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# NUMERICAL SIMULATION OF RELAXATION OF QUANTUM THERMAL FLUCTUATIONS

*A generalization of quantum-mechanical equations expressed in the hydrodynamic form by introducing terms that involve the diffusion velocity at zero and finite temperatures, as well as the diffusion pressure energy in a warm vacuum, into the Lagrangian density has been proposed. It is used as a basis for constructing a system of equations similar to the Euler equations, but making allowance for quantum-mechanical and thermal effects, for the model of one-dimensional hydrodynamics. The equations obtained generalize the equations of the Nelson stochastic mechanics. A numerical analysis of the solutions of this system allowed a conclusion to be drawn about its validity for the description of the relaxation of quantum thermal fluctuations.*

*Keywords*: $(\hbar, k)$-dynamics, quantum thermostat, cold and warm vacua, effective action, self-diffusion, diffusion pressure energy density, drift and diffusion velocities, numerical analysis.

## 1. Introduction

Thermal fluctuations in hydrodynamics have been taken into account for half a century. However, there is no consistent quantum-mechanical statistical theory till now that would make allowance for quantum-mechanical and thermal effects simultaneously [1]. In this paper, we describe an approach to the development of such a theory, by using the hydrodynamic form of quantum mechanics as a starting point. With this aim in view, we propose to generalize the theory by considering the quantum-thermal diffusion existing at zero and finite Kelvin temperatures, which reflects the stochastic character of the environmental influence, and the diffusion pressure energy density. In this case, we will be be based on the concept of thermal equilibrium generalized to the case where the stochastic influence of both the quantum-mechanical and thermal types are taken into account simultaneously. As a result, a system of hydrodynamic equations is obtained in the case of a one-dimensional model, which is analogous to a system of Euler equations, but differs from it by making allowance for quantum-mechanical and thermal effects.

As a rule, the hydrodynamic equations are derived from either statistical mechanics or kinetics. In both cases, specific ideas concerning the medium structure and the interactions between its components are used. Accordingly, hydrodynamic fluctuations are accounted for by including a random stress tensor, together with a regular one, into the hydrodynamic equations. For the former tensor, only a correlator is given on the basis of fluctuation-dissipation theorem.

At the same time, hydrodynamics is conceptually similar to the equilibrium thermodynamics, since it

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is also a model-free theory in principle. Therefore, we propose to consider the theory describing the relaxation of quantum thermal fluctuations of the density and the drift velocity, under the equilibrium condition with respect to the temperature, as a stochastic hydrodynamics. Then, the derivation of corresponding equations can be started from a generalization of the hydrodynamic form of quantum mechanics at zero temperature as a model-free theory onto the case where the self-diffusion in cold and warm vacua is taken into consideration explicitly. This approach allows the hydrodynamic form of quantum mechanics to be extended, for the first time, to the case of finite temperatures and makes it possible to include not only the self-diffusion, but also the diffusion pressure of a warm vacuum into consideration.

As a result, in the framework of a one-dimensional model, we will obtain a system of stochastic hydrodynamic equations, which is valid at an arbitrary temperature. Its specific feature consists in that the quantum and thermal fluctuations are taken into account nonadditively. Moreover, those equations can be written down in the form inherent to equations of the two-velocity hydrodynamics, which is a generalization of the Nelson stochastic mechanics.

In this research, we use the results obtained earlier in work [2] as a basis. In the cited work, the theory of $(\hbar, k)$-dynamics was developed, which made it possible to introduce a consistent quantum thermal description of the thermal equilibrium state that differs from both the standard equilibrium thermodynamics and the quantum statistical mechanics (QSM).

The idea of $(\hbar, k)$-dynamics is based on changing from the classical thermostat model with the distribution modulus $\theta_{\rm cl} = k_{\rm B}T$ to the adequate quantum-mechanical one (the quantum thermostat or the "warm" vacuum, which is a set of normal modes with all frequencies $\omega$) with the distribution modulus $\theta_{\rm qu} = k_{\rm B}\mathbb{T}$. Here, the quantity

$$\mathbb{T} \equiv \frac{\hbar\omega}{2k_{\rm B}}\coth\frac{\hbar\omega}{2k_{\rm B}T} = \varkappa\coth\left(\varkappa\frac{\omega}{T}\right) \qquad (1)$$

is called the effective temperature, and the notation

$$\varkappa = \hbar/2k_{\rm B}$$

is used for brevity. The advantage of the parameter $\mathbb{T}$ in comparison with the absolute (Kelvin) temperature $T$ consists in that the former never becomes zero. This circumstance makes it possible to consider a contact of the system with the environment at $T \geqslant 0$ from a common position, which is important, when fluctuations of both quantum-mechanical and thermal origins occur simultaneously. This quantity is accepted as a generalized criterion of thermal equilibrium for an object in contact with the quantum thermostat.

The main distinction of the $(\hbar, k)$-dynamics from QSM consists in that the state of an object in equilibrium with a quantum thermostat is not described by a density matrix, but by a complex wave function $\psi(q, \omega)$, whose amplitude and phase depend on the temperature. In the coordinate representation, this wave function looks like

$$\psi(q,\omega) = \left[2\pi(\Delta q)^2\right]^{-1/4}\exp\left\{-\frac{q^2}{4(\Delta q)^2}(1 - i\alpha)\right\}, \qquad (2)$$

where $(\Delta q)^2$ is the coordinate dispersion, and $\alpha$ a parameter determining the phase.

In the framework of the $(\hbar, k)$-dynamics, we simultaneously introduce a new macroscopic parameter, the effective influence exerted by the quantum thermostat on the system, which is calculated as the average of the quantum thermal influence operator, $\mathbb{J} = \bar{\hat{\jmath}}$:

$$\mathbb{J} = \frac{\hbar}{2}\sqrt{\alpha^2+1} = \mathbb{J}^0\sqrt{\alpha^2+1}. \qquad (3)$$

Here, $\mathbb{J}^0 = \hbar/2$ is the limiting $\mathbb{J}$-value at the absolute temperature $T \to 0$, which corresponds to a purely quantum influence. In this case, the phase factor $\alpha$ vanishes, which corresponds to the case of a real-valued wave function $\psi$. In the general case, the temperature dependence of the effective influence is contained in the radicand in Eq. (3). Taking into account that $\varkappa = \hbar/(2k_{\rm B})$ and the phase parameter $\alpha$ in formula (3) satisfies the equality

$$\alpha^2 \equiv \sinh^{-2}\left(\varkappa\frac{\omega}{T}\right),$$

Eq. (3) for the effective influence $\mathbb{J}$ reads

$$\mathbb{J} = \frac{\hbar}{2}\coth\left(\varkappa\frac{\omega}{T}\right) = \frac{\hbar}{2}\coth\frac{\hbar\omega}{2k_{\rm B}T}. \qquad (4)$$

## 2. Effective Influence as a Universal Characteristic of Transport Processes. Self-Diffusion Coefficient

First of all, we note that, in accordance with the results of work [2], the most important thermodynamic parameters in the equilibrium state can be expressed in terms of the effective influence $\mathbb{J}$. In particular, these are the effective temperature $\mathbb{T}$,

$$\mathbb{T} = \frac{\omega}{k_\mathrm{B}}\mathbb{J}, \tag{5}$$

the effective internal energy $\mathbb{U}$,

$$\mathbb{U} = \omega\mathbb{J}, \tag{6}$$

and the effective entropy $\mathbb{S}$,

$$\mathbb{S} = -k_\mathrm{B}\left(1 + \ln 2\frac{\mathbb{J}}{\hbar}\right).$$

Introducing the limiting values of $\mathbb{S}$ and $\mathbb{J}$ as $T \to 0$, namely, $\mathbb{S}^0 = k_\mathrm{B}$ and $\mathbb{J}^0 = \frac{\hbar}{2}$, which correspond to a purely quantum influence, the relation between the effective influence and the effective entropy can be found:

$$\mathbb{J} = \mathbb{S}^0\left(1 + \ln\frac{\mathbb{J}}{\mathbb{J}^0}\right). \tag{7}$$

