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DERIVATION OF THE DIRAC AND DIRAC-LIKE EQUATIONS OF ARBITRARY SPIN FROM THE CORRESPONDING RELATIVISTIC CANONICAL QUANTUM MECHANICS

The new relativistic equations of motion for the particles with spin s = 1, s = 3/2, and s = 2 and nonzero mass have been introduced. The description of the relativistic canonical quantum mechanics of the arbitrary mass and spin has been given. The link between the relativistic canonical quantum mechanics of the arbitrary spin and the covariant local field theory has been found. The manifestly covariant arbitrary-spin field equations that follow from the quantum mechanical equations have been considered. The covariant local field theory equations for a spin s = (1,1) particle-antiparticle doublet, spin s = (1,0,1,0) particle-antiparticle multiplet, spin s = (3/2,3/2) particle-antiparticle doublet, spin s = (2,2) particle-antiparticle doublet, spin s = (2,0,2,0) particle-antiparticle multiplet, and spin s = (2,1,2,1) particle-antiparticle multiplet have been introduced. The Maxwell-like equations for a boson with spin s = 1 and mass m > 0 have been introduced as well.

Keywords: relativistic quantum mechanics, Schödinger–Foldy equation, Dirac equation, Maxwell equations, arbitrary spin.

1. Introduction

The start of the relativistic quantum mechanics was given by Paul Dirac with his well-known equation for an electron [1]. More precisely, in this 4-component model, the spin $s = (1/2, 1/2)$ particle-antiparticle doublet of two fermions was considered (in particular, the electron-positron doublet). Nevertheless, the quantum-mechanical interpretation of the Dirac equation, which should be similar to the physical interpretation of the nonrelativistic Schrödinger equation, is not evident and is hidden deeply in the Dirac model. In order to visualize the quantum mechanical interpretation of the Dirac equation, a transformation to the canonical (quantum-mechanical) representation was suggested [2]. In this Foldy–Wouthuysen (FW) representation of the Dirac equation, the quantum-mechanical interpretation is much clearer. Nevertheless, the direct and evident quantum-mechanical interpretation of the spin $s = (1/2, 1/2)$ particle-antiparticle doublet can be fulfilled only within the framework of the relativistic canonical quantum mechanics (RCQM) (see, e.g., the consideration in [3]).

Note that here only the first-order particle and field equations (together with their canonical nonlocal pseudodifferential representations) are considered. The second-order equations (like the Klein–Gordon–Fock equation) are not the subject of this investigation.

Different approaches to the description of the particles of an arbitrary spin can be found in [4–13]. In this article, only the approach started in [13] is the basis for further applications. Other results given in [4–12] are not analyzed, not considered, and not used here. Note only one general deficiency of the known equations for arbitrary spin. For the spin $s > 1/2$, the existing equations should be complemented by some additional conditions. The equations suggested and considered here are free of this deficiency. The start of such consideration is taken from [13], where the main foundations of the RCQM are formulated. Below in the text, the results of [13] are generalized and extended. Note that the cases $s = 3/2$ and $s = 2$ were not presented in [13], especially in explicit demonstrative forms.

Contrary to the times of works [2, 13, 14], the RCQM today is the enough approbated and generally accepted theory. The spinless Salpeter equation has been introduced in [14]. The allusion on the RCQM and the first steps are given in [13], where the Salpeter equation for the $2s+1$-component wave function was considered, and the cases of $s = 1/2$ and $s = 1$ were presented as examples. In [15], L. Foldy (properly, László Földi) continued his investigations [13] by the consideration of the relativistic particle systems with interaction. The interaction was introduced by a specific group-theoretic method.

After that in the RCQM, the construction of mathematical foundations was realized, and the solution of specific quantum-mechanical problems for different potentials was executed. Some mathematical foundations and the spectral theory of pseudodifferential operator $\sqrt{\mathbf{p}^2 + m^2} - Ze^2/r$ were given in [16–19]. The application of the RCQM to the quark-antiquark bound state problem can be found in [20, 21]. The numerical solutions of the RCQM equation for arbitrary confining potentials were presented in [21]. In [22], the spinless Salpeter equation for the N particle system of spinless bosons with gravitational interaction was applied. In [23], a lower bound on the maximum mass of a boson star on the basis of the Hamiltonian $\sqrt{\mathbf{p}^2 + m^2} - \alpha/r$ was calculated. In [24], the results calculated by the author with the spinless Salpeter equation are compared with those obtained from the Schrödinger equation for heavy-quark systems, heavy-light systems, and light-quark systems. In each case, the Salpeter energies agree with experiment substantially better than the Schrödinger energies. Work [25] dealt with an investigation of exact numerical solutions. The spinless Salpeter equation with the Coulomb potential is solved exactly in the momentum space and is shown to agree very well with a coordinate-space calculation. In [26, 27], the problem of calculations of the spectrum of energy eigenvalues on the basis of the spinless Salpeter equation was considered. The spinless relativistic Coulomb problem was studied. It was shown how to calculate, by some special choices of basis vectors in the Hilbert space of solutions, for the rather large class of power-law potentials, at least upper bounds on these energy eigenvalues. The authors of works [26, 27] proved that, for the lowest-lying levels, this may be done even analytically. In work [28], the spinless Salpeter equation was rewritten into integral and integro-differential equations. Some analytical results concerning the spinless Salpeter equation and the action of the square-root operator have been presented. Further F. Brau constructed an analytical solution of the one-dimensional spinless Salpeter equation with a Coulomb potential supplemented by a hard core interaction, which keeps the particle in the x positive region [29]. In the context of RCQM based on the spinless Salpeter equation, it was shown [30] how to construct a large class of upper limits on the critical value, $g_c^{(\ell)}$, of the coupling constant, g, of the central potential, $V(r) = -gv(r)$. In [31], the lower bounds on the ground-state energy, in one and three dimensions, for the spinless Salpeter equation applicable to potentials, for which the attractive parts are in $\mathrm{L^p(R^n)}$ for some $p > n$ ($n = 1$ or 3), were found. An extension to confining potentials, which are not in $\mathrm{L^p(R^n)}$, was also presented. In work [32], the authors used the theory of fractional powers of linear operators to construct a general (analytical) representation theory for the square-root energy operator $\gamma^0\sqrt{\mathbf{p}^2 + m^2} + V$ of the FW canonical field theory, which is valid for all values of spin. The example of the spin $1/2$ case, considering a few simple still solvable and physically interesting cases, was presented in details in order to understand how to interpret the operator. Note that the corresponding results for the RCQM can be found from the FW canonical field theory results [32] (see Subsection 6.2 below). Using the momentum space representation, the authors of [33] presented an analytical treatment of the one-dimensional spinless Salpeter equation with a Coulomb interaction. The exact bound-state energy equation was determined. The results obtained were shown to agree very well with those of exact numerical calculations existing in the literature. In [34], an exact analytical treatment of the spinless Salpeter equation with a one-dimensional Coulomb interaction in the context of quantum mechanics with a modified

