doi: 10.15407/ujpe60.07.0664

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PACS 98.80.Qc, 98.80.Cq, 95.35.+d, 95.36.+x

THE EXPANDING UNIVERSE: CHANGE OF REGIME

The aim of the paper is to explain, on the basis of the strict equations of quantum geometrodynamics for a cosmological model with the Robertson–Walker metric, the possible change of a regime of the expansion of the universe, from acceleration to deceleration or vice versa. We show that the change of the rate of expansion can point to the existence of a particular type of forces acting in the universe. It is indicated that these forces have the quantum nature. The cause of the expansion and a change of its regimes is a special form of the effective potential well, in which the universe is moving as a whole.

Keywords: dark energy, dark matter, quantum geometrodynamics, cosmology.

1. Introduction

According to the standard cosmological model, the very early universe went through a period of accelerated exponential expansion, which followed by a period of deceleration. The expansion of the present-day universe is accelerating again [1, 2]. It is assumed that the overwhelming majority of matter in the universe (∼95%) is in the form of substances of the unknown origin called the dark energy and the dark matter. Concerning the physical properties of these substances, it is known that the dark matter is gravitating, while the dark energy is antigravitating. The competition between these two components of the dark sector of matter-energy in the universe determines the dynamics of the expansion which can change in the course of time from deceleration, when the dark matter dominates, to acceleration, when the repulsive action of the dark energy is predominant.

Since the physical nature of the dark matter and the dark energy remains unknown till now, numerous different models were proposed (see, e.g., Refs. [3–5]). These models are intended to reconcile the classical theory of gravity, based on general relativity, with current astrophysical data.

It is conceivable that the transition from the decelerating expansion to the accelerating expansion occurred at a redshift of ≈ 0.6, as well as the transition from the inflation to the radiation-domination, could be a reflection of the internal property of the universe. This demonstrates that the universe is a more complicated system than it is supposed in general relativity. For example, the universe could be a quantum object.

The aim of the present paper is to explain, on the basis of the strict equations of quantum geometrodynamics within a specific, rather simple, exactly solvable cosmological model, the possible change of a regime of the expansion of the universe. We show that the change of the rate of expansion can point to the existence of a particular type of forces acting in the universe. It is indicated that these forces have the quantum nature.

In Sect. 2, we shortly review the Hamiltonian formalism for the minisuperspace model based on the Dirac–Arnowitt–Deser–Misner approach to general relativity [6, 7], expounded in Refs. [8, 9]. The canonical quantization of matter and gravitational fields is given here. We demonstrate that, finally, the problem of the dynamics of the universe can be reduced to the problem of one-dimensional motion of an analog particle with arbitrary mass and zero total energy in the force field of the potential formed by the curvature of space, matter, and quantum additions to the energy density and the pressure of matter, which are calculated precisely. In Sect. 3, the specific quantum model of the universe with matter in the form of a dust is studied. The results obtained in the paper are summarized in Sect. 4.

Throughout the paper, the Planck system of units is used. As a result, all quantities in the equations become dimensionless. The length $l_{\rm P} = \sqrt{2G\hbar/(3\pi c^3)}$ is taken as a unit of length, and the $\rho_{\rm P} = 3c^4/(8\pi G l_{\rm P}^2)$ is used as a unit of energy density and pressure. The proper time τ is measured in units of length. An arc time (conformal time) T is expressed in radians. The scalar field is taken in $\phi_{\rm P} = \sqrt{3c^4/(8\pi G)}$, and so on. Here, G is Newton's gravitational constant.

2. Theory

2.1. Hamiltonian Formalism

In the present paper, we confine ourselves to a study of the isotropic cosmological model. The space-time is described by the Robertson–Walker metric

$$ds^2 = a^2[dT^2 - d\Omega_3^2], \tag{1}$$

where a is the cosmic scale factor, which is a function of time, T is the time variable connected with the proper time τ by the differential equation $d\tau = a dT$, T is the "arc-parameter measure of time": during the interval $d\tau$, a photon moving on a hypersphere of radius $a(\tau)$ covers an arc dT measured in radians [10]. The quantity $d\Omega_3^2$ is a line element on a unit three-sphere. Following the ADM formalism [6,7], one can extract the so-called lapse function N, which specifies the time reference scale, from the total differential dT: $dT = N d\eta$, where η is the "arc time" coinciding with T for $N = 1$ (cf. Refs. [10, 11]). In the general case, the function N plays the role of a Lagrange multiplier in the Hamiltonian formalism, and it should be taken into account in an appropriate way.

To be specific, we consider the cosmological system (universe) described by the Hamiltonian [8, 9]

$$H = \frac{N}{2}\left\{-\pi_a^2 - a^2 + a^4[\rho_\phi + \rho_\gamma]\right\} + \\ + \lambda_1\left\{\pi_\Theta - \frac{1}{2}\,a^3\rho_0 s\right\} + \lambda_2\left\{\pi_{\tilde\lambda} + \frac{1}{2}\,a^3\rho_0\right\}, \tag{2}$$

where π_a, π_Θ, $\pi_{\tilde\lambda}$ are the momenta canonically conjugate with the variables a, Θ, and $\tilde\lambda$, ρ_ϕ is the energy density of matter (the field ϕ), ρ_γ is the energy density of a perfect fluid, which defines a material reference frame [8, 12], and it is a function of the density of the rest mass ρ_0 and the specific entropy s [13], Θ is the thermasy, $\tilde\lambda$ is the potential for the specific free energy taken with an inverse sign (for details, see Ref. [8]), and N, λ_1, and λ_2 are the Lagrange multipliers.