Therefore, it seems natural that the universal constant $\varkappa$ is physically defined as the limit of the ratio between two fundamental macroscopic quantities, the effective influence $\mathbb{J}$ and the effective entropy $\mathbb{S}$,

$$\varkappa = \frac{\hbar}{2k_\mathrm{B}} \equiv \lim_{T\to 0}\frac{\mathbb{J}}{\mathbb{S}} = \frac{\mathbb{J}^0}{\mathbb{S}^0}. \tag{8}$$

However, this is not the end. As was shown in work [7], the transport coefficients typical of nonequilibrium thermodynamics can be expressed in terms of the effective influence $\mathbb{J}$, which demonstrates their stochastic origin. For instance, this is evident for the self-diffusion process in a medium with a nonuniform density after the equilibrium with respect to the temperature has been established.

Really, as was shown in the theory of Brownian motion at rather high temperatures ($t \gg \tau$) [3], the uncertainty relation

$$(\Delta p)(\Delta q) = mD_T \tag{9}$$

is satisfied. Here, $D_T$ is the coefficient of thermally driven diffusion. In particular, for a free microparticle, $D_T = k_\mathrm{B}T\tau/m$, where $\tau$ is the relaxation time, whereas $D_T = k_\mathrm{B}T/m\omega$ for a Brownian oscillator [4]. As was shown in work [5], the momentum–coordinate Heisenberg uncertainty relation for a quantum oscillator in the state of equilibrium with a warm vacuum looks like

$$(\Delta p)\;(\Delta q) = \mathbb{J} = \frac{\hbar}{2}\coth\left(\varkappa\frac{\omega}{T}\right). \tag{10}$$

Comparing Eqs. (9) and (10), this relation can be rewritten in the form

$$(\Delta p)(\Delta q) = m\mathbb{D}. \tag{11}$$

Then the quantity

$$\mathbb{D} = \frac{\hbar}{2m}\coth\left(\varkappa\frac{\omega}{T}\right) \equiv \frac{\mathbb{J}}{m} \tag{12}$$

is natural to be called the effective self-diffusion coefficient. Note that the quantity $\hbar/2m$ in the case of contact with a cold vacuum – or, in other words, in the absence of a thermal influence from the environment – was earlier called by Nelson [6] as the quantum diffusion coefficient $\hbar/2m = D_\mathrm{qu}$.

From Eq. (12), it follows that the coefficient $\mathbb{D}$ acquires the physical meaning of the effective influence per mass unit. The limiting $\mathbb{D}$-values at high and low absolute temperatures equal

$$\begin{aligned} &\mathbb{D} \to D_T = \frac{k_\mathrm{B}T}{m\omega} \quad \text{at } k_\mathrm{B}T \gg \hbar\omega/2, \\ &\mathbb{D} \to D_\mathrm{qu} = \frac{\hbar}{2m} \quad \text{at } k_\mathrm{B}T \ll \hbar\omega/2. \end{aligned} \tag{13}$$

As was demonstrated in work [7], the effective influence $\mathbb{J}$ can be used to introduce, on the basis of relation (12), other effective transport coefficients as well, such as the coefficients of heat conductivity, shear viscosity, and so on, which characterize nonequilibrium processes. Hence, the majority of transport coefficients can be expressed in terms of the effective self-diffusion coefficient $\mathbb{D}$, which can be, in principle, measured experimentally.

Concerning the constant $\varkappa$, it can be expressed in terms of observed transport coefficients, while analyzing specific experiments. Namely, relations of the following type are used:

$$\varkappa = \left(\frac{\mathbb{D}}{\mathbb{S}/m}\right)_\mathrm{min} = \left(\frac{\eta_\mathrm{ef}}{\mathbb{S}/V}\right)_\mathrm{min} = \ldots, \tag{14}$$

where $\mathbb{S}/m$ is the effective entropy per mass unit, $\mathbb{S}/V$ the effective entropy per volume unit, and $\eta_\mathrm{ef}$ the effective coefficient of shear viscosity.

## 3. Standard Quantum Mechanics in the Hydrodynamic Form

The nonrelativistic field form of standard quantum mechanics (at $T = 0$) can be obtained by nullifying the variation of action functional [8]

$$\mathcal{S} = \int_{t_1}^{t_2} dt \int dq\, \mathcal{L}_0[\psi^*; \psi]. \tag{15}$$

Here, $\mathcal{L}_0[\psi^*; \psi]$ is the Lagrangian density for a spinless particle at $T = 0$, whereas $\psi(q, t)$ and $\psi^*(q, t)$ are the wave function and its complex conjugate, respectively, which have the meaning of independent nonrelativistic fields. Below, the consideration is restricted to the one-dimensional case.

It is evident that, in the general case, the functional $\mathcal{L}_0[\psi^*; \psi]$ has to be chosen in the form

$$\mathcal{L}_0[\psi^*; \psi] = \psi^*(q, t) \left( i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} \right) \psi(q, t) - \\ - \psi^*(q, t) U(q) \psi(q, t), \tag{16}$$

where the parentheses on the right-hand side contain the Klein–Gordon operator in the nonrelativistic limit, and the potential energy operator $U(q)$ characterizes the energy of regular influence. The independent variation of the action in form (16) with respect to the field $\psi^*$ brings about the condition

$$\int_{t_1}^{t_2} dt \int dq \frac{\delta \mathcal{L}_0[\psi^*; \psi]}{\delta \psi} = \\ = \int_{t_1}^{t_2} dt \int dq \left( i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial q^2} - U(q)\psi \right) = 0 \tag{17}$$

that gives rise to the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial q^2} + U(q)\psi. \tag{18}$$

Accordingly, its complex conjugate is obtained by varying the action in form (16) with respect to $\psi$. The result differs from formula (18) by the substitutions of $-i$ for $i$ and $\psi^*$ for $\psi$. We would like to emphasize that the Schrödinger equations for the complex wave functions $\psi$ and $\psi^*$ have the sense of Euler–Lagrange equations, with the wave functions being always complex-valued in the full-scale quantum mechanics.

Let us written the wave function in the form

$$\psi(q, t) = \sqrt{\rho(q, t)} \exp\{i\theta(q, t)\}, \tag{19}$$

where $\rho(q, t) = |\psi(q, t)|^2$. This expression, as well as its complex conjugate, can be substituted directly into the Schrödinger equations for $\psi$ and $\psi^*$ to obtain a system of equations for the functions $\rho(q, t)$ and $\theta(q, t)$, which has been known long ago in the literature as quantum mechanics in the hydrodynamic form [8, 9]. Since we aim at constructing a modified hydrodynamics on the basis of the microscopic description, we propose another approach to the problem. It dictates to develop the theory in the framework of the Lagrange formulation from the very beginning. Therefore, let us begin from the transformation of the Lagrangian density $\mathcal{L}_0$ to variables that are the most suitable for the hydrodynamic description. As the functional arguments of the Lagrangian density, we select two independent real-valued functions, the probability density $\rho$ and the phase $\theta$, rather than the complex wave functions $\psi$ and $\psi^*$. In essence, they are close to the mass density $\rho_m$ and the drift velocity $v \sim \frac{\partial \theta}{\partial q}$, which are typical of standard hydrodynamics.