Heisenberg algebra implying the existence of a minimal length was presented. The problem was tackled in the momentum space representation. The bound-state energy equation and the corresponding wave functions were exactly obtained. The probability current for a quantum spinless relativistic particle was introduced [35] on the basis of the Hamiltonian dynamics approach, by using the spinless Salpeter equation. The correctness of the presented formalism was illustrated by examples of exact solutions to the spinless Salpeter equation including the new ones. Thus, the following partial wave packet solutions of this equation were presented in [35]: the solutions for free massless and massive particles on a line, for a massless particle in a linear potential, plane wave solution for a free particle (these solution is given here in formula (8) for N-component case), and the solution for a free massless particle in three dimensions. Further, other time-dependent wave packet solutions of the free spinless Salpeter equation were given in [36]. With regard for the relation of such wave packets to the Lévy process, the spinless Salpeter equation (in the one-dimensional space-time) was called the Lévy–Schrödinger equation in [36]. The several examples of the characteristic behavior of such wave packets have been shown, in particular, the multimodality arising in their evolutions: a feature at variance with the typical diffusive unimodality of both the corresponding Lévy process densities and usual Schrödinger wave functions. A generic upper bound is obtained [37] for the spinless Salpeter equation with two different masses. Analytical results are presented for systems relevant for hadronic physics: Coulomb and linear potentials when a mass is vanishing. A detailed study for the classical and the quantum motion of a relativistic massless particle in an inverse square potential has been presented recently in [38]. The quantum approach to the problem was based on the exact solution of the corresponding spinless Salpeter equation for bound states. Finally in [38], the connection between the classical and the quantum descriptions via the comparison of the associated probability densities for the momentum has been made. The goal of the recent paper [39] is a comprehensive analysis of the intimate relationship between jump-type stochastic processes (e.g., Lévy flights) and nonlocal (due to integro-differential operators involved) quantum dynamics. In [39], a special attention is paid to the spinless Salpeter (here, $m \geq 0$) equation and the evolution of various wave packets, in particular, to their radial expansion in 3D. Foldy's synthesis of "covariant particle equations" is extended to encompass the free Maxwell theory, which, however, is devoid of any "particle" content. Links with the photon wave mechanics are explored. The authors of [39] considered our results [40] presented also in more earlier preprint (see the last reference in [39]).

In works [3, 40], where we started our investigations in the RCQM, this relativistic model for the test case of the spin $s = (1/2, 1/2)$ particle-antiparticle doublet was formulated. In [40], this model was considered as a system of axioms on the level of the von Neumann monograph [41], where the mathematically well-defined consideration of the nonrelativistic quantum mechanics was given. Furthermore, in [3, 40], the operator link between the spin $s = (1/2, 1/2)$ particle-antiparticle doublet RCQM and the Dirac theory was given, and Foldy's synthesis of "covariant particle equations" was extended to the start from the RCQM of the spin $s = (1/2, 1/2)$ particle-antiparticle doublet.

Below, the same procedure is fulfilled for the spin $s = (1, 1)$, $s = (1, 0, 1, 0)$, $s = (3/2, 3/2)$, $s = (2, 2)$, $s = (2, 0, 2, 0)$, and spin $s = (2, 1, 2, 1)$ RCQM. The corresponding equations, which follow from the RCQM for the covariant local field theory, are introduced.

In other words, the so-called Foldy synthesis [13] of the covariant particle equations is extended here by starting from the RCQM of arbitrary spin, and the related equations of the covariant local field theory are the final step of such program. The canonical representation of the field equations (an analog of the FW representation) is the intermediate step in this method.

Therefore, I am not going to formulate here a new relativistic quantum mechanics! The foundations of the RCQM based on the spinless Salpeter equation were already formulated in [13–40].

The relativistic quantum mechanics under consideration is called canonical due to three main reasons. (i) The model under consideration has direct link with the nonrelativistic quantum mechanics based on the nonrelativistic Schrödinger equation. The principles of heredity and correspondence with other models of physical reality lead directly to the nonrelativistic Schrödinger quantum mechanics. (ii) The FW model is already called by many authors as the canonical representation of the Dirac equation or a canonical field model, see, e.g., work [13]. The difference between the field model given by the FW model and the RCQM is minimal – in corresponding equations it is only the presence or the absence of the beta matrix. (iii) The list of relativistic quantum-mechanical models is long. The Dirac model and the FW model are called by the "old" physicists as the relativistic quantum mechanics as well (one of my tasks in this paper is to show in a visual and demonstrative way that these models have only weak quantum-mechanical interpretation). Further, the fractional relativistic quantum mechanics and the proper-time relativistic quantum mechanics can be listed, *etc.* Therefore, in order to avoid a confusion, the model under consideration must have its proper name. Due to reasons (i)–(iii) above, the best name for it is RCQM.

The results are presented on three linking levels.

Covariant local field theory ← Canonical field theory ← Relativistic canonical quantum mechanics

Here, the standard relativistic concepts, definitions, and notations in the form convenient for our consideration are chosen. For example, in the Minkowski space-time,

$$\begin{aligned} &\mathrm{M}(1,3) = \{x \equiv (x^\mu) = (x^0 = t,\, \mathbf{x} \equiv (x^j))\}; \\ &\mu = \overline{0,3},\;\; j = 1,2,3, \end{aligned} \tag{1}$$

x^μ denotes the Cartesian (covariant) coordinates of the points of the physical space-time in the arbitrary fixed inertial reference frame (IRF). We use the system of units with $\hbar = c = 1$. The metric tensor is given by

$$\begin{aligned} &g^{\mu\nu} = g_{\mu\nu} = g^\mu_\nu,\; (g^\mu_\nu) = \mathrm{diag}\,(1,-1,-1,-1)\,; \\ &x_\mu = g_{\mu\nu}x^\mu, \end{aligned} \tag{2}$$

and the summation over the twice repeated indices is implied.

2. Foundations of the Relativistic Canonical Quantum Mechanics of a Particle with Nonzero Mass and Arbitrary Spin *s*

The equation of motion is the Schrödinger–Foldy equation

$$i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x) \tag{3}$$

for the N-component wave function

$$f \equiv \mathrm{column}(f^1, f^2, ..., f^{\mathrm{N}}), \quad \mathrm{N} = 2s+1. \tag{4}$$

Equation (3) is a direct sum of one-component spinless Salpeter equations. This equation has been introduced in formula (21) of [13].

The suggestion to call the main equation of the RCQM as the Schrödinger–Foldy equation was given in [40] and [3]. Our motivation was as follows. In works [13, 15], the-two component version of Eq. (3) is called the Schrödinger equation. Moreover, the one-component version of Eq. (3) was suggested in [14] and is called in the literature as the spinless Salpeter equation (see, e.g., [25–31, 33–35, 37, 38] and references therein). Nevertheless, in view of L. Foldy's contribution to the construction of the RCQM and his proof of the principle of correspondence between the RCQM and nonrelativistic quantum mechanics, we propose *to call the N-component equations of this type as the Schrödinger-Foldy equations.*

The space of the states is taken as a rigged Hilbert space

$$\mathrm{S}^{3,\mathrm{N}} \equiv \mathrm{S}(\mathrm{R}^3) \times \mathrm{C}^{\mathrm{N}} \subset \mathrm{H}^{3,\mathrm{N}} \subset \mathrm{S}^{3,\mathrm{N}*}. \tag{5}$$

Here, $\mathrm{S}^{3,\mathrm{N}}$ is the N-component Schwartz test function space over the space $\mathrm{R}^3 \subset \mathrm{M}(1,3)$, and $\mathrm{H}^{3,\mathrm{N}}$ is the Hilbert space of the N-component square-integrable functions over the $x \in \mathrm{R}^3 \subset \mathrm{M}(1,3)$:

$$\begin{aligned} &\mathrm{H}^{3,\mathrm{N}} = \mathrm{L}_2(\mathrm{R}^3) \otimes \mathrm{C}^{\otimes \mathrm{N}} = \{f = (f^{\mathrm{N}}) : \mathrm{R}^3 \to \mathrm{C}^{\otimes \mathrm{N}}; \\ &\textstyle\int d^3x |f(t,\mathbf{x})|^2 < \infty\}, \end{aligned} \tag{6}$$

where d^3x is the Lebesgue measure in the space $\mathrm{R}^3 \subset \mathrm{M}(1,3)$ of the eigenvalues of the position operator $\mathbf{x}$ of the Cartesian coordinate of the particle in an arbitrary fixed IRF. Further, $\mathrm{S}^{3,\mathrm{N}*}$ is the space of the N-component Schwartz generalized functions. The space $\mathrm{S}^{3,\mathrm{N}*}$ is conjugated to that of the Schwartz test functions $\mathrm{S}^{3,\mathrm{N}}$ by the corresponding topology (see, e.g. [42]).