Hamiltonian (2) is a linear combination of constraints and thus weakly vanishes, $H \approx 0$. The variations of the Hamiltonian with respect to N, λ_1, and λ_2 give three constraint equations,

$$-\pi_a^2 - a^2 + a^4[\rho_\phi + \rho_\gamma] \approx 0, \\ \pi_\Theta - \frac{1}{2}\,a^3\rho_0 s \approx 0, \quad \pi_{\tilde\lambda} + \frac{1}{2}\,a^3\rho_0 \approx 0. \tag{3}$$

From the conservation of these constraints in time, it follows that the number of particles of a perfect fluid in the proper volume[1] $\frac{1}{2}a^3$ and the specific entropy conserve: $E_0 \equiv \frac{1}{2}a^3\rho_0 = \text{const}$, $s = \text{const}$. With regard for these conservation laws and the vanishing of the momenta conjugate with the variables ρ_0 and s, one can discard the degrees of freedom corresponding to these variables and convert the second-class constraints into first-class constraints in accordance with Dirac's proposal [8, 14].

It is convenient to choose the perfect fluid with the density ρ_γ in the form of relativistic matter (radiation). Then one can put $a^4\rho_\gamma \equiv E = \text{const}$ in Eq. (3). The matter field with the energy density ρ_ϕ and the pressure p_ϕ can be taken for definiteness in the form of a uniform scalar field ϕ,

$$\rho_\phi = \frac{2}{a^6}\,\pi_\phi^2 + V(\phi), \quad p_\phi = \frac{2}{a^6}\,\pi_\phi^2 - V(\phi), \tag{4}$$

where $V(\phi)$ is the potential of this field, and π_ϕ is the momentum conjugate with ϕ. After the averaging with respect to appropriate quantum states, the scalar field turns into the effective matter fluid (see Ref. [9], and below).

The equation of motion for the classical dynamical variable $\mathcal{O} = \mathcal{O}(a, \phi, \pi_a, \pi_\phi, ...)$ has the form

$$\frac{d\mathcal{O}}{dT} \approx \{\mathcal{O}, \frac{1}{N}H\}, \tag{5}$$

where H is Hamiltonian (2), $\{.,.\}$ are the Poisson brackets.

[1] This volume is equal to $2\pi^2 a^3$, where a is taken in units of length.

2.2. Quantization

In quantum theory, the first-class constraint equations (3) become constraints on the state vector Ψ [14] and, in this way, define the space of physical states, which can be turned into a Hilbert space (cf. Ref. [15]). Passing from classical variables in Eqs. (2)–(4) to the corresponding operators, using the conservation laws, and introducing the noncoordinate co-frame

$$hd\tau = sd\Theta - d\tilde{\lambda}, \quad hdy = sd\Theta + d\tilde{\lambda}, \tag{6}$$

where $h = \frac{\rho_\gamma + p_\gamma}{\rho_0}$ is the specific enthalpy, p_γ is the pressure of radiation, and y is a supplementary variable, we obtain [8, 9]

$$\begin{aligned} &\left(-\partial_a^2 + a^2 - 2a\hat{H}_\phi - E\right)|\Psi\rangle = 0, \quad \partial_y|\Psi\rangle = 0, \\ &\left(-i\partial_T - \frac{2}{3}E\right)|\Psi\rangle = 0, \end{aligned} \tag{7}$$

where

$$\hat{H}_\phi = \frac{1}{2}a^3\hat{\rho}_\phi \tag{8}$$

is the Hamiltonian operator of the scalar field ϕ, the operator $\hat{\rho}_\phi$ is described by Eq. (4) with $\pi_\phi = -i\partial_\phi$. From Eq. (7), it follows that the evolution of the state vector Ψ in time is described by the exponential multiplier as follows:

$$\Psi(T) = \mathrm{e}^{i\frac{2}{3}E(T-T_0)}\Psi(T_0), \tag{9}$$

so that the arc-parameter T appears to be the most natural time variable in quantum theory as well. Here, T_0 is an arbitrary constant taken as a time reference point. The vector $\Psi(T_0) \equiv |\psi\rangle$ is defined in the space of two variables a and ϕ. According to Eqs. (7), it is annihilated by the constraint equation

$$\left(-\partial_a^2 + a^2 - 2a\hat{H}_\phi - E\right)|\psi\rangle = 0. \tag{10}$$

By substituting the Poisson brackets with the commutators of operators $\hat{\mathcal{O}} = \{a, -i\partial_a\}$ and $\frac{1}{N}\hat{H}$, we obtain the quantum analog of Eq. (5) for the operator of momentum $\pi_a = -i\partial_a$ and its time derivative

$$\langle\psi| - i\partial_a|\psi\rangle = \langle\psi| - \frac{da}{dT}|\psi\rangle, \tag{11}$$

$$\langle\psi| - i\frac{d}{dT}\partial_a|\psi\rangle = \langle\psi|a - \hat{H}_\phi + 3\hat{L}_\phi|\psi\rangle, \tag{12}$$

where

$$\hat{L}_\phi = \frac{1}{2}a^3\hat{p}_\phi \tag{13}$$

is the Lagrangian operator of the scalar field, and $\hat{p}_\phi$ is given by Eq. (4) with $\pi_\phi = -i\partial_\phi$.

The operator on the left-hand side of Eq. (10) is not the Hamiltonian of the system (it has the dimensions of [energy] $\times$ [length] in physical units). Whether this operator is self-adjoint depends on the behavior of the vector $|\psi\rangle$ and its first derivatives with respect to the scale factor and field variables on the boundaries of the ranges of their values. In this connection, we consider the Hamiltonian $\hat{H}_\phi$, which can be diagonalized by means of some state vectors $\langle x|u_k\rangle$ of a quantum scalar field in the representation of the generalized variable $x = x(\frac{1}{2}a^3, \phi)$. The explicit form of x is determined by the form of the potential $V(\phi)$ taken as a real function [9]. Assuming that the vectors $|u_k\rangle$ satisfy the completeness condition, $\sum_k |u_k\rangle\langle u_k| = 1$, and that they are orthonormalized, $\langle u_k|u_{k'}\rangle = \delta_{kk'}$, we guarantee the self-adjointness of the operator $\hat{H}_\phi$ and the reality of the function $M_k(a)$ in the equation