With this purpose in view, let us change the arguments in the Lagrangian density (16) by substituting expression (19) for $\psi$ and the corresponding expression for $\psi^*$. Then we obtain

$$\mathcal{L}_0[\psi; \psi^*] = \mathcal{L}_0[\rho; \theta] = \\ = -\hbar \frac{\partial \theta}{\partial t} \rho - \frac{\hbar^2}{2m} \left( \frac{\partial \theta}{\partial q} \right)^2 \rho - \frac{\hbar^2}{8m} \left( \frac{\partial \rho}{\partial q} \right)^2 \frac{1}{\rho} - \\ - U(q)\rho + i \frac{\hbar}{2} \frac{\partial \rho}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial}{\partial q} \left( \frac{1}{2} \frac{\partial \rho}{\partial q} + i\rho \frac{\partial \theta}{\partial q} \right). \tag{20}$$

Here, the term containing $\frac{\partial \rho}{\partial t}$ can be neglected, because it gives a zero contribution, when the action $\mathcal{S}$ in form (15) is varied with respect to $\theta$ or $\rho$. The last term in Eq. (20) is the total derivative with respect to $q$, so that it can also be excluded from the definition of $\mathcal{L}_0[\rho; \theta]$. Therefore, the ultimate expression for the Lagrangian density $\mathcal{L}_0[\rho; \theta]$ looks like

$$\mathcal{L}_0[\rho; \theta] = -\hbar \frac{\partial \theta}{\partial t} \rho - \frac{\hbar^2}{2m} \left( \frac{\partial \theta}{\partial q} \right)^2 \rho - \\ - \frac{\hbar^2}{8m} \left( \frac{\partial \rho}{\partial q} \right)^2 \frac{1}{\rho} - U(q)\rho. \tag{21}$$

Varying the action $\mathbb{S}$ written in form (15), in which $\mathcal{L}_0[\rho; \theta]$ is taken from Eq. (21), with respect to the variables $\theta$ and $\rho$, we obtain the following equations for the real functions $\rho(q,t)$ and $\theta(q,t)$:

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial q}\left(\rho\frac{\hbar}{m}\frac{\partial\theta}{\partial q}\right)=0, \tag{22}$$

$$\hbar\frac{\partial\theta}{\partial t}+\frac{\hbar^2}{2m}\left(\frac{\partial\theta}{\partial q}\right)^2+U(q)-$$
$$-\frac{\hbar^2}{8m}\left[\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial q}\right)^2+2\frac{\partial}{\partial q}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial q}\right)\right]=0. \tag{23}$$

They coincide with the equations that could be obtained for the functions $\rho$ and $\theta$ directly from the Schrödinger equations. However, now it is clear that they have a sense of the Lagrange–Euler equations for the action $\mathbb{S}$ of type (15) written with the use of the variables $\rho$ and $\theta$.

Equation (22) is conventionally assumed to be a continuity equation for the function $\rho(q,t)$. At the same time, taking into account that the quantity $\hbar\theta(q,t)$ has the dimensionality of action, Eq. (23) is considered as an analog of the Hamilton–Jacobi equation. In so doing, the term in the brackets in formula (23) is sometimes interpreted as an additional energy of the quantum origin, $U_{\rm qu}(q)$, which vanishes in the semiclassical limit ($\hbar\to 0$).

Surely, Eqs. (22) and (23) for $\rho$ and $\theta$, on the one hand, and the Schrödinger equations for $\psi$ and $\psi^*$, on the other hand, are formally equivalent. However, the derivation of the quantum mechanical equations in the hydrodynamic form (Eqs. (22) and (23)) immediately from the least-action principle is physically more preferable from the viewpoint of constructing the stochastic hydrodynamics. At the same time, in order to obtain the sought result, we must solve the problem of the Lagrangian density form. In our opinion, this form has to involve consistently the stochastic influence of the environment (the quantum thermostat).

## 4. Quantum Self-Diffusion in a "Cold" Vacuum

In order to reveal the capabilities of a required generalization for $\mathcal{L}_0[\rho;\theta]$, let us first consider the case of cold vacuum. For this purpose, let us determine the physical meaning of the second and third terms on the right-hand side of expression (21). In accordance with the terminology introduced by Kolmogorov for Markovian processes in the general theory of stochastic processes [10] and used by Nelson in his stochastic mechanics [6], the quantity

$$v\equiv\frac{\hbar}{m}\frac{\partial\theta}{\partial q} \tag{24}$$

will be called the drift velocity. Accordingly, the quantity

$$u\equiv -D_{\rm qu}\frac{1}{\rho}\frac{\partial\rho}{\partial q}=-\frac{\hbar}{2m}\frac{1}{\rho}\frac{\partial\rho}{\partial q} \tag{25}$$

will be called the diffusion velocity in a cold vacuum. We would like to emphasize its stochastic quantum origin.

In terms of the velocities $v$ and $u$, formulas (21)–(23) read

$$\mathcal{L}_0[\rho,\theta]=-\hbar\frac{\partial\theta}{\partial t}\rho-\frac{m}{2}(v^2+u^2)\rho-U\rho, \tag{21a}$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial q}(\rho v)=0, \tag{22a}$$

$$\hbar\frac{\partial\theta}{\partial t}+\frac{m}{2}v^2+U-\frac{m}{2}\left[u^2-\frac{\hbar}{m}\frac{\partial u}{\partial q}\right]=0, \tag{23a}$$

which makes the generalization possible. From formula (22a), it follows that the standard continuity equation (22) has a semiclassical character, because the corresponding probability flux density depends only on the drift velocity $v$, whereas the diffusion velocity $u$ generated by the stochastic influence of a cold vacuum is not taken into account.

In this connection, we recall that the Fokker–Planck equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial q}(\rho V)=0 \tag{26}$$

including the total velocity of the probability flux density

$$V=v+u \tag{27}$$

is, according to Kolmogorov [10], the most general form of a continuity equation. We will demonstrate that this equation makes it possible to describe the approach to the thermal equilibrium state by means of the self-diffusion, with the case of cold vacuum including.

Attention should be focused on the fact that the combination $\frac{m}{2}(v^2+u^2)$ entering expression (21a) for $\mathcal{L}_0[\rho,\theta]$ is a sum of independent contributions made by the kinetic energies of drift and diffusion motions. At the same time, the probability flux depends on the total velocity (27). Hence, the natural substitution of $v^2 + u^2$ by $V^2$ suggests itself in expression (21a) in order to obtain the Fokker–Planck equation. As a result, the total expression for the kinetic energy associated with the probability flux will be taken into consideration. Thus, even the standard quantum mechanics (at $T = 0$) can be generalized.