In general, the mathematical correctness of consideration demands to make the calculations in the space $\mathrm{S}^{3,\mathrm{N}*}$ of generalized functions, i.e. with the application of cumbersome functional analysis. Nevertheless, one can consider the properties of the Schwartz test function space $\mathrm{S}^{3,\mathrm{N}}$ in triple (5). The space $\mathrm{S}^{3,\mathrm{N}}$ is dense both in the quantum-mechanical space $\mathrm{H}^{3,\mathrm{N}}$ and in the space of generalized functions $\mathrm{S}^{3,\mathrm{N}*}$.

Therefore, any physical state $f \in \mathrm{H}^{3,\mathrm{N}}$ can be approximated with an arbitrary precision by the corresponding elements of the Cauchy sequence in $\mathrm{S}^{3,\mathrm{N}}$, which converges to the given $f \in \mathrm{H}^{3,\mathrm{N}}$. Further, with regard for the requirement to measure the arbitrary value of the quantum-mechanical model with nonabsolute precision, this means that all specific calculations can be fulfilled within the Schwartz test function space $\mathrm{S}^{3,\mathrm{N}}$.

Furthermore, the mathematical correctness of the consideration demands to determine the domain of definitions and the range of values for any used operator and for the functions of operators. Note that if the kernel space $\mathrm{S}^{3,\mathrm{N}} \subset \mathrm{H}^{3,\mathrm{N}}$ is taken as the common domain of definitions of the generating operators $\mathbf{x} = (x^j)$, $\widehat{\mathbf{p}} = (\widehat{p}^j)$, $\mathbf{s} \equiv \left(s^j\right) = (s_{23}, s_{31}, s_{12})$ of coordinate, momentum, and spin, respectively, then this space appears to be also the range of their values. Moreover, the space $\mathrm{S}^{3,\mathrm{N}}$ appears to be the common domain of definitions and values for the set of all below-mentioned functions from the 9 operators $\mathbf{x} = (x^j)$, $\widehat{\mathbf{p}} = (\widehat{p}^j)$, $\mathbf{s} \equiv \left(s^j\right)$ (for example, for the generators $(\widehat{p}_\mu, \widehat{j}_{\mu\nu})$ of the irreducible unitary representations of the Poincaré group $\mathcal{P}$ and for different sets of commutation relations). Therefore, in order to guarantee the realization of the principle of correspondence between the results of cognition and the instruments of cognition in the given model, it is sufficient to take the algebra $\mathrm{A_S}$ of the all sets of observables of the given model in the form of Hermitian power series of the 9 generating operators $\mathbf{x} = (x^j)$, $\widehat{\mathbf{p}} = (\widehat{p}^j)$, $\mathbf{s} \equiv \left(s^j\right)$ converged in $\mathrm{S}^{3,\mathrm{N}}$.

Note that the Schrödinger–Foldy equation (3) has generalized solutions, which do not belong to the space $\mathrm{H}^{3,\mathrm{N}}$ (6). Therefore, the application of the rigged Hilbert space $\mathrm{S}^{3,\mathrm{N}} \subset \mathrm{H}^{3,\mathrm{N}} \subset \mathrm{S}^{3,\mathrm{N}*}$ (5) is necessary.

Some other details of motivations of the choice of spaces (5) and (6) (and all necessary notations) are given in [3], where the corresponding 4-component spaces are considered.

The operator of particle spin is chosen in the complete matrix form and is associated with the SU(2) group. The orthonormalized diagonal Cartesian basis, in which the third component of the spin has the diagonal form, is necessary. The corresponding generators of the SU(2) group irreducible representations are chosen to be the spin operators of the corresponding particle states.

Hence, the spin operator is given as

$$\mathbf{s} \equiv \left(s^j\right) = (s_{23}, s_{31}, s_{12}) : \left[s^j, s^l\right] = i\varepsilon^{jln} s^n, \qquad (7)$$

where ε^{jln} is the Levi-Civita tensor and $s^j = \varepsilon^{j\ell n} s_{\ell n}$ are the Hermitian $\mathrm{N} \times \mathrm{N}$ matrices – the generators of the N-dimensional representation of the spin group SU(2) (universal covering of the SO(3)⊂SO(1,3) group). Below, in Sections 3–6, the fixed specific representations of the SU(2) group are associated with the fixed particular spin values.

The general solution of the equation of motion (3) is given by

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx} a^{\mathrm{N}}(\mathbf{k})\, \mathrm{d_N}, \qquad (8)$$

where the notations

$$kx \equiv \omega t - \mathbf{kx}, \quad \omega \equiv \sqrt{\mathbf{k}^2 + m^2}, \qquad (9)$$

are used. The orts of the N-dimensional Cartesian basis have the form

$$\mathrm{d}_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{vmatrix}, \ \mathrm{d}_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{vmatrix}, \ ...., \ \mathrm{d_N} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{vmatrix}. \qquad (10)$$

Solution (8) is associated with the stationary complete set of the operators ($\mathbf{p}$, $s^3 = s_z$, g) of momentum, spin projection, and sign of the charge (in the case of charged particles). It is easy to see from Sections 3–6 that, for the different N, the spin projection operators are different. The stationary complete set of operators is the set of all functionally independent mutually commuting operators, each of which commutes with the operator of energy (in this model with the operator $\sqrt{m^2 - \Delta}$).

The interpretation of the amplitudes $a^{\mathrm{N}}(\mathbf{k})$ follows from the equations for the eigenvalues of the operators ($\mathbf{p}$, $s^3 = s_z$, g). The functions $a^{\mathrm{N}}(\mathbf{k})$ are the quantum-mechanical momentum-spin amplitudes of a single particle with corresponding momentum, spin, and charge values (in the case of charged particle), respectively.

The relativistic invariance of the model under consideration requires, as a first step, the consideration of its invariance with respect to the proper orthochronous Lorentz $\mathrm{L}^{\uparrow}_{+}$ =SO(1,3)=$\{\Lambda = (\Lambda^\mu_\nu)\}$ and

Poincaré $P_+^{\uparrow} = T(4)\times)L_+^{\uparrow} \supset L_+^{\uparrow}$ groups. This invariance in an arbitrary relativistic model is the implementation of Einstein's relativity principle in the special relativity form. Note that the mathematical correctness requires the invariance mentioned above to be considered as the invariance with respect to the universal coverings $\mathcal{L} = SL(2,C)$ and $\mathcal{P} \supset \mathcal{L}$ of the groups $L_+^{\uparrow}$ and $P_+^{\uparrow}$, respectively.