$$\langle u_k|\hat{H}_\phi|u_{k'}\rangle = M_k(a)\delta_{kk'}, \tag{14}$$

where the index of the state k can take both discrete and continuous values (in the latter case, the condition of orthogonality of the state vectors $|u_k\rangle$ is written by means of the Dirac delta function), and $M_k(a) = \frac{1}{2}a^3\rho_m$ is the proper energy of matter in the volume $\frac{1}{2}a^3$. The energy density and the pressure of matter [2],

$$\rho_m = \langle u_k|\hat{\rho}_\phi|u_k\rangle, \quad p_m = \langle u_k|\hat{p}_\phi|u_k\rangle \tag{15}$$

have the form

$$\rho_m = \frac{2M_k(a)}{a^3}, \quad p_m = w_m\rho_m, \tag{16}$$

where

$$w_m = -\frac{1}{3}\frac{d\ln M_k(a)}{d\ln a} \tag{17}$$

is the equation of state parameter. In the model $V(\phi) = \lambda_\alpha\phi^\alpha$, where λ_α is the coupling constant and $\alpha \geq 0$, matter reduces to a barotropic fluid with the parameter $w_m = \frac{\alpha-2}{\alpha+2}$. For $\alpha = 0$, the barotropic fluid

[2] The index k is omitted.

takes the form of the vacuum of the scalar field in the k-th state. The value $\alpha = 1$ corresponds to the strings. Matter in the form of a dust is reproduced by $\alpha = 2$, whereas $\alpha = 4$ leads to the relativistic matter and so on. The so-called stiff Zel'dovich matter is obtained in the limiting case $\alpha = \infty$.

In the general case, the proper energy $M_k(a)$ depends on a. It describes a classical source (as a mass-energy) of the gravitational field in k-th state. In principle, it may contain the contribution from both luminous and dark matters.

Using Eq. (14), one can integrate Eqs. (10)–(12) with respect to the matter field variable. Let us express the vector $|\psi\rangle$ in the form of the expansion in the complete set of states $|u_k\rangle$,

$$|\psi\rangle = \sum_k |u_k\rangle\langle u_k|\psi\rangle. \tag{18}$$

Then Eq. (10) yields the equation for the function $\langle a|f_k\rangle \equiv \langle u_k|\psi\rangle$,

$$\left(-\partial_a^2 + a^2 - 2aM_k(a)\right)|f_k\rangle = E|f_k\rangle. \tag{19}$$

This equation can be considered as an eigenvalue equation. Its solution $|f_k\rangle$ is an eigenfunction corresponding to the eigenvalue E. The function $|f_k\rangle$ describes the geometrical properties of the quantum universe filled with matter, whose mass-energy is $M_k(a)$.

In order to turn to the classical observables (such as the Hubble expansion rate and the deceleration parameter), we extract the amplitude and the phase $S_k(a)$ in the function $|f_k\rangle$,

$$\langle a|f_k\rangle = \frac{C_k}{\sqrt{\partial_a S_k(a)}}\, e^{iS_k(a)}, \tag{20}$$

where C_k is the constant determined by the boundary condition on the function $\langle a|f_k\rangle$, e.g., on the asymptotics $a \to \infty$. If the function $|f_k\rangle$ is real, then it is expressed through the Euclidean phase $S_E = -iS_k$. If the phase S_k is a real function, then Eq. (20) will describe the outgoing or incoming wave propagating in the space of the scale factor a. In this case, the general solution of Eq. (19) will be a superposition of $|f_k\rangle$- and $\langle f_k|$-states separately describing the expanding or contracting quantum universe (cf. Ref. [16]).

Substituting expression (20) into Eq. (19) and taking into account that $\langle a|f_k\rangle$ is nontrivial, we obtain the non-linear equation for the phase $S_k(a)$

$$(\partial_a S_k)^2 + a^2 - 2aM_k(a) - E =$$

$$= \frac{3}{4}\left(\frac{\partial_a^2 S_k}{\partial_a S_k}\right)^2 - \frac{1}{2}\frac{\partial_a^3 S_k}{\partial_a S_k}. \tag{21}$$

Using expansion (18) and representation (20), Eq. (11) can be rewritten in the form

$$\langle\Psi(T_0)|\left(\partial_a S_k + \frac{i}{2}\frac{\partial_a^2 S_k}{\partial_a S_k} + \frac{da}{dT}\right)|\Psi(T_0)\rangle = 0. \tag{22}$$

Since the instant of time T_0 is arbitrary, one gets the relation between the classical momentum $\pi_a = -\frac{da}{dT}$ and the phase $S_k(a)$

$$\partial_a S_k + \frac{i}{2}\frac{\partial_a^2 S_k}{\partial_a S_k} = -\frac{da}{dT}, \tag{23}$$

where the second term on the left-hand side follows from the amplitude of function (20) and has the quantum nature (it is proportional to $l_{\rm P}^2$ in ordinary units). In the classical limit, the right-hand side of Eq. (21) vanishes and this equation turns into the Hamilton–Jacobi equation for the action $S_k(a)$.

Using Eq. (23), one can reduce Eq. (21) to the form

$$\frac{1}{2}\left(\frac{da}{dT}\right)^2 + U(a) = 0, \tag{24}$$

where

$$U(a) = \frac{1}{2}\left[a^2 - 2aM_k(a) - Q_k(a) - E\right]. \tag{25}$$

The function

$$Q_k(a) = i\partial_a^2 S_k + \frac{1}{2}\left[\left(\frac{\partial_a^2 S_k}{\partial_a S_k}\right)^2 - \frac{\partial_a^3 S_k}{\partial_a S_k}\right] \tag{26}$$

determines the quantum correction ρ_Q to the energy density of matter in the form

$$\rho_Q = \frac{Q_k(a)}{a^4} \equiv \frac{2M_Q(a)}{a^3}, \tag{27}$$

where $M_Q(a) = \frac{1}{2}a^3\rho_Q$ is the proper energy of the quantum source of the gravitational field. The pressure produced by the quantum source is

$$P_Q = w_Q\rho_Q, \tag{28}$$

where

$$w_Q = \frac{1}{3}\left(1 - \frac{d\ln Q_k(a)}{d\ln a}\right) \qquad (29)$$

is the equation of state parameter, in which the first term is a correction for relativity, while the second term comes from the quantum dynamics of the system.