Now, let us generalize the Lagrangian density $\mathcal{L}_0[\rho,\theta]$ in form (21a) by carrying out the corresponding substitution. Then we obtain

$$\tilde{\mathcal{L}}_0[\rho,\theta] = -\hbar\,\frac{\partial\theta}{\partial t}\,\rho - \frac{m}{2}V^2\rho - U\rho =$$
$$= \mathcal{L}_0[\rho;\theta] - mvu\rho = \mathcal{L}_0[\rho;\theta] + \frac{\hbar^2}{2m}\frac{\partial\theta}{\partial q}\frac{\partial\rho}{\partial q}. \tag{28}$$

The variation of the action functional $\mathcal{S}$ in form (15) with $\tilde{\mathcal{L}}_0[\rho,\theta]$ with respect to $\theta$ automatically brings us to the Fokker–Planck equation with the quantum diffusion coefficient $D_{\mathrm{qu}}$,

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial q}(\rho V) = \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial q}\left(\rho\frac{\hbar}{m}\frac{\partial\theta}{\partial q}\right) - D_{\mathrm{qu}}\frac{\partial^2\rho}{\partial q^2} = 0. \tag{29}$$

At the same time, the variation of $S$ with respect to $\rho$ almost does not change the Hamilton–Jacobi equation, in which there appears an additional, in comparison with Eq. (23a), insignificant term $\frac{\hbar}{2}\frac{\partial v}{\partial q}$. As a result, the analog of Eq. (23a) looks like

$$\hbar\frac{\partial\theta}{\partial t} + \frac{m}{2}v^2 + U - \frac{m}{2}\left(u^2 - \frac{\hbar}{m}\frac{\partial u}{\partial q}\right) + \frac{\hbar}{2}\frac{\partial v}{\partial q} = 0. \tag{30}$$

The obtained equations (29) and (30) generalize Eqs. (22a) and (23a) and allow the quantum stochastic influence of a cold vacuum to be consistently taken into account.

## 5. Self-Diffusion in the Quantum Thermostat at $T = 0$

Let us apply the approach developed above to the description of the self-diffusion, when the quantum and thermal effects are taken into account simultaneously. For this purpose, let us introduce the temperature-dependent Lagrangian density $\tilde{\mathcal{L}}_0[\rho,\theta]$ and require that it should transform into the expression for $\tilde{\mathcal{L}}_0[\rho,\theta]$ of type (28) obtained as $T \to 0$. To satisfy this condition, it is enough to substitute the diffusion coefficient $D_{\mathrm{qu}}$ in expression (28) for the diffusion velocity by $\mathbb{D}$ in form (8) and to introduce an additional term $U_T(q)\rho$, which makes allowance for the density of the diffusion pressure energy as a result of the thermal stochastic influence of the environment, into the expression for the Lagrangian density.

In our opinion, the expression for $U_T$ has to possess a form that is analogous to the factor $-mu^2/2$ in expression (21a) for a cold vacuum. However, it must be modified so that $U_T \to 0$ as $T \to 0$. The corresponding expression looks like

$$U_T(q) = -\frac{m}{2}\left[\frac{\alpha}{\Upsilon}\right]^2 u_{\mathrm{ef}}^2 = -\frac{\hbar^2}{8m}\,\alpha^2\left(\frac{1}{\rho}\frac{\partial\rho}{\partial q}\right)^2, \tag{31}$$

where we use the notation

$$\alpha^2 \equiv \sinh^{-2}\varkappa\frac{\omega}{T}, \quad \Upsilon = \coth\left(\varkappa\frac{\omega}{T}\right),$$

and

$$u_{\mathrm{ef}} \equiv -\mathbb{D}\frac{1}{\rho}\frac{\partial\rho}{\partial q}$$

is the effective diffusion velocity in a warm vacuum. This quantity is defined analogously to the velocity $u$ in Eq. (25), but now it is expressed in terms of the effective diffusion coefficient $\mathbb{D}$ of type (12). Hence, as the Lagrangian density at $T = 0$, we choose the expression

$$\tilde{\mathcal{L}}_T(\rho,\theta) = -\hbar\frac{\partial\theta}{\partial t}\rho - \frac{m}{2}(v + u_{\mathrm{ef}})^2\rho - U\rho - U_T\rho. \tag{32}$$

For the variation operation to be convenient, let us rewrite expression (32) in terms of the random functions $\theta$ and $\rho$ in the explicit form:

$$\tilde{\mathcal{L}}_T(\rho,\theta) = -\hbar\frac{\partial\theta}{\partial t}\rho -$$
$$-\left\{\frac{\hbar^2}{2m}\left(\frac{\partial\theta}{\partial q}\right)^2\rho - \frac{\hbar^2}{2m}\Upsilon\frac{\partial\theta}{\partial q}\frac{\partial\rho}{\partial q} + \frac{\hbar^2}{8m}\Upsilon^2\frac{1}{\rho}\left(\frac{\partial\rho}{\partial q}\right)^2\right\} -$$
$$-U\rho - \frac{\hbar^2}{8m}\alpha^2\frac{1}{\rho}\left(\frac{\partial\rho}{\partial q}\right)^2. \tag{33}$$

Varying the action $\mathcal{S}$ with $\widetilde{\mathcal{L}}_T$ written in form (33) with respect to $\theta$ brings us again to the Fokker–Planck equation, which is similar to Eq. (29) with an accuracy to the substitution of $D_{\mathrm{qu}}$ by the effective diffusion coefficient $\mathbb{D}_{\mathrm{ef}}$,

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial q}\left(\rho\frac{\hbar}{m}\frac{\partial\theta}{\partial q}\right)-\mathbb{D}\frac{\partial^2\rho}{\partial q^2}=0. \tag{34}$$

Analogously, varying $\mathcal{S}$ with respect to $\rho$ leads to the Hamilton–Jacobi equation generalized to the case of the stochastic influence of a warm vacuum,

$$\hbar\frac{\partial\theta}{\partial t}+\frac{\hbar^2}{2m}\left(\frac{\partial\theta}{\partial q}\right)^2+\frac{\hbar^2}{2m}\Upsilon\,\frac{\partial^2\theta}{\partial q^2}+U(q)-$$
$$-\frac{\hbar^2}{8m}\Xi_T\left[\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial q}\right)^2+2\frac{\partial}{\partial q}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial q}\right)\right]=0, \tag{35}$$

where the notation

$$\Xi_T=2\Upsilon^2-1=2\coth^2\left(\varkappa\frac{\omega}{T}\right)-1 \tag{36}$$

was introduced for convenience. Note that $\Xi_T(T==0)\equiv\Xi_0=1$.

The obtained equations (34) and (35), in turn, generalize Eqs. (29) and (30) and make it possible to consistently take the stochastic influence of a warm vacuum into account. The latter is indirectly contained in the quantities $\mathbb{D}$, $\Xi_T$, and $\Upsilon$ entering those equations and depending on the world constants $\hbar$ and $k_{\mathrm{B}}$. From the physical viewpoint, this means that both types of stochastic environmental influence are taken into account simultaneously: the quantum-mechanical one characterized by Planck's constant $\hbar$ and the thermal one characterized by the Boltzmann constant $k_{\mathrm{B}}$.

Certainly, the system of the Fokker–Planck equation (34) and the Hamilton–Jacobi one (35) is a nontrivial generalization of the Schrödinger equation. There are two ways of their further application. First, one can directly solve this system to find the functions $\rho$ and $\theta$ of different kinds. As was showed by us recently [12], this procedure makes it possible to calculate nonequilibrium wave functions with temperature-dependent amplitudes and phases; then they can be used to calculate macroparameters in nonequilibrium states. On the other hand, those equations can also be modified by transforming them to the form inherent to the equations of the two-velocity stochastic hydrodynamics for the characteristic velocities $v$ and $u$.

## 6. One-Dimensional Model of Two-Velocity Stochastic Hydrodynamics

To modify the system of equations (34) and (35), let us express them in the form of equations depending on the variables of the same kind, the velocities $v$ and $u_{\mathrm{ef}}$, which are typical of any Markovian process. As a result, we obtain a system of equations of two-velocity stochastic hydrodynamics, which generalizes the equations of the Nelson stochastic mechanics to the case of quantum thermal influence from the environment.