For the group $\mathcal{P}$, we choose real parameters $a = (a^{\mu}) \in M(1,3)$ and $\varpi \equiv (\varpi^{\mu\nu} = -\varpi^{\nu\mu})$ with well-known physical meaning. For the standard $\mathcal{P}$ generators $(p_\mu, j_{\mu\nu})$, we use commutation relations in the manifestly covariant form

$$[p_\mu, p_\nu] = 0,\ [p_\mu, j_{\rho\sigma}] = ig_{\mu\rho}p_\sigma - ig_{\mu\sigma}p_\rho,$$
$$[j_{\mu\nu}, j_{\rho\sigma}] = -i\,(g_{\mu\rho}j_{\nu\sigma} + g_{\rho\nu}j_{\sigma\mu} + \tag{11}$$
$$+ g_{\nu\sigma}j_{\mu\rho} + g_{\sigma\mu}j_{\rho\nu}).$$

The following assertion should be noted. Not a matter of fact that noncovariant objects such as the Lebesgue measure d^3x and noncovariant generators of algebras are explored, the model of RCQM of arbitrary spin is a relativistic invariant in the following sense. The Schrödinger–Foldy equation (3) and the set of its solutions $\{f\}$ (8) are invariant with respect to the irreducible unitary representation of the group $\mathcal{P}$, the $N \times N$ matrix-differential generators of which are given by the following nonlocal operators:

$$\widehat{p}_0 = \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \widehat{p}_\ell = i\partial_\ell,$$
$$\widehat{j}_{\ell n} = x_\ell\widehat{p}_n - x_n\widehat{p}_\ell + s_{ln} \equiv \widehat{m}_{\ell n} + s_{\ell n}, \tag{12}$$

$$\widehat{j}_{0\ell} = -\widehat{j}_{\ell 0} = t\widehat{p}_\ell - \frac{1}{2}\{x_\ell, \widehat{\omega}\} - \left(\frac{s_{\ell n}\widehat{p}_n}{\widehat{\omega} + m} \equiv \breve{s}_\ell\right), \tag{13}$$

where the orbital parts of the generators are not changed under the transition from one spin to another. Under such transitions, only the spin parts (7) of expressions (12) and (13) are changed. Indeed, the direct calculations visualize that generators (12) and (13) commute with the operator of Eq. (3) and satisfy the commutation relations (11) of the Lie algebra of the Poincaré group $\mathcal{P}$. In formulae (12) and (13), the SU(2)-spin generators $s^{\ell n}$ have particular specific forms for each representation of the SU(2) group (see the examples in Sections 3–6).

Note that generators (12) and (13) are known from formulae (B-25)—(B-28) in [13].

Note also that, together with generators (12) and (13), another set of 10 operators commutes with the operator of Eq. (3), satisfies the commutation relations (11) of the Lie algebra of Poincaré group $\mathcal{P}$, and, therefore, can be chosen as the Poincaré symmetry of the model under consideration. This second set is given by the generators $\widehat{p}^0$, $\widehat{p}^\ell$ from (12) together with the orbital parts of the generators $\widehat{j}^{\ell n}$, $\widehat{j}^{0\ell}$ from (13).

Thus, the irreducible unitary representation of the Poincaré group $\mathcal{P}$ in space (5), with respect to which the Schrödinger–Foldy equation (3) and the set of its solutions $\{f\}$ (8) are invariant, is given by a series converging in this space

$$(a, \varpi) \to U(a, \varpi) =$$
$$= \exp\left(-ia^0\widehat{p}_0 - i\mathbf{a}\widehat{\mathbf{p}} - \frac{i}{2}\varpi^{\mu\nu}\widehat{j}_{\mu\nu}\right), \tag{14}$$

where the generators $(\widehat{p}^\mu, \widehat{j}^{\mu\nu})$ are given in (12) and (13) with the arbitrary values of the SU(2) spins $\mathbf{s} = (s^{\ell n})$ (7).

The validity of this assertion is verified by the following three steps. (i) The calculation that the $\mathcal{P}$-generators (12) and (13) commute with the operator $i\partial_0 - \widehat{\omega}$ of the Schrödinger–Foldy equation (3). (ii) The verification that the $\mathcal{P}$-generators (12) and (13) satisfy the commutation relations (11) of the Lie algebra of the Poincaré group $\mathcal{P}$. (iii) The proof that generators (12) and (13) realize the spin $s(s+1)$ representation of this group. Therefore, the Bargmann–Wigner classification on the basis of the calculation of corresponding Casimir operators should be given. These three steps can be made by direct and noncumbersome calculations.

Expression (14) is well known, but rather formal. In fact, in the case of non-Lie operators the transition from a Lie algebra to a finite group of transformations is a rather nontrivial action. The mathematical justification of (14) can be fulfilled in the framework of the Schwartz test function space and will be given in a next special publication.

The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu\widehat{p}_\mu = m^2 I_N, \tag{15}$$

$$W = w^\mu w_\mu = m^2\mathbf{s}^2 = s(s+1)m^2 I_N, \tag{16}$$

where I_N is the $N \times N$ unit matrix, and $s = 1/2, 1, 3/2, 2, \ldots$.

Below in the next sections, the particular examples of spins $s = 1/2$, 1, 3/2, 2 singlets and spins $s = (1, 1)$, (1,0), (1,0,1,0) multiplets are considered briefly.

The partial cases $s = 1/2$, 1, 3/2, 2 and $s = (1, 1)$, $s = (1, 0)$, $s = (1, 0, 1, 0)$, $s = (3/2, 3/2)$, $s = (2, 2)$, $s = (2, 0, 2, 0)$, $s = (2, 1, 2, 1)$ can be presented on the level of the axiomatic approach [43]. The way of such consideration was demonstrated in [3] on the test example of the spin $s = (1/2, 1/2)$ particle-antiparticle doublet of fermions.

3. On the Relativistic Canonical Quantum Mechanics of the Arbitrary Mass and Spin Particle Multiplets

In view of the step-by-step consideration of the different partial examples, which are given in [44] in Sections 7–17, the generalization for the particle multiplet of arbitrary spin can be formulated.

The theory is completely similar to that given above in Section 2, where the RCQM of the particle singlet of arbitrary spin is presented. The specification is only in the application of the reducible representation of the SU(2) and the Poincaré $\mathcal{P}$ groups.

Furthermore, the simple particle multiplets are constructed as the ordinary direct sum of the corresponding particle characteristics. The particle-antiparticle doublets and multiplets are constructed as the specific direct sum of the particle and antiparticle characteristics, in which the antiparticle is considered as the mirror reflection of the particle and the information about equal and positive masses of the particle and the antiparticle is inserted. According to the Pauli principle, the third component of the antiparticle spin has the opposite sign to the third component of the particle spin. The partial examples are given in [44] in formulas (64), (91), (153), (154), (172), (178), (179), (190), (191), (201), (202), (212), and (213).

Thus, for the general form of the arbitrary particle-antiparticle doublet, the Schrödinger–Foldy equation has the form $i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x)$, where $f \equiv \text{column}(f^1, f^2, ..., f^{2\text{N}})$, $\text{N} = 2\text{s} + 1$. The corresponding $2\text{N} \times 2\text{N}$ spin operator is given by $\mathbf{s}_{2\text{N}} = \left|\begin{smallmatrix} \mathbf{s}_\text{N} & 0 \\ 0 & -C\mathbf{s}_\text{N}C \end{smallmatrix}\right|$. The 2N-dimensional rigged Hilbert space and the ordinary Hilbert space are the corresponding direct sums of spaces (5) and (6), respectively. Other necessary formulas follow from formulas (7)–(16) of Section 2 after the substitutions of 2N instead of N and the above-given $2\text{N} \times 2\text{N}$ spin operators instead of $\text{N} \times \text{N}$ spin operators from Section 2.