According to Eqs. (25)–(29), all quantum corrections to the energy density and the pressure of ordinary matter in the universe are collected in the gravitational quantum source function $Q_k(a)$.

Passing to dimensional physical units, we find [9] that the first term in Q_k is proportional to $l_{\rm P}^2$, while the term with higher derivatives of the phase S_k in the square brackets of Eq. (26) is proportional to $l_{\rm P}^4$. Therefore, one can conclude that the quantum corrections make contributions $\sim\hbar$ and $\sim\hbar^2$ to the dynamics of the expanding universe.

From Eq. (24) after the differentiation with respect to T, we obtain

$$\frac{d^2a}{dT^2} = -\frac{dU(a)}{da}. \qquad (30)$$

The formulae (24) and (30) allow us to draw an analogy with the equations of classical mechanics describing the conservation of energy of a particle moving in the potential well (25). These relations may be interpreted as the equations that describe the motion of a particle, an analog of the universe, with an arbitrary mass and the zero total energy under the action of the force

$$F(a) = -\frac{dU(a)}{da} = -a + M_k(a) + + a\frac{dM_k(a)}{da} + \frac{1}{2}\frac{dQ_k(a)}{da}. \qquad (31)$$

In addition to the space curvature effect and the mass term, this force involves the gradients (pressures) of classical and quantum gravitational sources.

It should be pointed out that relations (24) and (30) only formally coincide with the equations of classical mechanics. They describe the universe, in which, in addition to the classical source of a gravitational field in the form of matter with the mass $M_k(a)$, there is a relativistic quantum source with the mass $M_Q(a)$, which can have, under specific conditions, a serious influence on the dynamics of the universe. These conditions depend on the relation between the masses $M_k(a)$ and $M_Q(a)$. The mass $M_k(a)$ is given by the Hamiltonian $\hat{H}_\phi$ (8), i.e., in the end, by the potential $V(\phi)$ chosen from model arguments. The mass $M_Q(a)$ is defined by the quantum source function $Q_k(a)$, whose form (26) is totally determined by the solution of the quantum problem (21).

Passing to the proper time τ, one reduces Eqs. (24) and (30) to

$$\left(\frac{\dot{a}}{a}\right)^2 = \rho_{\rm tot} - \frac{1}{a^2}, \quad \frac{\ddot{a}}{a} = -\frac{1}{2}\left(\rho_{\rm tot} + 3p_{\rm tot}\right), \qquad (32)$$

where the dots denote the derivatives with respect to τ, and

$$\rho_{\rm tot} = \rho_m + \rho_\gamma + \rho_Q, \quad p_{\rm tot} = p_m + p_\gamma + P_Q. \qquad (33)$$

The deceleration parameter $q = -\frac{a\ddot{a}}{\dot{a}^2}$ in the model under consideration is reduced to the expression

$$q = 1 - \frac{a}{2U}\frac{dU}{da}. \qquad (34)$$

In the approximation $Q_k = 0$, relations (32) and (33) reduce to the ordinary Einstein–Friedmann equations, which describe the closed universe filled with matter with the density ρ_m and radiation with the density ρ_γ. The quantum correction to the pressure of matter is stipulated by the fact that the state vector $|\psi\rangle$ (18) is a superposition of all possible states of the classical source of the gravitational field $M_k(a)$ [9].

Equations (24) and (30) are exact. From these equations, it follows that, in general case, the force (31) can perform both the positive work on the universe, which is similar to the work of the repulsive forces of the dark energy, and the negative work analogous to the work of the attractive forces of the dark matter. The kind of work, which is performed on the universe, depends on the sign and behavior of the potential well $U(a)$ in Eq. (24).

The influence of a gravitational quantum source on the dynamics of the expanding universe depends on the value and the sign of the energy density ρ_Q (27) and the pressure P_Q (28).

If there exists the domain, where the function $Q_k(a) > 0$, and $\ln Q_k(a)$ depends on $\ln a$, so that w_Q can be parametrized in the form $w_Q = -\frac{1}{3}\delta$, where δ is an arbitrary positive or negative constant, then the quantum corrections can imitate, for example, the contribution from the de Sitter vacuum

($\delta = 3$), domain walls ($\delta = 2$), strings ($\delta = 1$), dust ($\delta = 0$), radiation ($\delta = -1$), or perfect gas ($\delta = -2$). In such a model, the quantum source is $Q_k(a) \sim a^{\delta+1}$. Identifying the energy density $\rho_Q > 0$ with the energy density of the dark energy, one finds that the case $\delta = 3$ reproduces the cosmological constant [3], the values $1 < \delta < 3$ correspond to the quintessence [17], whereas the phantom field [18] is described by the values $\delta > 3$.

However, it is possible that the quantum effects will generate the quantum corrections, for which the function $Q_k(a) < 0$, and the corresponding energy density is negative. This case is not extraordinary. According to quantum field theory, for instance, the vacuum fluctuations make a negative contribution to the field energy per unit area (the Casimir effect). As was shown in Ref. [19], the quantum correction ρ_Q takes a negative value near the initial cosmological singularity.

In order to find out the impact of the mass $M_k(a)$ and the quantum source function $Q_k(a)$ on the evolution of the universe, we consider a specific exactly solvable quantum problem.

3. An Exactly Solvable Model

Let matter be represented by a dust ($p_m = 0$). Such a type of matter is reproduced by the scalar field model with the potential $V(\phi) = \lambda\phi^2$, where the field ϕ oscillates near the point of its true vacuum, and λ is the coupling constant [9].