Now, let us demonstrate that Eqs. (34) and (35) allow the equations of stochastic hydrodynamics to be obtained in the most convenient form. Taking the aforesaid into account, the matter concerns only the one-dimensional model. In order to perform the corresponding transformation, let us first transform the continuity equation (34) into an equation for the diffusion velocity. The latter, in accordance with Eq. (25), can be written in the form

$$u_{\mathrm{ef}}=-\mathbb{D}\frac{\partial\ln\rho}{\partial q}.$$

First, let us transform Eq. (34) by explicitly introducing $v$ and $u_{\mathrm{ef}}$ into it and multiply the result by $-\mathbb{D}/\rho$:

$$-\frac{\mathbb{D}}{\rho}\frac{\partial\rho}{\partial t}-\frac{\mathbb{D}}{\rho}\left[\rho\frac{\partial(v+u_{\mathrm{ef}})}{\partial q}+\frac{\partial\rho}{\partial q}(v+u_{\mathrm{ef}})\right]=0. \tag{37}$$

Now, we should differentiate Eq. (37) with respect to $q$, change the order of differentiation in the first term, take into account that $\frac{1}{\rho}\frac{\partial\rho}{\partial q}=\frac{\partial}{\partial\rho}\ln\rho$, and introduce $u_{\mathrm{ef}}$ everywhere. As a result, we obtain

$$\frac{\partial u_{\mathrm{ef}}}{\partial t}+\frac{\partial}{\partial q}(vu_{\mathrm{ef}})+\frac{\partial}{\partial q}u_{\mathrm{ef}}^2-\mathbb{D}\frac{\partial^2}{\partial q^2}(v+u_{\mathrm{ef}})=0. \tag{38}$$

This equation can be expressed in a more elegant form by using the substantial derivative of the diffusion velocity $u_{\mathrm{ef}}$, which is typical of hydrodynamics:

$$\frac{du_{\mathrm{ef}}}{dt}\equiv\frac{\partial u_{\mathrm{ef}}}{\partial t}+u_{\mathrm{ef}}\frac{\partial u_{\mathrm{ef}}}{\partial q}=$$
$$=-\frac{\partial}{\partial q}(vu_{\mathrm{ef}})-\frac{\partial}{\partial q}\frac{u_{\mathrm{ef}}^2}{2}+\mathbb{D}_{\mathrm{ef}}\frac{\partial^2}{\partial q^2}(v+u_{\mathrm{ef}}). \tag{39}$$

In order to express Eq. (35 in the explicit hydrodynamic form, let us also rewrite it in terms of the variables $v$ and $u_{\mathrm{ef}}$,

$$\hbar\frac{\partial\theta}{\partial t}+\frac{m}{2}v^2+\frac{\hbar}{2}\Upsilon\frac{\partial v}{\partial q}+U(q)-$$
$$-\frac{m}{2}\Xi_T\left(u_{\mathrm{ef}}^2-\frac{\hbar}{m}\frac{\partial u_{\mathrm{ef}}}{\partial q}\right)=0. \tag{40}$$

To exclude the function $\theta$, we can differentiate Eq. (40) with respect to $q$, change the order of differentiation in the first term, and explicitly introduce the function $v$ in it, by following Eq. (24). After forming the substantial derivative of the drift velocity $v$, we obtain

$$\frac{dv}{dt} \equiv \left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial q}\right) = -\frac{1}{m}\frac{\partial U}{\partial q} + \\ + \Xi_T \frac{\partial}{\partial q}\frac{u_{\mathrm{ef}}^2}{2} - \frac{\hbar}{2m}\left(\Xi_T \frac{\partial^2 u_{\mathrm{ef}}}{\partial q^2} + \Upsilon \frac{\partial v^2}{\partial q^2}\right). \tag{41}$$

Recall that the role of velocity $v$ in those equations is played only by the quantity $\Delta v$ generated by the stochastic influence. In the case concerned, as well as at $T = 0$, the expressions for $\rho$ and $\theta$ are related to the wave functions of thermal correlated-coherent states, in which the argument of the exponential function depends on $q^2$. Whence it follows that the last terms in Eqs. (39) and (41) with the second derivatives of $v \equiv \Delta v$ and $u_{\mathrm{ef}}$ with respect to $q$ equal zero.

As a result, the system of equations for the one-dimensional model of two-velocity stochastic hydrodynamics has the following general form:

$$\frac{du_{\mathrm{ef}}}{dt} \equiv \frac{\partial u_{\mathrm{ef}}}{\partial t} + \frac{\partial}{\partial q}\frac{u_{\mathrm{ef}}^2}{2} = -\frac{\partial}{\partial q}(vu_{\mathrm{ef}}) - \frac{\partial}{\partial q}\frac{u_{\mathrm{ef}}^2}{2}, \tag{42a}$$

$$\frac{dv}{dt} = -\frac{1}{m}\frac{\partial U}{\partial q} + \Xi_T \frac{\partial}{\partial q}\frac{u_{\mathrm{ef}}^2}{2}. \tag{42b}$$

Note that, in the general case, the proposed equations (42) are valid for any temperature. Each equation of this system makes allowance for the self-diffusion in a warm vacuum characterized by the coefficient $D_{\mathrm{ef}}$ in $u_{\mathrm{ef}}$. In addition, the right-hand side of Eq. (42b) includes the gradient of the classical potential $U(q)$ and the gradient of the diffusion pressure energy density. The latter is associated with the stochastic influence of a quantum thermostat, actual at $T = 0$ as well. An analogous contribution of the diffusion pressure energy of the quantum thermostat is also contained in the right-hand side of Eq. (42a). This contribution does not vanish even if $v = 0$.

In order to compare the system obtained with the equations of the Nelson stochastic mechanics,

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial q}(vu), \tag{43a}$$

$$\frac{dv}{dt} = -\frac{1}{m}\frac{\partial U}{\partial q} + \frac{\partial}{\partial q}\frac{u^2}{2}, \tag{43b}$$

which are valid only at $T = 0$, let us consider the system of equations (42) in the case of cold vacuum ($T = 0$). Then $u_{\mathrm{ef}} = u$. In addition, to make the comparison convenient, we restore the partial derivative with respect to the time in Eq. (42a); for this purpose, we combine similar terms in formula (42). As a result, we obtain

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial q}(vu) - \frac{\partial}{\partial q}u^2, \tag{44a}$$

$$\frac{dv}{dt} = -\frac{1}{m}\frac{\partial U}{\partial q} + \frac{\partial}{\partial q}\frac{u^2}{2}. \tag{44b}$$

One can see that Eqs. (43b) and (44b) are identical. However, the corresponding systems of equations differ substantially from each other. It becomes noticeable, when comparing Eqs. (43a) and (44a). The difference is expectedly associated with the fact that our theory takes into account the self-diffusion, which occurs even in a cold vacuum. As a consequence, the equation for the diffusion velocity includes the gradient of the diffusion pressure energy density.

Below, we compare the solutions of the systems of equations (43) and (44) in order to establish important physical distinctions between them that occur at $T = 0$. Actually, it is of interest to elucidate how the account for the self-diffusion in Eqs. (44) in the case of cold vacuum (at $T = 0$) affects the form of obtained solutions in comparison with those for the system of equations (43).

## 7. Analysis of the Solutions of Eqs. (43) and (44)

Note that, as was shown in Sec. 6, the system of equations (42) was derived taking the self-diffusion at an arbitrary temperature into account. At the same time, the system of equations (44) is valid only at $T = 0$. Therefore, a correct comparison can be done only between the system of equations (43), which is a particular case of system of equations (42) at $T = 0$, and the system of equations (44).