The general description for other arbitrary multiplets considered in Sections 12, 13, 16, and 17 of [44] is formulated by similar minimal efforts.

4. On the Transition to the Nonrelativistic Canonical Quantum Mechanics of the Arbitrary Mass and Spin

In the nonrelativistic limit, the Schrödinger–Foldy equation (3) is transformed into the ordinary Schrödinger equation $\left(i\partial_0 - \frac{\mathbf{p}^2}{2m}\right)\psi(x) = 0$ for the N-component wave function $\psi \equiv \text{column}(\psi^1, \psi^2, ..., \psi^\text{N})$, $\text{N} = 2\text{s} + 1$.

Each of the equations considered in [44] of the RCQM (2-, 3-, 4-, 5-, 6-, 8-, 10-, 12-, and 16-component ones) is transformed into the ordinary Schrödinger equation with the corresponding number of components. Moreover, here, the SU(2) spin operator is the same as in the RCQM and is given by formulae (7). Therefore, here as in the RCQM, the SU(2) generators for the spin $s = 1/2$ are given in [44] in formulae (19) and (20), for the spin $s = 1$ in (29), for the spin $s = 3/2$ in (39), for the spin $s = 2$ in (48) of [44], *etc.* for the multiplet SU(2) spins.

Therefore, the equation of motion of nonrelativistic quantum mechanics is invariant with respect to the same representations of the SU(2) group, with respect to which the relativistic equation (3) is invariant. The difference is in the application of the Galilean group and its representations instead of the Poincaré $\mathcal{P}$ group and its representations.

For the models with interaction, it is much more easier to solve the ordinary Schrödinger equation with interaction potential $V(x)$ instead of the pseudodifferential equation (3). Moreover, the solutions of a nonrelativistic equation with interaction can be useful for obtaining the corresponding solutions of Eq. (3).

Thus, the nonrelativistic equation can be useful not only by itself, but for various approximations of the relativistic equation (3) as well.

5. The Example of Relativistic Canonical Quantum Mechanics of the Spin $s = (1/2, 1/2)$ Particle-Antiparticle Doublet

The main example of the RCQM of the arbitrary mass and spin particle multiplet is the model of the spin

$s = (1/2, 1/2)$ particle-antiparticle doublet (e^-e^+-doublet in partial case). In [44], this example was formulated on the level of modern axiomatic approaches to quantum field theory. The *axioms of this relativistic model* are considered on the level of correctness of von Neumann's nonrelativistic approach [41]. The list of main assertions is given as the following axioms: *on the space of states, on the time evolution of the state vectors, on the fundamental dynamical variables, on the external and internal degrees of freedom,* which are considered in the form similar to that in [45], *on the algebra of observables, on the relativistic invariance of the theory,* with the modern definition of the symmetry of partial differential equations [46], *on the main and additional conservation laws, on the stationary complete sets of operators, on the solutions of the Schrödinger–Foldy equation, on the Clifford–Dirac algebra, on the dynamic and kinematic aspects of the relativistic invariance, on the mean value of the operators of observables, on the principles of heredity and the correspondence, the physical interpretation.*

Here, the definition of the pseudodifferential (nonlocal) operator from Eq. (3), which was given in [44] in the consideration of the axiom **on the time evolution of the state vectors**, can be clarified as follows.

The action (see, e.g., [35]) of the pseudodifferential (nonlocal) operator

$$\widehat{\omega} \equiv \sqrt{\widehat{\mathbf{p}}^2 + m^2} = \sqrt{-\Delta + m^2} \geq m > 0; \quad \widehat{\mathbf{p}} \equiv (\widehat{p}^j) = -i\nabla, \quad \nabla \equiv (\partial_\ell), \tag{17}$$

in the coordinate representation is given by

$$\widehat{\omega} f(t, \mathbf{x}) = \int d^3y K(\mathbf{x} - \mathbf{y}) f(t, \mathbf{y}), \tag{18}$$

where the function $K(\mathbf{x} - \mathbf{y})$ has the form $K(\mathbf{x} - \mathbf{y}) = -\frac{2m^2 K_2(m|\mathbf{x}-\mathbf{y}|)}{(2\pi)^2|\mathbf{x}-\mathbf{y}|^2}$ and $K_\nu(z)$ is the modified Bessel function (Macdonald function), $|\mathbf{a}|$ designates the norm of the vector $\mathbf{a}$. Further, the integral form

$$(\widehat{\omega} f)(t, \mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{i\mathbf{k}\mathbf{x}} \widetilde{\omega} \widetilde{f}(t, \mathbf{k}); \quad \widetilde{\omega} \equiv \sqrt{\mathbf{k}^2 + m^2}, \quad \widetilde{f} \in \widetilde{\mathrm{H}}^{3,4}, \tag{19}$$

of the operator $\widehat{\omega}$ is used often (see, e.g., [13]), where f and $\widetilde{f}$ are linked by the 3-dimensional Fourier transformations

$$f(t, \mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{i\mathbf{k}\mathbf{x}} \widetilde{f}(t, \mathbf{k}) \Leftrightarrow \Leftrightarrow \widetilde{f}(t, \mathbf{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3x e^{-i\mathbf{k}\mathbf{x}} \widetilde{f}(t, \mathbf{x}), \tag{20}$$

(in (20), $\mathbf{k}$ belongs to the spectrum $\mathrm{R}^3_{\mathbf{k}}$ of the operator $\widehat{\mathbf{p}}$, and the parameter $t \in (-\infty, \infty) \subset \mathrm{M}(1,3)$).

Further, the axiom **on the Clifford–Dirac algebra** is updated here to the following form.

The Clifford–Dirac algebra of the γ-matrices must be introduced into the FW representations. The reasons are as follows.

A part of the Clifford–Dirac algebra operators is directly related to the spin 1/2 doublet operators ($\frac{1}{2}\gamma^2\gamma^3$, $\frac{1}{2}\gamma^3\gamma^1$, $\frac{1}{2}\gamma^1\gamma^2$) (in the anti-Hermitian form). In the FW representation for the spinor field [2], these spin operators commute with the Hamiltonian and with the operator of the equation of motion $i\partial_0 - \gamma^0\widehat{\omega}$. In the Pauli–Dirac representation, these operators do not commute with the Dirac equation operator. Only the sums of the orbital operators and such spin operators commute with the Diracian. So, *if we want to relate the orts γ^μ of the Clifford–Dirac algebra with the actual spin, we must introduce this algebra into the FW representation.*

In the quantum-mechanical representation (i.e., in the space {f} of 4-component solutions (8) of the 4-component Schrödinger–Foldy equation (3)), the γ-matrices are obtained by the transformation v given in formula (22) below.

Moreover, we use a generalized algebra of the Clifford–Dirac type over the field of real numbers. This algebra was introduced in [47–51]. The use of 29 orts of this *real algebra SO(8)* gives the additional possibilities in comparison with only 16 elements of the standard Clifford–Dirac algebra SO(1,5) (see, e.g., [47–51]).