Really, if one introduces the variable $x = \left(\frac{\lambda a^6}{2}\right)^{1/4}\phi$, then the Hamiltonian $\hat{H}_\phi$ (8) takes the form

$$\hat{H}_\phi = \left(\frac{\lambda}{2}\right)^{1/2}\left(-\partial_x^2 + x^2\right). \quad (35)$$

We introduce the state vectors $\langle x|u_k\rangle$, which satisfy the equation

$$\left(-\partial_x^2 + x^2 - \epsilon_k\right)|u_k\rangle = 0, \quad (36)$$

where ϵ_k is an eigenvalue. This equation describes the quantum oscillator with $\epsilon_k = 2k+1$, $k = 0, 1, 2, \ldots$. From Eqs. (14), (35), and (36), it follows that

$$M_k(a) = \sqrt{2\lambda}\left(k + \frac{1}{2}\right) \equiv M. \quad (37)$$

Here, M is the total mass of k non-interacting identical particles with the masses $\sqrt{2\lambda}$.

It is convenient to introduce a new variable $z = a - M$, which describes a deviation of a from its "equilibrium" value at the point, where[3] $a = M$. Then Eqs. (19) and (21) take the form (the index k is omitted)

$$\left[-\partial_z^2 + z^2 - (2n+1)\right]|f\rangle = 0, \quad (38)$$

$$(\partial_z S)^2 + z^2 - (2n+1) = \frac{3}{4}\left(\frac{\partial_z^2 S}{\partial_z S}\right)^2 - \frac{1}{2}\frac{\partial_z^3 S}{\partial_z S}, \quad (39)$$

where $n = 0, 1, 2, \ldots$ is the quantum number, which numerates the discrete states of the universe, $E + M^2 = 2n + 1$, in the potential well z^2.

The potential well (25) in Eq. (24) reduces to

$$U = \frac{1}{2}\left[z^2 - (2n+1) - Q(z)\right]. \quad (40)$$

The quantities $|f\rangle$, S, and U are the functions of z. In addition, they depend on the free indices k and n, which are omitted here and below, when these indices are inessential.

Both Eqs. (38) and (39) have two solutions

$$\begin{aligned}\langle z|f\rangle_1 &= H_n(z)e^{-z^2/2},\\ \langle iz|f\rangle_2 &= H_{-n-1}(iz)e^{z^2/2},\end{aligned} \quad (41)$$

and

$$\partial_z S_1(z) = i\frac{e^{z^2}H_n^{-2}(z)}{2\int_0^z dx\, e^{x^2}H_n^{-2}(x)}, \quad (42)$$

$$\partial_z S_2(iz) = -\frac{e^{-z^2}H_{-n-1}^{-2}(iz)}{2\int_0^{iz} dx\, e^{x^2}H_{-n-1}^{-2}(x)}, \quad (43)$$

respectively, where $H_\nu(y)$ is the Hermitian polynomial. According to (20) and (41), the function $|f\rangle_1$ is real and expressed through the Euclidean phase $S_E = -i\,S_1$. The second solution $|f\rangle_2$ of Eq. (38) is complex. The corresponding phase S_2 appears to be complex. Usually, the solution $|f\rangle_2$ is discarded as unphysical. However, in quantum cosmology, both solutions should be considered. Indeed, only in such an approach, one can obtain nontrivial results about the topological properties of the universe as an essentially quantum system and can clarify the nature of the dark matter and the dark energy.

[3] In dimensional units, we have $a = \frac{2}{3\pi}\frac{G}{c^2}M$ (cf. Refs. [10, 11]).

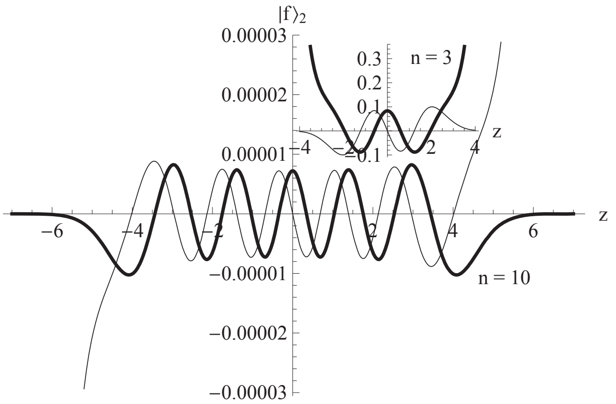

Fig. 1. Real (boldface curve) and imaginary (thin curve) parts of the function $|f\rangle_2$ from (41) *versus* the deviation z for $n = 10$ and $n = 3$ (in the inset)

Let us consider the quantum universe described by the wavefunction $|f\rangle_2$. The solution $|f\rangle_2$ from (41) as a function of $z = a - M$, where a is a real variable, is shown in Fig. 1 for $n = 10$ and $n = 3$ (in the inset). The quantum n-th state $|f\rangle_2$ is determined by the mass M of the universe in accordance with the condition of quantization: $2n + 1 = M^2 + E$. For example, the observed part of our universe is characterized by the parameters $M \sim 10^{61}$ ($\sim 10^{80}$ GeV) and $E \sim 10^{118}$ ($\rho_\gamma \approx 10^{-10}$ GeV/cm^3) [10, 20]. From the viewpoint of the model under consideration, it is in the state with $n \sim 10^{122}$ (up to $\sim 10^{-4}$). This estimate practically coincides with the estimate given by Hartle and Hawking [16]. Considering $n = 10$ as a number large enough to put $E = 0$, we obtain $M = 4.58$. For the case $n = 3$, we use the approximation $E = M^2$, so that $M = 1.87$.