### *7.1. Method of determination of the class of an equation*

Note that the total derivative of the velocity equals

$$\frac{dv}{dt} = \frac{\partial v}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial v}{\partial t} = v\frac{\partial v}{\partial q} + \frac{\partial v}{\partial t}. \tag{45}$$

Let us rewrite Eqs. (43) and (44) in terms of partial derivatives. For the sake of convenience, we introduce the notation $\frac{1}{m}\frac{\partial U}{\partial q} = \alpha(q)$. Then the systems of equations (43) and (44) look like

$$\begin{aligned} &\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial q} + u\frac{\partial v}{\partial q} = 0, \\ &\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial q} - u\frac{\partial u}{\partial q} = \alpha(q), \end{aligned} \tag{46}$$

and

$$\begin{aligned} &\frac{\partial u}{\partial t} + (v+2u)\frac{\partial u}{\partial q} + u\frac{\partial v}{\partial q} = 0, \\ &\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial q} - u\frac{\partial u}{\partial q} = \alpha(q), \end{aligned} \tag{47}$$

respectively, where $u$ is the diffusion velocity, and $v$ the drift one. The main difference between systems (46) and (47) consists in that, when deriving the latter, the assumption about the self-diffusion was used.

In this work, we analyze only the homogeneous systems of equations (46) and (47), so that we put $\alpha(q) = 0$. The most important issue that determines the procedure of solution of those equations is the establishment of the type of an equation: elliptic, hyperbolic, or parabolic one. It is known that, while solving hyperbolic equations, the concept of a characteristic, i.e. an integral of a certain characteristic equation, is used. The elliptic operator has no characteristics in the real region, and, generally speaking, elliptic differential equations describe stationary equilibrium states in physics. Hence, the establishment of the class, which the corresponding system belongs to, allows a conclusion about the character of solutions of the equation to be drawn and a certain physical interpretation for them to be given.

Proceeding from the aforesaid, let us begin our analysis. Equations (46) and (47) compose a system of quasilinear partial differential equations of the first order for two unknown functions $u(t,q)$ and $v(t,q)$. Therefore, following [14], let us express each of the equations in systems (46) and (47) in the form

$$\begin{aligned} L_1 = A_1u_t + B_1u_q + C_1v_t + D_1v_q, \\ L_2 = A_2u_t + B_2u_q + C_2v_t + D_2v_q, \end{aligned} \tag{48}$$

where $A_i$, $B_i$, $C_i$, and $D_i$ $(i = 1, 2)$ are known functions of the variables $t$, $q$, $u$, and $v$. Let all the functions under consideration be continuous and possess continuous derivatives of required orders. It is well known that the linear combination $af_x + bf_y$ of the partial derivatives of a function of two variables, $f(x,y)$, is a derivative in the direction determined by the relation $\frac{dx}{dy} = \frac{a}{b}$. If the functions $x(l)$ and $y(l)$ parametrize a certain curve and if $\frac{x_l}{y_l} = \frac{a}{b}$, the combination $af_x + bf_y$ is a derivative of the function $f(x,y)$ along this curve. This fact allows us to change from the analysis of the systems of partial differential equations (46) and (47) to the analysis of algebraic equations.

Consider such functions $u(t,q)$ and $v(t,q)$, for which the coefficients in the differential equations (48) depend only on $t$ and $q$. Let us find a linear combination

$$L = \lambda_1L_1 + \lambda_2L_2 \tag{49}$$

such that its differential would contain derivatives only along a single direction. Such a direction depending on the point $(t,q)$, as well as on the values of functions $u(t,q)$ and $v(t,q)$ at this point, is called a characteristic. Let this direction be given by the ratio $t_l : x_l$. Then, as was mentioned above, the condition that the functions $u(t,q)$ and $v(t,q)$ in the differential expression $L$ are differentiated in the same direction looks like

$$\frac{\lambda_1A_1 + \lambda_2A_2}{\lambda_1B_1 + \lambda_2B_2} = \frac{\lambda_1C_1 + \lambda_2C_2}{\lambda_1D_1 + \lambda_2D_2} = \frac{t_l}{x_l}, \tag{50}$$

since the coefficients of the $u_t$, $u_x$, $v_t$,, and $v_x$ derivatives in the expression $L$ are determined by the corresponding terms in proportions (50). Multiplying expression (49) by $t_l$, we obtain

$$\begin{aligned} &Lt_l = (\lambda_1A_1 + \lambda_2A_2)u_tt_l + (\lambda_1B_1 + \lambda_2B_2)u_qt_l + \\ &+ (\lambda_1C_1 + \lambda_2C_2)v_tt_l + (\lambda_1D_1 + \lambda_2D_2)v_qt_l = \\ &= (\lambda_1A_1 + \lambda_2A_2)(u_l - u_qq_l) + (\lambda_1B_1 + \lambda_2B_2)u_qt_l + \\ &+ (\lambda_1C_1 + \lambda_1C_2)(v_l - v_qq_l) + (\lambda_1D_1 + \lambda_2D_2)v_qt_l = \\ &= (\lambda_1A_1 + \lambda_2A_2)u_l + (\lambda_1C_1 + \lambda_2C_2)v_l - \\ &- [(\lambda_1A_1 + \lambda_2A_2)q_l - (\lambda_1B_1 + \lambda_2B_2)t_l]u_q - \\ &- [(\lambda_1C_1 + \lambda_2C_2)q_l - (\lambda_1D_1 + \lambda_2D_2)t_l]v_q = \\ &= (\lambda_1A_1 + \lambda_2A_2)u_l + (\lambda_1C_1 + \lambda_2C_2)v_l, \end{aligned} \tag{51}$$

because, owing to Eq. (48),

$$\begin{aligned} &(\lambda_1A_1 + \lambda_2A_2)q_l - (\lambda_1B_1 + \lambda_2B_2)t_l = \\ &= (\lambda_1C_1 + \lambda_2C_2)q_l - (\lambda_1D_1 + \lambda_2D_2)t_l = 0. \end{aligned} \tag{52}$$

Analogously, when multiplying $L$ by $q_l$, we obtain

$$Lq_l = (\lambda_1 B_1 + \lambda_2 B_2)u_l + (\lambda_1 D_1 + \lambda_2 D_2)v_l. \tag{53}$$

If the functions $u(t,q)$ and $v(t,q)$ are solutions of system (48), and if the expression $L$ has a derivative in the direction $l$ determined by the ratio $t_l : q_l$, Eq. (50) yields easily a system of two linear homogeneous algebraic equations for the parameters $\lambda_1$ and $\lambda_2$,

$$\begin{aligned}&\lambda_1(A_1 q_l - B_1 t_l) + \lambda_2(A_2 q_l - B_2 t_l) = 0,\\ &\lambda_1(C_1 q_l - D_1 t_l) + \lambda_2(C_2 q_l - D_2 t_l) = 0.\end{aligned} \tag{54}$$

System (54) has a nontrivial solution if its determinant equals zero, i.e.

$$\begin{vmatrix} A_1 q_l - B_1 t_l & A_2 q_l - B_2 t_l \\ C_1 q_l - D_1 t_l & C_2 q_l - D_2 t_l \end{vmatrix} = 0, \tag{55}$$

which is convenient to be written as a quadratic form

$$a t_l^2 - 2b t_l q_l + c q_l^2 = 0, \tag{56}$$

where $a = [BD]$, $2b = [AD] + [BC]$, $c = [AC]$, and $[XY] = X_1 Y_2 - X_2 Y_1$.

Depending on the determinant sign of form (56), the equations can be classed as follows.

1. If $b^2 - ac < 0$, the quadratic form (566) differs from zero at any real $t_l$ and $q_l$. Consequently, there is no real characteristic direction, and the system of differential equations belongs to the elliptic type.