The definitions of spin matrices (64) in [44] *de facto* determine the so-called "quantum-mechanical" representation of the Dirac matrices

$$\bar{\gamma}^{\hat{\mu}} : \bar{\gamma}^{\hat{\mu}}\bar{\gamma}^{\hat{\nu}} + \bar{\gamma}^{\hat{\nu}}\bar{\gamma}^{\hat{\mu}} = 2g^{\hat{\mu}\hat{\nu}}; \quad \bar{\gamma}_0^{-1} = \bar{\gamma}_0, \; \bar{\gamma}_l^{-1} = -\bar{\gamma}_l, \quad g^{\hat{\mu}\hat{\nu}} \equiv (+----), \; \hat{\mu} = 0,1,2,3,4. \tag{21}$$

The matrices $\bar{\gamma}^\mu$ (21) of this representation are linked with the Dirac matrices $\gamma^{\hat{\mu}}$ in the standard Pauli–

Dirac (PD) representation:

$$\bar\gamma^0=\gamma^0,\quad \bar\gamma^1=\gamma^1 C,\quad \bar\gamma^2=\gamma^0\gamma^2 C,$$
$$\bar\gamma^3=\gamma^3 C,\quad \bar\gamma^4=\gamma^0\gamma^4 C; \tag{22}$$
$$\bar\gamma^{\hat\mu}=v\gamma^{\hat\mu}v,\quad v\equiv\begin{vmatrix} I_2 & 0\\ 0 & CI_2\end{vmatrix}=v^{-1},$$

where the standard Dirac matrices $\gamma^{\hat\mu}$ are given by

$$\gamma^0=\begin{vmatrix} I_2 & 0\\ 0 & -I_2\end{vmatrix},\quad \gamma^k=\begin{vmatrix} 0 & \sigma^k\\ -\sigma^k & 0\end{vmatrix},$$
$$\gamma^4\equiv\gamma^0\gamma^1\gamma^2\gamma^3,\quad \gamma^0\gamma^1\gamma^2\gamma^3\gamma^4=-I_4, \tag{23}$$

C is the operator of complex conjugation.

Note that, in terms of $\bar\gamma^\mu$ matrices (22), the RCQM spin operator (64) of [44] has the form

$$\mathbf{s}=\frac{i}{2}(\bar\gamma^2\bar\gamma^3,\,\bar\gamma^3\bar\gamma^1,\,\bar\gamma^1\bar\gamma^2). \tag{24}$$

Therefore, the complete analogy with the particle-antiparticle doublet spin in the FW representation exists

$$\mathbf{s}_{\rm FW}=\frac{i}{2}(\gamma^2\gamma^3,\,\gamma^3\gamma^1,\,\gamma^1\gamma^2). \tag{25}$$

The $\bar\gamma^\mu$ matrices (22) together with the matrix $\bar\gamma^4\equiv\bar\gamma^0\bar\gamma^1\bar\gamma^2\bar\gamma^3$, imaginary unit $i\equiv\sqrt{-1}$, and operator C of complex conjugation in $H^{3,4}$ generate the quantum-mechanical representations of the generalized algebras of the Clifford–Dirac type over the field of real numbers, which were put into consideration in [47] (see also [48–51]).

Recall that, in [47–51] for the purposes of finding the links between the fermionic and bosonic states, not 5 (as in (21)–(23)), but 7 generating γ matrices were used. In addition to γ^1, γ^2, γ^3, γ^4 matrices from (23), 3 new γ matrices were introduced. Therefore, the set of 7 generating γ matrices is given by

$$\gamma^1,\,\gamma^2,\,\gamma^3,\,\gamma^4,\,\gamma^5\equiv\gamma^1\gamma^3 C,\,\gamma^6\equiv i\gamma^1\gamma^3 C,$$
$$\gamma^7\equiv i\gamma^0;\quad \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6\gamma^7=I_4. \tag{26}$$

Matrices (26) generate the 64-dimensional $Cl^{\rm R}(0,6)$ algebra over the field of real numbers.

Here, in the quantum-mechanical representation, these γ matrices (in terms of the standard γ matrices) have the form

$$\bar\gamma^1,\,\bar\gamma^2,\,\bar\gamma^3,\,\bar\gamma^4\equiv\bar\gamma^0\bar\gamma^1\bar\gamma^2\bar\gamma^3,\,\bar\gamma^5\equiv\gamma^1\gamma^3 C,$$
$$\bar\gamma^6\equiv -i\gamma^2\gamma^4 C,\,\bar\gamma^7\equiv i; \tag{27}$$
$$\bar\gamma^1\bar\gamma^2\bar\gamma^3\bar\gamma^6\bar\gamma^5\bar\gamma^6\bar\gamma^7=I_4,$$

and satisfy the anticommutation relations in the form

$$\bar\gamma^{\rm A}\bar\gamma^{\rm B}+\bar\gamma^{\rm B}\bar\gamma^{\rm A}=-2\delta^{\rm AB},\quad {\rm A}=\overline{1,7}. \tag{28}$$

The Clifford–Dirac anticommutation relations for matrices (26) are similar.

The γ matrices (26), (27) generate also the representation of 29 dimensional real algebra SO(8) (the dimension of the standard Clifford–Dirac algebra SO(1,5) is 16), which was introduced in [47–51]. Note also that SO(8) from [47–51] is the algebra with complex elements over the field of real numbers. Both the fundamental representation $S^{\rm AB}=\frac{1}{4}[\gamma^{\rm A},\gamma^{\rm B}]$ and the RCQM representation $\bar S^{\rm AB}=\frac{1}{4}[\bar\gamma^{\rm A},\bar\gamma^{\rm B}]$ are generated by matrices (26) and (27), respectively. As for the structure, subalgebras, and different representations of the $Cl^{\rm R}(0,6)$ algebra, see, e.g., [51].

The additional possibilities, which are open by the 29 orts of the algebra SO(8) in comparison with 16 orts of the standard Clifford–Dirac algebra SO(1,5), are principal in the description of Bose states in the framework of the Dirac theory [47–51]. The algebra SO(8) includes two independent SU(2) subalgebras $1/2(\gamma^2\gamma^3,\,\gamma^3\gamma^1,\,\gamma^1\gamma^2)$ and $1/2(\gamma^5\gamma^6,\,\gamma^6\gamma^4,\,\gamma^4\gamma^5)$, when the standard Clifford–Dirac algebra includes only one given by the elements $(\gamma^2\gamma^3,\,\gamma^3\gamma^1,\,\gamma^1\gamma^2)$.

The subalgebra SO(6) of the algebra SO(8) (see the details in [51]) has all characteristic properties of the Clifford algebra and includes both above-mentioned SU(2) subalgebras. Moreover, all elements of SO(6) commutes with the operator of the equation of motion.

Therefore, the quantum-mechanical representation of the algebra SO(8) over the field of real numbers should be taken as the extended algebra of γ matrices of the RCQM and its subalgebra SO(6) can be useful in the role of the Clifford–Dirac algebra.

Other axioms of the relativistic description of the spin $s=(1/2,1/2)$ particle-antiparticle doublet can be found in [44].

6. General Description of the Arbitrary Spin Field Theory

6.1. Transition from the nonlocal relativistic canonical quantum mechanics to the covariant local relativistic field theory

The above-formulated RCQM has the independent meaning as a useful model for elementary particle physics. However, another application of the RCQM model has the important meaning as well. Each model of the quantum-mechanical particle singlet or multiplet considered above can be formulated also in the framework of the covariant local relativistic field theory. Moreover, it is not difficult to find the link between the RCQM and the covariant local relativistic field theory. For the partial case of spin $s = (1/2, 1/2)$ particle-antiparticle doublet, such link was already given in [44] by 4×4 matrix-differential operators v (102) multiplied by V (119) or resulting W (120)–(123) of [44]. Further, in [44] for the case of higher spins doublets and multiplets, the corresponding 6×6, 8×8, 10×10, 12×12, and 16×16 transition operators were found. Therefore, the relationship between the Schrödinger–Foldy equations for different multiplets and the Dirac equation (or the Dirac-like equations) for these multiplets has been introduced in [44]. On this basis, the general method of derivation of the different equations of the covariant local relativistic field theory is formulated in what follows. The start of such derivation is given from the corresponding Schrödinger–Foldy equations of the RCQM. Note that the derivation of the covariant particle equations from equations in the FW representation (the so-called Foldy synthesis [13]) is well known. Here, the new possibilities to fulfill the derivation of the covariant particle equations starting from the RCQM (therefore, not from the FW representation) are open. The new covariant equations of the local relativistic field theory for the spin $s = (3/2, 3/2)$, $s = (1, 0, 1, 0)$, $s = (2, 0, 2, 0)$, $s = (2, 1, 2, 1)$ particle-antiparticle doublets and multiplets are found here by this method (the explicit forms were demonstrated in [44]). The new equations for the spin $s = (1, 1)$ and $s = (2, 2)$ particle-antiparticle doublets in the FW representation are introduced as well (see [44] for details). In the general form embracing the arbitrary spin, these results are presented below.