The real $\mathrm{Re}\,|f\rangle_2$ and imaginary $\mathrm{Im}\,|f\rangle_2$ parts oscillate in the interval $|z| < M$ and are shifted in the phase with respect to each other by $\frac{\pi}{2}$. For $n = 10$, the function $\mathrm{Re}\,|f\rangle_2$ decreases exponentially outside this interval, while $\mathrm{Im}\,|f\rangle_2$ diverges exponentially as $|z| \to +\infty$. For $n = 3$, we have the inverse picture. In any case, in the interval bounded by the values $|z| \leq M$, the function $|f\rangle_2$ can be normalized. The normalization constant will depend on the quantum number n.

Using Eq. (43), we obtain the expression for Q (26) as a function of iz,

$$Q(iz) = -(2n+1) + \\ + 2(n+1)\frac{H_{-n-2}(iz)H_{-n}(iz)}{H^2_{-n-1}(iz)}. \quad (44)$$

The potential well (25) takes the form

$$U(z) = \frac{1}{2}z^2 - (n+1)\frac{H_{-n-2}(iz)H_{-n}(iz)}{H^2_{-n-1}(iz)}. \quad (45)$$

It is a complex function of the form

$$U(z) = U_R(z) + iU_I(z), \quad (46)$$

where $U_R(z)$ and $U_I(z)$ are real functions.

The evolution of the universe with the complex potential well (46) can be described in terms of the formalism with complex scale factor

$$a = a_R + i\,a_I, \quad (47)$$

where a_R and a_I are real functions of time T. The possibility of the introduction of a complex metric tensor and its relation to the real physical gravitational field was studied, e.g., in Refs. [21, 22] (see also references therein). Taking the common point of view, we assume that the physical gravitational field is described by the real part of metric (1) (the real line element). In our model, the necessity to pass to the complex variable a is related to the complexity of the wavefunction $|f\rangle_2$. Since the real and imaginary parts of this function vanish at different points (see Fig. 1), the real physical quantities, such as the kinetic energy, potential well, and deceleration parameter appear to be free of discontinuities, which are typical of the real function $|f\rangle_1$ in the region $-M < z < M$ [23].

The energy conservation law (24) can be rewritten in the form of two conditions

$$\begin{aligned} &\frac{1}{2}\left[\left(\frac{da_R}{dT}\right)^2 - \left(\frac{da_I}{dT}\right)^2\right] + U_R = 0, \\ &\frac{da_R}{dT}\,\frac{da_I}{dT} + U_I = 0. \end{aligned} \quad (48)$$

Hence, it follows that there are two solutions for the parts of the kinetic energy related to a change of a_R and a_I with time T,

$$\begin{aligned} &\left(\frac{da_R}{dT}\right)^2_\pm = -U_R \pm \sqrt{U_R^2 + U_I^2}, \\ &\left(\frac{da_I}{dT}\right)^2_\pm = U_R \pm \sqrt{U_R^2 + U_I^2}. \end{aligned} \quad (49)$$

The potential well (46) depends only on the real part of the scale factor. Therefore instead of $z = a -$

$-M$, one must take $z = a_R - M$ in Eq. (45). From Eqs. (48) and (49), it follows that the imaginary part a_I must be expressed in terms of the real part a_R,

$$a_I = \int_0^{a_R} dx \frac{U_I(x)}{U_R(x) \mp \sqrt{U_R^2(x) + U_I^2(x)}}, \tag{50}$$

with the boundary condition $a_I(a_R = 0) = 0$. Thus, in such a model, all elements of the complex spacetime are expressed via one real parameter $a_R = a_R(T)$.

The complexity of the spacetime metric leads to the interference of the kinetic energies $K^{\pm}_{R,I} \equiv \frac{1}{2}\left(\frac{da_{R,I}}{dT}\right)^2_{\pm}$ described by Eqs. (49). This interference smoothes out behavior of these energies near the points $|z| = z_0$, where $U_R = 0$. In this case, the motion is always realized in the real time T, since the domain with the Euclidean signature is found to be inaccessible.

The real and imaginary parts of the energy (45) as functions of z are plotted in Fig. 2 for the quantum numbers $n = 10$ and $n = 3$. The general behavior of U_R and U_I with respect to z is not changed for arbitrary values of the quantum number n, from $n \sim \sim 1$ up to $n \gg 1$. The same is true for other physical parameters (see Figs. 3 and 4).

The points, where U_R and U_I have extrema or vanish, are determined by n and M. So, we can conclude that, in the interval $|z| < M$, the energy $U_R(z)$ is well approximated by the expression: $U_R = \frac{1}{2}z^2 - \left(n + \frac{1}{2}\right)$. It vanishes at the points $z_0 \approx \pm\sqrt{2(n+1)}$. The imaginary part U_I has extrema at these points. It vanishes at the points $z = 0$ and $|z| = +\infty$. The real part of the potential energy is negative, $U_R < 0$, in the region $|z| < z_0$ and positive, $U_R > 0$, for $|z| > z_0$. We have $U_R \to +\infty$ as $|z| \to +\infty$. The imaginary part of the potential energy is positive, $U_I > 0$, at $z < 0$ and negative, $U_I < 0$, on the semiaxis $z > 0$. The inequality $|U_R| \gg |U_I|$ holds in the whole range of z, except the points near $|z| = z_0$.

