{Sp}\left(G^2\right) - p_3 \mathrm{Sp}(G)\Big].$$

From whence, we obtain the following representations for the invariants p_i:

$$p_1 = \mathrm{Sp}(G), \quad p_2 = \frac{1}{2}\mathrm{Sp}\left(G^2\right) - \frac{1}{2}\mathrm{Sp}^2(G),$$
$$p_3 = \frac{1}{3}\Big[\mathrm{Sp}\left(G^3\right) - \mathrm{Sp}(G)\mathrm{Sp}\left(G^2\right) -$$
$$- \frac{1}{2}\left(\mathrm{Sp}\left(G^2\right) - \mathrm{Sp}^2(G)\right)\mathrm{Sp}(G)\Big] =$$
$$= \frac{1}{3}\mathrm{Sp}\left(G^3\right) - \frac{1}{2}\mathrm{Sp}\left(G^2\right)\mathrm{Sp}(G) + \frac{1}{6}\mathrm{Sp}^3(G),$$
$$p_4 = \frac{1}{4}\Big[\mathrm{Sp}\left(G^4\right) - \mathrm{Sp}(G)\mathrm{Sp}\left(G^3\right) -$$
$$- \frac{1}{2}\mathrm{Sp}^2\left(G^2\right) + \frac{1}{2}\mathrm{Sp}^2(G)\mathrm{Sp}\left(G^2\right) -$$
$$- \frac{1}{3}\mathrm{Sp}\left(G^3\right)\mathrm{Sp}(G) + \frac{1}{2}\mathrm{Sp}\left(G^2\right)\mathrm{Sp}^2(G) -$$
$$- \frac{1}{6}\mathrm{Sp}^3(G)\mathrm{Sp}(G)\Big];$$

finally, we find the expression

$$p_4 = \frac{1}{4}\Bigg[\mathrm{Sp}\left(G^4\right) - \frac{4}{3}\mathrm{Sp}(G)\mathrm{Sp}\left(G^3\right) -$$
$$- \frac{1}{2}\mathrm{Sp}^2\left(G^2\right) + \mathrm{Sp}^2(G)\mathrm{Sp}\left(G^2\right) - \frac{1}{6}\mathrm{Sp}^4(G)\Bigg].$$

In the case where the matrix G is antisymmetric, we have the equality

$$\tilde{G} = -G, \quad p_1 = \mathrm{Sp}G = 0,$$
$$\tilde{G^3} = -G^3, \quad \mathrm{Sp}\left(G^3\right) = 0,$$
$$p_1 = 0, \quad p_2 = \frac{1}{2}\mathrm{Sp}\left(G^2\right), \tag{68}$$
$$p_3 = 0, \quad p_4 = \frac{1}{4}\mathrm{Sp}\left(G^4\right) + \frac{1}{8}\mathrm{Sp}^2\left(G^2\right).$$

The characteristic equation (65) takes the form

$$G^4 - p_2 G^2 - p_4 = 0, \tag{69}$$

this case is realized in the construction of the characteristic polynomial for the electromagnetic tensor. In this case, relations (66) become simpler:

$$\begin{aligned}
\lambda_0 &= \frac{\mu^3 - \mu\,\lambda^2\,p_2}{\mu^4 - \mu^2\,\lambda^2\,p_2 - \lambda^4\,p_4},\\
\lambda_1 &= \frac{-\mu^2\,\lambda + \lambda^3\,p_2}{\mu^4 - \mu^2\,\lambda^2\,p_2 - \lambda^4\,p_4},\\
\lambda_2 &= \frac{\mu\,\lambda^2}{\mu^4 - \mu^2\,\lambda^2\,p_2 - \lambda^4\,p_4},\\
\lambda_3 &= \frac{-\lambda^3}{\mu^4 - \mu^2\,\lambda^2\,p_2 - \lambda^4\,p_4}.
\end{aligned} \tag{70}$$

For the additional verification, we consider the simple case without the electromagnetic tensor, when, additionally, the space-time is described by the Ricci tensor of the following simple form (elementary examples are the de Sitter spaces):

$$\begin{aligned}
&G_{\alpha\beta} = \frac{R}{4} g_{\alpha\beta}, \quad G_\alpha^\beta = \frac{R}{4}\delta_\alpha^\beta,\\
&\mathrm{Sp}G = R, \quad \mathrm{Sp}\left(G^2\right) = \frac{1}{4}R^2,\\
&\mathrm{Sp}\left(G^3\right) = \frac{1}{4^2}R^3, \quad \mathrm{Sp}\left(G^4\right) = \frac{1}{4^3}R^4,
\end{aligned} \tag{71}$$