6.2. The canonical (FW type) model of the arbitrary spin particle-antiparticle field

The step-by-step consideration of the different partial examples in Sections 21–27 of [44] enabled us to rewrite them in the general form, which is valid for arbitrary spins. Therefore, the generalization of the consideration given in Sections 21–27 of [44] leads to the general formalism of the arbitrary spin fields. The formalism presented below in this section is valid for an arbitrary particle-antiparticle multiplet in general and for the particle-antiparticle doublet in particular.

The operator, which transforms the RCQM of the arbitrary spin particle-antiparticle multiplet (Section 3) into the corresponding canonical particle-antiparticle field and *vice versa*, has the form

$$v_{2\mathrm{N}} = \begin{vmatrix} \mathrm{I_N} & 0 \\ 0 & C\mathrm{I_N} \end{vmatrix}, \quad \mathrm{N} = 2s + 1, \tag{29}$$

where C is the operator of complex conjugation. As it was explained already in Section 9 of [44], the transition with the help of operator (29) is possible for the anti-Hermitian operators. It was demonstrated [47–51] that the prime (anti-Hermitian) generators play a special role in the group-theoretic approach to quantum theory and symmetry analysis of the corresponding equations. It is due to the anti-Hermitian generators of the groups under consideration that the additional bosonic properties of the FW and Dirac equations have been found in [47–51]. The mathematical correctness of the appealing to the anti-Hermitian generators is considered in [52, 53] in detail.

Formulae (30)–(34) below are found from the corresponding formulas of the RCQM with the help of operator (29). The way of the generalization of the RCQM to the $2N$-component particle-antiparticle multiplet is given in Section 3.

For the general form of an arbitrary spin canonical particle-antiparticle field, the equation of motion of the FW type is given by

$$\left(i\partial_0 - \Gamma^0_{2\mathrm{N}}\widehat{\omega}\right)\phi(x) = 0, \; \Gamma^0_{2\mathrm{N}} \equiv \sigma^3_{2\mathrm{N}} = \begin{vmatrix} \mathrm{I_N} & 0 \\ 0 & -\mathrm{I_N} \end{vmatrix}, \tag{30}$$

The general solution is given by

$$\begin{gathered} \phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \times \\ \times \left[e^{-ikx} a^{\mathrm{N}}(\mathbf{k}) \mathrm{d_N} + e^{ikx} a^{*\check{\mathrm{N}}}(\mathbf{k}) \mathrm{d}_{\check{\mathrm{N}}}\right], \\ \mathrm{N} = 1, 2, ..., \mathrm{N}, \quad \check{\mathrm{N}} = \mathrm{N}+1, \mathrm{N}+2, ..., 2\mathrm{N}, \end{gathered} \tag{31}$$

where $a^{\mathrm{N}}(\mathbf{k})$ are the quantum-mechanical momentum-spin amplitudes of the particle, and $a^{\check{\mathrm{N}}}(\mathbf{k})$ are the quantum-mechanical momentum-spin amplitudes of the antiparticle, $\{\mathrm{d}\}$ is the $2N$-component Cartesian basis (10).

The spin operator has the form

$$\mathbf{s}_{2\mathrm{N}} = \begin{vmatrix} \mathbf{s}_\mathrm{N} & 0 \\ 0 & \mathbf{s}_\mathrm{N} \end{vmatrix}, \quad \mathrm{N} = 2s + 1, \tag{32}$$

where $\mathbf{s}_\mathrm{N}$ are the $\mathrm{N} \times \mathrm{N}$ generators of arbitrary spin irreducible representations of the SU(2) algebra, which satisfy the commutation relations (7).

The generators of the reducible unitary representation of the Poincaré group $\mathcal{P}$, with respect to which the canonical field equation (30) and the set $\{\phi\}$ of its solutions (31) are invariant, are given by

$$\widehat{p}^0 = \Gamma^0_{2\mathrm{N}}\widehat{\omega} \equiv \Gamma^0_{2\mathrm{N}}\sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell, \\ \widehat{j}^{\ell n} = x^\ell \widehat{p}^n - x^n \widehat{p}^\ell + s^{\ell n}_{2\mathrm{N}} \equiv \widehat{m}^{\ell n} + s^{\ell n}_{2\mathrm{N}}, \tag{33}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \\ - \frac{1}{2}\Gamma^0_{2\mathrm{N}}\left\{x^\ell, \widehat{\omega}\right\} + \Gamma^0_{2\mathrm{N}}\frac{(\mathbf{s}_{2\mathrm{N}} \times \mathbf{p})^\ell}{\widehat{\omega} + m}, \tag{34}$$

where arbitrary spin SU(2) generators $\mathbf{s}_{2\mathrm{N}} = (s^{\ell n}_{2\mathrm{N}})$ have the form (32), $\Gamma^0_{2\mathrm{N}}$ is given in (30).

Note that, together with generators (33) and (34), another set of 10 operators commutes with the operator of Eq. (30), satisfies the commutation relations (11) of the Lie algebra of the Poincaré group $\mathcal{P}$, and, therefore, can be chosen as the Poincaré symmetry of the model under consideration. This second set is given by the generators $\widehat{p}^0$, $\widehat{p}^\ell$ from (33) together with the orbital parts of the generators $\widehat{j}^{\ell n}$, $\widehat{j}^{0\ell}$ from (33) and (34).

The calculation of the Casimir operators $p^2 = \widehat{p}^\mu \widehat{p}_\mu$, $W = w^\mu w_\mu$ (w^μ is the Pauli–Lubanski pseudovector) for the fixed value of spin completes the brief description of the model.

6.3. The locally covariant model of the arbitrary spin particle-antiparticle field

The operator, which transforms the canonical (FW type) model of the arbitrary spin particle-antiparticle field into the corresponding locally covariant particle-antiparticle field, is a generalized FW operator and is given by

$$V^\mp = \frac{\mp \boldsymbol{\Gamma}_{2\mathrm{N}} \cdot \mathbf{p} + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega} + m)}}, \quad V^- = (V^+)^\dagger, \\ V^- V^+ = V^+ V^- = \mathrm{I}_{2\mathrm{N}}, \quad \mathrm{N} = 2s + 1, \tag{35}$$

where

$$\Gamma^j_{2\mathrm{N}} = \begin{vmatrix} 0 & \Sigma^j_{\mathrm{N}} \\ -\Sigma^j_{\mathrm{N}} & 0 \end{vmatrix}, \quad j = 1, 2, 3, \tag{36}$$

where Σ^j_{N} are the $\mathrm{N} \times \mathrm{N}$ Pauli matrices. Of course, for the matrices $\Gamma^\mu_{2\mathrm{N}}$ (30) and (36), the relations

$$\Gamma^\mu_{2\mathrm{N}}\Gamma^\nu_{2\mathrm{N}} + \Gamma^\nu_{2\mathrm{N}}\Gamma^\mu_{2\mathrm{N}} = 2g^{\mu\nu} \tag{37}$$

are valid.