In Fig. 3, the real K^+_R and imaginary K^+_I parts of the kinetic energy (49) of type (+) for $n = 10$ and $n = 3$ are shown. In the whole range of the deviation $z = a_R - M$, both these energies are positive and describe the motion in the real time. At the points $|z| = z_0$, they equal each other in accordance with Eqs. (48). At the point $z = 0$, the energy K^+_R has a maximum equal to $\left(n + \frac{1}{2}\right)$, while the energy K^+_I vanishes. In the interval $|z| < z_0$, the condition $K^+_R \gg K^+_I$ is satisfied. In the domain $|z| > z_0$, where

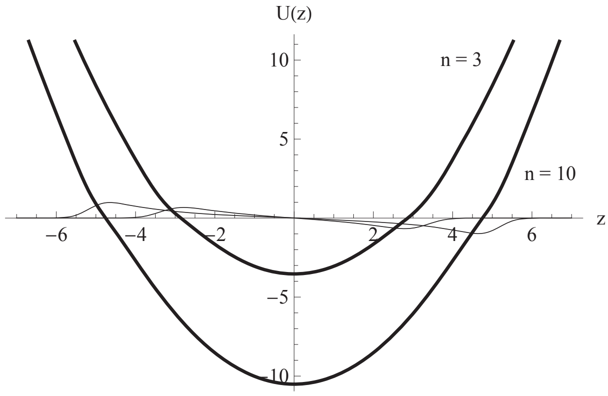

Fig. 2. Real (boldface curve) and imaginary (thin curve) parts of the potential well $U(z)$ (45) *versus* the deviation $z = a_R - M$ for $n = 10$ and $n = 3$

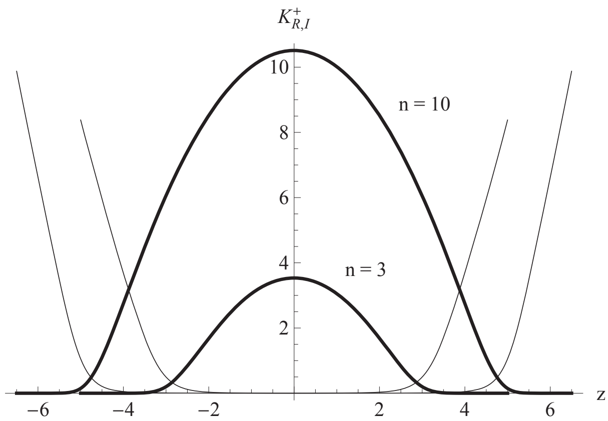

Fig. 3. Real K^+_R (boldface curve) and imaginary K^+_I (thin curve) parts of the kinetic energy (49) *versus* the deviation $z = a_R - M$ for $n = 10$ and $n = 3$

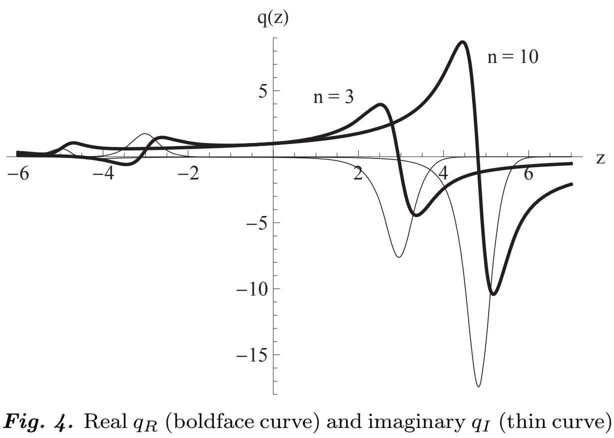

Fig. 4. Real q_R (boldface curve) and imaginary q_I (thin curve) parts of the deceleration parameter (34) *versus* the deviation $z = a_R - M$ for the potential energy (45) with $n = 10$ and $n = 3$

$U_R > 0$, the energy $K_R^+ \to 0$ as $|z| \to +\infty$, while the energy $K_I^+ \to U_R$.

The kinetic energy K_R^+ increases in the interval $-\infty < z < 0$. This means that the internal forces perform the positive work on the universe accelerating the expansion. This work is analogous to the work of the forces of the dark energy. On the contrary, the kinetic energy K_I^+ decreases in this interval, the negative work is done on the universe. As a result, the expansion decelerates. This is equivalent to the work of the attractive forces of the dark matter.

The energy K_R^+ decreases in the interval $0 < z < < +\infty$, demonstrating that the work performed on the universe is negative. In this way, the presence of an additional source of gravitational attraction is imitated. Inversely, the energy K_I^+ increases in this interval, the performed work is positive, as under the action of the forces of the dark energy.

So, the kinetic energies K_R^+ and K_I^+ show the work the universe should do in order to overcome the action of the internal forces of repulsion and attraction, which exist simultaneously and compete with each other at all stages of the evolution of the universe. Whether the expansion of the universe is accelerating or decelerating depends on the relation between the forces performing the work, causing the acceleration or the deceleration.

The plots of the real K_R^- and imaginary K_I^- parts of the kinetic energy (49) of type $(-)$ would be the mirror images of the plots in Fig. 3 with regard to the substitutions $K_R^+ \to -K_I^-$ and $K_I^+ \to -K_R^-$. Both energies are negative in the whole range of the deviation z and the motion can be described in the imaginary time $\xi = -iT$. The analysis of the solution of type $(+)$ given above remains valid for the solution of type $(-)$ after the formal substitution $T \to \xi$. This means that the gravitational and antigravitational forces, which perform work on the universe analogous to the dark matter and the dark energy, can exist in the spacetime with the Euclidean-signature metric as well.

Thus, the presence of the imaginary part U_I in the potential well (46) and in Eqs. (48) indicates that the processes of absorption and release of the energy pumping over between the states with an effective attraction and repulsion of matter are running in the system.

In Fig. 4, the real q_R and imaginary q_I parts of the deceleration parameter (34) are shown as functions of the deviation z for the potential well (45) with $n = 10$ and $n = 3$. In the region $|z| \leq M$, where $|q_R| \gg |q_I|$ (i.e. $|q_I/q_R|_{z=0} \approx 0.02$), the contribution from q_I can be neglected. In this stage, the universe expands with deceleration, since the antigravitational action of the forces performing the positive work is not enough to overcome the attraction of the ordinary and dark matters. The value $q_R(z = 0) = 1$ reproduces the results of general relativity [24]. At the point $z = 0$, we have $a_R = M$. In the region $a_R \approx 2M$, the redistribution of the energy takes place in the universe, as demonstrated by the peaks on the curves q_R and q_I in Fig. 4. The forces of attraction and repulsion compete with each other at $a_R < 2M$, where $q_R > 0$ and $q_I < 0$. At reaching the region $z > M$, where $a_R > 2M$, both parts of the deceleration parameter become negative, by demonstrating that the expansion of the universe is accelerating. Starting from the point $z \simeq 1.5M$ ($z = 6$ for $n = 10$), the parameter q_I vanishes and the rate of expansion is described only by the real part $q_R < 0$. In the limit $z \to +\infty$, the forces of attraction and repulsion will exactly compensate each other.