2. If $b^2 - ac > 0$, there are two characteristic directions at every point. The directions are given by the ratio $t_l : q_l$ and correspond to two different roots $\lambda_1$ and $\lambda_2$ of the quadratic form (56). In this case, the system of differential equations belongs to the hyperbolic type.

3. In the case $b^2 - ac = 0$, expression (56) has a single root of multiplicity two. There is a degenerate direction corresponding to this root. The system of differential equations belongs to the parabolic type.

### *7.2. Analysis of Eqs. (46) and (47)*

On the basis of aforesaid, let us analyze the system of Nelson equations (46). The corresponding coefficients are

$$\begin{aligned}&A_1 = 1,\ B_1 = v,\ C_1 = 0,\ D_1 = u,\\ &A_2 = 0,\ B_2 = -u,\ C_2 = 1,\ D_2 = v.\end{aligned} \tag{57}$$

From whence, it follows that

$$b^2 - ac = -u^2 < 0.$$

Therefore, the hydrodynamic system of Nelson equations is elliptic and cannot be used to analyze the evolution of fluctuations in the framework of a quantum-mechanical description of the system.

For system (47), the coefficient $B_1$ is different, and

$$\begin{aligned}&A_1 = 1,\ B_1 = 2u + v,\ C_1 = 0,\ D_1 = u,\\ &A_2 = 0,\ B_2 = -u,\ C_2 = 1,\ D_2 = v.\end{aligned} \tag{58}$$

As a result,

$$b^2 - ac = 0,$$

which testifies that the system of equations (47) is parabolic. In other words, the system is of the evolution type and therefore can be used to describe the evolution of perturbations emerging at fluctuations.

Let us determine the characteristic direction for system (47). Following to what was said above, we compose a linear combination of two equations from this system and require that it should contain derivatives of the functions $u(q,t)$ and $v(q,t)$ only in a single direction $(u_l, v_l)$ given by $(t_l, q_l)$:

$$L = u_t + (2u + v - \lambda u)u_q + \lambda v_t (u + \lambda v)v_q = 0. \tag{59}$$

Then, according to Eq. (50), the conditions determining the direction $(t_l, q_l)$ look like

$$\begin{aligned}&q_l = (2u + v - \lambda u)t_l,\\ &\lambda q_l = (u + \lambda v)t_l.\end{aligned} \tag{60}$$

From whence,

$$(\lambda - 1)^2 = 0. \tag{61}$$

This means that there is one characteristic direction in problem (47), and it is determined by the condition

$$q_l = (u + v)t_l. \tag{62}$$

From Eqs. (60)–(62), we obtain the following characteristic equation for $u$ and $v$:

$$u_l + v_l = 0. \tag{63}$$

The meaning of Eq. (62) consists in that the characteristic in the $(q,t)$-plane represents the motion of probable perturbations with a velocity that is a sum of the drift and diffusion components:

$$\frac{dq}{dt} = u + v. \tag{64}$$

While applying this system, the following circumstance should be born in mind. The fluctuations of macroscopic parameters, e.g., the temperature, density, or pressure, inevitably perturb the variables $u$ and $v$ in hydrodynamic equations. The evolution of those perturbations can be described and examined, by using the system of equations (47).

### *7.3. Numerical simulation of solutions of Eqs. (46) and (47)*

Our considerations presented above will be illustrated by numerically simulating the solutions of systems (46) and (47). A model equation for those systems is the transport equation written in the vector form

$$\frac{\partial y}{\partial t} + A(y)\frac{\partial y}{\partial q} = f, \tag{65}$$

where $y = (u, v)^{\mathrm{T}}$, $A \in \mathbb{R}^{2\times 2}$ is the matrix of the system, and $f \in \mathbb{R}^2$. Let us consider the Cauchy problem on the real axis for systems (46) and (47):

$$y(q,0) = y_0(q),$$
$$q \in \mathbb{R}, \quad t > 0.$$

To solve the problem in the $(q,t)$ plane, a mesh with the step $h$ along the $q$-coordinate and the step $\tau$ along the $t$-one was used:

$$\omega_{h\tau} = \omega_h \times \omega_\tau,$$
$$\omega_h = \{q_k = kh,\ k = 0, \pm 1, \pm 2, ...\},$$
$$\omega_\tau = \{t_n = n\tau,\ n = 0, 1, 2, ...\}.$$

The solution of the problem was analyzed with the help of the implicit three-layer scheme

$$\frac{3y_k^{n+1} - 4y_k^n + y_k^{n-1}}{2\tau} = A(y)\frac{y_{k+1}^{n+1} - y_{k-1}^{n+1}}{2h} + \varphi \tag{66}$$

characterized by the approximation order $O(\tau^2 + h^2)$. Problem (65) was solved, by using the iteration method. Let us demonstrate that scheme (66) is absolutely stable. For the homogeneous scalar equation

$$\frac{\partial y}{\partial t} + a(y)\frac{\partial y}{\partial q} = 0,$$

the solution of problem (66) is sought in the form

$$y_k^n = \eta^n e^{ikh\theta}, \tag{67}$$

where $i = \sqrt{-1}$ and $\theta \in \mathbb{R}$. Substituting Eq. (67) into the equation

$$\frac{3y_k^{n+1} - 4y_k^n + y_k^{n-1}}{2\tau} = a(y)\frac{y_{k+1}^{n+1} - y_{k-1}^{n+1}}{2h},$$

and making simple transformations, we obtain the equation for $\eta$:

$$\mu\eta^2 - 4\eta + 1 = 0, \tag{68}$$

where $\mu = 3 + 2a\gamma i \sin\theta$ and $\gamma = \frac{\tau}{h}$.

Let us determine the set $G$ of points in the complex plane, $\mu = r + is$, for which the roots of Eq. (68) do not exceed unity by absolute value. The boundary of the region $G$ consists of points $\mu$, for which $|\eta| < 1$. Let us use Eq. (68) to express the parameter $\mu$ in terms of the variable $\eta$:

$$\mu = \frac{4}{\eta} - \frac{1}{\eta^2}.$$

It is evident that if $|\eta| = 1$, then, putting $\eta = e^{-i\varphi}$, we obtain

$$\mu = 4e^{i\varphi} - e^{2i\varphi}.$$

When the argument $\varphi$ varies from 0 to $2\pi$, the point $\mu$ describes a closed curve $\Gamma$, whose equation in the complex plane $\mu = r + is$ is convenient to be represented in the parametric form,

$$\begin{aligned} r &= 4\cos\varphi - \cos 2\varphi, \\ s &= 4\sin\varphi - \sin 2\varphi. \end{aligned} \tag{69}$$

From Eqs. (69), one can see that the curve $\Gamma$ is symmetric with respect to the real axis $r$. It intersects this axis at the points $\mu(0) = 3$ and $\mu(\pi) = -5$, where the derivatives

$$\frac{ds}{dr} = \frac{\cos 2\varphi - \cos 2\varphi}{2\sin\varphi - \sin 2\varphi}, \quad \frac{d^2 s}{dr^2} = \frac{6\sin^2\frac{\varphi}{2}}{(2\sin\varphi - \sin 2\varphi)^3}$$

are not determined. One can see that the second derivative of the curve $\Gamma$ is negative at $0 < \varphi < \pi$ and positive at $\pi < \varphi < 2\pi$. This fact testifies that the closed curve $\Gamma$ is convex upward in the upper half-plane $\mu$ and convex downward in the lower one. Therefore, the region bounded by the closed curve $\Gamma$ is convex (Fig. 1). The straight line $\mu = 3 + is$ is tangent to the curve $\Gamma$ at the point $\mu = 3$. The other points of this straight line are located in the region outside the curve $\Gamma$. Let us demonstrate that, in this region, the condition $|\eta| < 1$ is satisfied, so that the locus of points outside the curve $\Gamma$ is a stability region for scheme (66).