that is

$$\begin{aligned}
&p_1 = R, \quad p_2 = \frac{1}{2}\frac{1}{4}R^2 - \frac{1}{2}R^2 = -\frac{3}{8}R^2,\\
&p_3 = \frac{1}{3}\frac{1}{16}R^3 - \frac{1}{2}\frac{1}{4}R^2 R + \frac{1}{6}R^3 = \frac{1}{16}R^3,\\
&p_4 = \frac{1}{4}\left[\frac{1}{16\cdot 4}R^4 - \frac{4}{3}R\frac{1}{16}R^3 - \frac{1}{2}\frac{1}{16}R^4 + \right.\\
&\left. + R^2\frac{1}{4}R^2 - \frac{1}{6}R^4\right] = -\frac{1}{4^4}R^4.
\end{aligned} \tag{72}$$

The expressions for p_i correspond to the following characteristic equation

$$G = (G_\alpha^\beta), \quad \left(G - \frac{R}{4}\right)^4 = 0. \tag{73}$$

Note that, in the presence of a geometric background,

$$\begin{aligned}
&D_\beta\Phi - \lambda\left(F_{\beta\alpha} + i\frac{\hbar c}{e}\frac{R}{4}g_{\beta\alpha}\right)\Phi^\alpha - \mu\Phi_\beta = 0,\\
&D^\beta\Phi_\beta - \mu\Phi = 0;
\end{aligned} \tag{74}$$

and in the absence of an external electromagnetic field, the system of equations (74) reads

$$D^\beta\Phi_\beta = Mc\Phi, \quad D_\beta\Phi = \left(Mc + i\lambda\frac{\hbar c}{e}\frac{R}{4}\right)\Phi_\beta. \tag{75}$$

In particular, in the case of the de Sitter spaces ($R(x) = R$), the effective additive (with a plus or minus) to the mass of a particle is

$$D^\beta\Phi_\beta = Mc\Phi, \quad D_\beta\Phi = \left(Mc + i\lambda\frac{\hbar c}{e}\frac{R}{4}\right)\Phi_\beta. \tag{76}$$

6. Conclusion

Thus, Cox's theory for a scalar particle with a larger set of tensor functions (the set of a scalar, 4-vector, antisymmetric and (irreducible) symmetric tensors) is generalized against the background of the Minkowski space and an arbitrary Riemannian space in the presence of external magnetic and electric fields.

For a special class of curved metrics allowing for the existence of nonrelativistic wave equations, a generalized Schrödinger-type quantum mechanical equation for Cox's particle is derived. This generally covariant formalism is specified in the presence of external magnetic and electric fields. It is shown that, in the most general form, the extended Proca-like first-order system of tensor equations contains non-minimal interaction terms through the electromagnetic tensor $F_{\alpha\beta}$ and the Ricci tensor $R_{\alpha\beta}$.

Thus, the general conclusion can be done: the effects of the large-scale structure of the Universe depend greatly on the form of the basic equations for elementary particles, and their modifications will lead to new physical phenomena due to the non-Euclidean geometry background.

The construction of the explicit solutions of generalized wave equations in the presence of magnetic and electric fields for the flat Minkowski space and simple curved backgrounds, spherical Riemann and hyperbolic Lobachevsky, will be given in separate papers – see [6, 7].

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СКАЛЯРНА ЧАСТИНКА КОКСА З ВНУТРІШНЬОЮ СТРУКТУРОЮ: ЗАГАЛЬНИЙ АНАЛІЗ У ЗОВНІШНІХ ЕЛЕКТРОМАГНІТНИХ І ГРАВІТАЦІЙНИХ ПОЛЯХ

Р е з ю м е

Релятивістська теорія Кокса для скалярної неточечної частинки з внутрішньою структурою в підході Прока розвинена в присутності зовнішніх однорідних магнітних і електричних полів в просторі Мінковського. Отримано узагальнене рівняння Клейна–Гордона–Фока в присутності однорідних магнітних і електричних полів. Виконано узагальнення цього формалізму на випадок довільного ріманова простору-часу. Для спеціального класу метрик, що допускають існування нерелятивістських хвильових рівнянь, отримано узагальнене квантово-механічне рівняння типу Шредінгера для частинки Кокса. Цей загальноковаріантний формалізм є застосовним за наявності зовнішніх магнітних і електричних полів. Показано, що узагальнена система Прока тензорних рівнянь містить члени немінімальної взаємодії через тензор електромагнітного поля $F_{\alpha\beta}$ і тензор Річчі $R_{\alpha\beta}$.