Again, as in the case of the complex metric tensor, one can accept that only the real part of the deceleration parameter q_R is a physically measurable quantity. The imaginary part q_I plays a role of a regularizing factor, which allows one to exclude the discontinuities caused by the vanishing of the real part of the function $|f\rangle_2$ at isolated points.

Let us compare the predictions of the quantum model under consideration with the observations in our universe. In the modern era, the scale factor (radius of the universe) $a_R \sim 10^{28}$ cm and the mass $M \sim 10^{80}$ GeV of matter in the observed part of our universe are estimated up to a coefficient less than $O(10)$. Such a radius a_R roughly coincides with the Hubble radius, while the mass M is estimated by the quantity of matter with the critical density $\rho_c \approx 10^{-5}$ GeV/cm^3 contained in the Hubble volume $\approx 2\pi^2 a_R^3$. In dimensionless units, which are used in this paper, these parameters prove to be of the same order of magnitude, $a_R \sim 10^{61}$ and $M \sim 10^{61}$. According to observations and theoretical estimations, the transition from the matter-dominated phase to the dark-energy-dominated universe takes place at a redshift of $\approx$0.6. This does not contradict the obtained condition $2 < a_R/M < 10$ (see Fig. 4), which determines the stage of transition to the phase of accelerating expansion of the universe.

4. Conclusion

In this paper, the evolution of the universe is studied in the exactly solvable dynamical quantum model with the Robertson–Walker metric. It is shown that the equation of motion, which describes the expansion or contraction of the universe, can be represented in the form of the zero total energy conservation law (24) for a particle being an analog of the universe. The analog particle has an arbitrary mass and moves in the potential well (25) under the action of the internal force (31), which involves the curvature of space, mass term, and gradients (pressures) of classical and quantum gravitational sources. The quantum source (26) emerges as a result of the evolution of the phase $S_k(a)$ of the state vector (20), which describes the geometrical properties of the quantum universe, in the space of the scale factor. Equation (21) for the phase $S_k(a)$ is non-linear, and contains the information about the curvature of space and quantum states of matter in the universe.

In a particular case of matter in the form of dust, this non-linear equation has the analytical solutions of two types: (i) real solution for the Euclidean phase $S_E = -iS_1$ (42) which corresponds to the real state vector $|f\rangle_1$ from (41); (ii) complex solution S_2 (43) for the state vector $|f\rangle_2$ from (41) in the space of complex scale factor.

The motion of the analogue particle as a mathematical equivalent of the evolving universe, described by the state vector $|f\rangle_2$, is characterized by two types of possible solutions for the real and imaginary parts of the kinetic energy (49) of types (+) and (−). The solution of type (+) describes the motion of the analogue particle in real time T, while the solution of type (−) corresponds to imaginary time $\xi = -iT$. The changes of the real and imaginary parts of the kinetic energy of one type during the evolution of the universe demonstrate that the internal forces simultaneously perform both the positive work on the universe (e.g., the energy K_R^+ increases as in Fig. 3), which is analogous to the work of the forces of dark energy, and the negative work (the energy K_I^+ decreases), which is similar to the work of the attractive forces of the dark matter. The general character of the expansion of the universe at a definite instant of time (parametrized by the deviation $z = a_R - M$ in Fig. 4) depends on which of the works dominates. The expansion of the universe becomes accelerating after reaching the region $a_R > 2M$. This result does not contradict the data on the expansion of our universe in the modern era and predicts that the forces of attraction and repulsion will exactly compensate each other in the infinite future ($a \to +\infty$).

In the approach under consideration, the change of the regimes of the expansion of the universe reflects a quantum nature of the universe. The equations of quantum theory (24) and (30) are transformed into the Einstein–Friedmann equations of general relativity (32) without dark energy in the limit $Q_k(a) \to 0$.

Thus, it appears that the quantum universe is such that, during its expansion, it decelerates, then accelerates, or *vice versa*, spontaneously. The cause of the expansion and the change of its regimes is a special form of the potential well (25), in which the universe is moving as a whole.

From Fig. 4, it follows that the main properties of the behavior of the deceleration parameter are invariable with the change of the quantum number n. This demonstrates that the model under consideration can explain the accelerating expansion (inflation) in the early universe (the domain of comparatively small values of quantum numbers) and the later transition from the decelerating expansion to the accelerating expansion of the universe (the domain of the very large values of quantum numbers) within a single approach. In both cases, a period of decelerating expansion is succeeded by a period of accelerating expansion. This change of the regime is caused by the behavior of the quantum correction P_Q to the total pressure (33) in the universe. The change of the pressure P_Q gives rise to additional fluctuations of the energy density $\rho_{\rm tot}$, which influence the formation of the large-scale structure in the universe.

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Received 18.11.14

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ВСЕСВІТ, ЩО РОЗШИРЮЄТЬСЯ:
ЗМІНА РЕЖИМУ

Р е з ю м е

Метою роботи є пояснення, на основі точних рівнянь квантової геометродинаміки для космологічної моделі з метрикою Робертсона–Вокера, можливої зміни режиму розширення всесвіту, з прискореного на уповільнений та навпаки. Показано, що зміна темпу розширення всесвіту може свідчити про наявність сил певного виду, що діють у всесвіті. Звертається увага на те, що природа цих сил є квантовою. Причиною розширення всесвіту та зміни його режиму слугує особлива форма ефективної потенціальної ями, в якій всесвіт рухається як ціле.