Really, let us consider the solution of the equation

$$(3 + is)\eta^2 - 4\eta + 1 = 0,$$

provided that $0 < s < 1$. Then one of the roots,

$$\eta = \frac{2 + \sqrt{1 - is}}{3 + is},$$

which corresponds to the maximum of the absolute value $|\eta|$, can be written in the form

$$\eta = \frac{\left(3 - \frac{is}{2}\right)(3 - is)}{9 + s^2} + O(s).$$

Its absolute value equals

$$|\eta| = \frac{\sqrt{81 + \frac{45s^2 + s^4}{4}}}{9 + s^2} =$$

$$= \sqrt{1 - \frac{27s^2 + s^4}{4(9 + s^2)}} = 1 - O(s^2) < 1.$$

Therefore, all the points $\mu = 3 + 2a\gamma i \sin\theta$ are located in the stability region, where $|\eta| \leq 1$. Hence, scheme (66) is absolutely stable and does not depend on the quantity $\gamma = \frac{\tau}{h}$. The stability region $G$ is illustrated in Fig. 1.

While carrying out the numerical simulation, we analyzed the solution obtained, when the initial condition for one of the variables was subjected to a certain perturbation in a vicinity of the point $q = 0$ in comparison with the reference homogeneous initial condition. This perturbation of the initial condition mimics the fluctuation of a macroscopic parameter in the physical system. The numerical calculations carried out according to the proposed implicit scheme gave rise to the following result. An arbitrary perturbation, even an infinitesimally small one, in the system of Nelson equations results in an unconfined growth of the variables $u$ and $v$. This is a consequence of the elliptic character of the Nelson equations, which are intended for the description of stationary processes, such as a flow around a body, electrostatic problems, stationary problems in the gravitation theory, and others.

The solution obtained for the system of equations (47) corresponds to a perturbation wave running along the spatial coordinate. In addition, the evolution of the perturbation shape is also observed. In Fig. 2, the numerical solutions of system (47) are shown for various time moments $t = 0$ ($a$), $\tau$ ($b$), $20\tau$ ($c$), and $50\tau$ ($d$), where $\tau$ is the step of integration over the time.

## 8. Discussion of Results and Conclusions

The idea to use the Lagrangian density $\mathcal{L}[\rho; \theta]$ in quantum mechanics seems to be suggested for the first time by Fenyes [11]. The expression proposed by him yields the Fokker–Planck equation containing the total velocity of a probability flux $V$ with the diffusion coefficient equal to either $D_{\text{qu}}$ or $D_T$. However, the generalized diffusion coefficient $D_{\text{ef}}$ was not introduced. Note that, in the cited work, a different equation of motion of the Hamilton–Jacobi type was derived. Moreover, the obtained system of equations for the functions $\rho$ and $\theta$ was asserted to be completely equivalent to the Schrödinger equations for the functions $\psi$ and $\psi^*$, despite that the Fokker–Planck equation, unlike the Schrödinger one, leads to the irreversibility.

In our approach, in contrast to that used in work [11], quantum thermal fluctuations and the diffusion pressure energy density associated with the stochas-

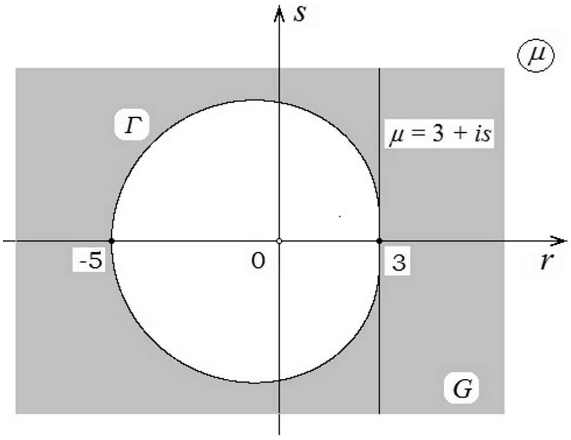

***Fig. 1.*** Boundary of the stability region for scheme (66)

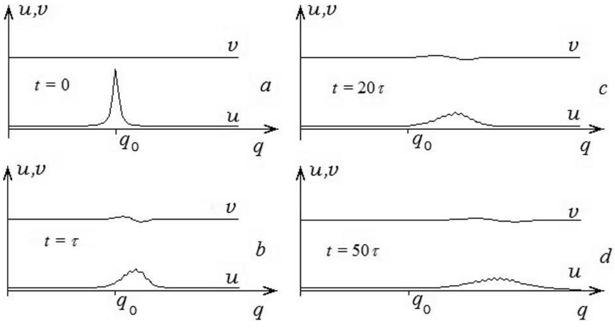

***Fig. 2.*** Results of numerical studies of the system of equations (47)

tic influence of the environment (the quantum thermostat) at $T \geqslant 0$ were consistently taken into account. Ultimately, we have represented the Fokker–Planck and Hamilton–Jacobi equations as the system of equations (42) for a one-dimensional model of the two-velocity stochastic hydrodynamics. The corresponding self-diffusion coefficient is governed by the effective influence of the environment depending on the fundamental constant $\varkappa = \frac{\hbar}{2k_\mathrm{B}}$.

In our opinion, this way can be used to construct a full-scale stochastic hydrodynamics, which involves not only the self-diffusion, but also the shear viscosity, and to apply it to the description of such interesting media as nearly perfect fluids (NPFs). For this purpose, it is necessary to change from Eq. (42b) for the drift velocity to an equation that generalizes the Navier–Stokes equation to the case with the self-diffusion.

Our analysis testified that the self-diffusion coefficient $\mathbb{D}_\mathrm{ef} = \mathbb{J}/m$ may be, probably, the most adequate characteristic of transport phenomena and can play an important role in the description of dissipative processes in NPFs. At present, there appears a possibility to determine $\mathbb{D}_\mathrm{ef}$ experimentally by studying the diffusion of massive quarks in a quark-gluon plasma obtained at the collision of heavy ions.

The numerical analysis of the solutions obtained for the particular case of system (42) in form (44), which is valid at $T = 0$, showed that those equations describe the relaxation of perturbations. Hence, the self-diffusion can be considered as a hydrodynamic mechanism driving the relaxation of quantum thermal fluctuations. The behavior of the solutions of system (42) in the general case will be done elsewhere.

Thus, we suppose that the hydrodynamic approach, which is proposed in this work, to the quantum-mechanical theory allows one to study, in principle, quantum thermal fluctuations on the basis of the obtained hydrodynamic equations.

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*Our duty is to indicate that the initial plan, formulation of the problem, and key ideas of this work were proposed by our co-author A.D. Sukhanov. This paper is devoted to his blessed memory.*

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## ЧИСЕЛЬНЕ МОДЕЛЮВАННЯ ПРОЦЕСУ РЕЛАКСАЦІЇ КВАНТОВО-ТЕПЛОВИХ ФЛУКТУАЦІЙ

Резюме

Пропонується узагальнення рівнянь квантової механіки в гідродинамічній формі шляхом введення в густину лагранжиана членів, що враховують дифузійну швидкість при нульовій і скінченних температурах, а також енергію дифузійного тиску теплого вакууму. На цій основі для моделі одновимірної гідродинаміки будується система рівнянь, аналогічних рівнянням Ейлера, але з урахуванням квантових і теплових ефектів. Вони являють собою узагальнення рівнянь стохастичної механіки Нельсона. Чисельний аналіз поведінки рішень даної системи дозволяє зробити висновок про її придатність для опису процесу релаксації квантово-теплових флуктуацій.