Note that, in formulas (35)–(37) and before the end of the section, the values of N are only even. Therefore, the canonical field equation (30) describes the larger number of multiplets than the generalized Dirac equation (38).

Formulas (38)–(42) below are found from the corresponding formulas (30)–(34) of the canonical field model on the basis of operator (35).

For the general form of an arbitrary spin locally covariant particle-antiparticle field, the Dirac-like equation of motion follows from Eq. (30) after transformation (35) and is given by

$$\left[i\partial_0 - \Gamma^0_{2\mathrm{N}}(\boldsymbol{\Gamma}_{2\mathrm{N}} \cdot \mathbf{p} + m)\right]\psi(x) = 0. \tag{38}$$

The general solution has the form

$$\psi(x) = V^-\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k \times \\ \times \left[e^{-ikx}a^{\mathrm{N}}(\mathbf{k})\mathrm{v}^-_{\mathrm{N}}(\mathbf{k}) + e^{ikx}a^{*\breve{\mathrm{N}}}(\mathbf{k})\mathrm{v}^+_{\breve{\mathrm{N}}}(\mathbf{k})\right], \tag{39}$$

where amplitudes and the notation $\breve{\mathrm{N}}$ are the same as in (31); $\left\{\mathrm{v}^-_{\mathrm{N}}(\mathbf{k}), \mathrm{v}^+_{\breve{\mathrm{N}}}(\mathbf{k})\right\}$ are $2N$-component Dirac basis spinors [54, 55] (for $N = 8$, see formulae (357) in [44]).

The spin operator is given by

$$\mathbf{s}_{\mathrm{D}} = V^- \mathbf{s}_{2\mathrm{N}} V^+. \tag{40}$$

where the operator $\mathbf{s}_{2\mathrm{N}}$ is known from (32). The explicit forms of few spin operators (40) are given in [44] in formulae (259)–(261), (284)–(286), and (359) for the particle-antiparticle multiplets $s = (1, 0, 1, 0)$, $s = (3/2, 3/2)$, $s = (2, 1, 2, 1)$, respectively.

The generators of the reducible unitary representation of the Poincaré group $\mathcal{P}$, with respect to which the covariant field equation (38) and the set $\{\psi\}$ of its solutions (39) are invariant, have the form

$$\widehat{p}^0 = \Gamma^0_{2\mathrm{N}}(\boldsymbol{\Gamma}_{2\mathrm{N}} \cdot \mathbf{p} + m), \quad \widehat{p}^\ell = -i\partial_\ell, \\ \widehat{j}^{\ell n} = x^\ell_{\mathrm{D}} \widehat{p}^n - x^n_{\mathrm{D}} \widehat{p}^\ell + s^{\ell n}_{\mathrm{D}} \equiv \widehat{m}^{\ell n} + s^{\ell n}_{\mathrm{D}}, \tag{41}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \\ - \frac{1}{2}\left\{x^\ell_{\mathrm{D}}, \widehat{p}^0\right\} + \frac{\widehat{p}^0(\mathbf{s}_{\mathrm{D}} \times \mathbf{p})^\ell}{\widehat{\omega}(\widehat{\omega} + m)}, \tag{42}$$

where the spin matrices $\mathbf{s}_{\mathrm{D}} = (s^{\ell n}_{\mathrm{D}})$ are given in (40), and the operator $\mathbf{x}_{\mathrm{D}}$ has the form

$$\mathbf{x}_{\mathrm{D}} = \mathbf{x} + \frac{i\boldsymbol{\Gamma}_{2\mathrm{N}}}{2\widehat{\omega}} - \frac{\mathbf{s}^\Gamma_{2\mathrm{N}} \times \mathbf{p}}{\widehat{\omega}(\widehat{\omega} + m)} - \frac{i\mathbf{p}(\boldsymbol{\Gamma}_{2\mathrm{N}} \cdot \mathbf{p})}{2\widehat{\omega}^2(\widehat{\omega} + m)}, \tag{43}$$

where the spin matrices $\mathbf{s}^\Gamma_{2\mathrm{N}} = \frac{i}{2}(\Gamma^2_{2\mathrm{N}}\Gamma^3_{2\mathrm{N}}, \Gamma^3_{2\mathrm{N}}\Gamma^1_{2\mathrm{N}}, \Gamma^1_{2\mathrm{N}}\Gamma^2_{2\mathrm{N}})$.

The last step in the brief description of the model is the calculation of the Casimir operators $p^2 = \widehat{p}^\mu \widehat{p}_\mu$, $W = w^\mu w_\mu$ (w^μ is the Pauli–Lubanski pseudovector) for the fixed value of spin.

6.4. The example of spin $s = (0, 0)$ particle-antiparticle doublet

The completeness of the consideration of the simplest spin multiplets and doublets [44] is achieved by the presentation of this example. The formalism follows from the general formalism of arbitrary spin after the substitution $s = 0$.

The Schrödinger–Foldy equation of the RCQM is the same as in (17) of [44]. The solution is the same as in (22) of [44]. The generators of the Poincaré group $\mathcal{P}$, with respect to which Eq. (17) from [44] for $s = (0, 0)$ is invariant, are given by (12) and (13) taken in the form of 2×2 matrices with spin terms equal to zero.

The corresponding FW-type equation of the canonical field theory is given by

$$\left(i\partial_0 - \sigma^3\widehat{\omega}\right)\phi(x) = 0, \quad \sigma^3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}. \tag{44}$$

The general solution is given by

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k \times \\ \times \left[e^{-ikx}a^1(\mathbf{k})\mathrm{d}_1 + e^{ikx}a^{*2}(\mathbf{k})\mathrm{d}_2\right]. \tag{45}$$

The generators of the Poincaré group $\mathcal{P}$, with respect to which Eq. (44) and the set $\{\phi\}$ of its solutions (45) are invariant, have the form

$$\widehat{p}^0 = \sigma^3\widehat{\omega} \equiv \sigma^3\sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell, \\ \widehat{j}^{\ell n} = x^\ell\widehat{p}^n - x^n\widehat{p}^\ell, \tag{46}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0\widehat{p}^\ell - \frac{1}{2}\sigma^3\left\{x^\ell, \widehat{\omega}\right\}. \tag{47}$$

Generators (46) and (47) are the partial 2×2 matrix form of operators (33) and (34) taken with the spin terms equal to zero.

Below, the validity of the general formalism under consideration is demonstrated on the important example of spin $s = 3/2$.

7. A Brief Scheme of the Relativistic Canonical Quantum Mechanics of the Single Spin $s = 3/2$ Fermion

The Schrödinger–Foldy equation is given by

$$i\partial_t f(x) = \sqrt{m^2 - \Delta}f(x), \quad f = \begin{vmatrix} f^1 \\ f^2 \\ f^3 \\ f^4 \end{vmatrix}. \tag{48}$$

The space of states is as follows $\mathrm{S}^{3,4} \subset \mathrm{H}^{3,4} \subset \mathrm{S}^{3,4*}$. The generators of the SU(2)-spin in the most spread explicit form are given by

$$s^1 = \frac{1}{2}\begin{vmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{vmatrix}, \\ s^2 = \frac{i}{2}\begin{vmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{vmatrix}, \tag{49}$$

$s^3 = \frac{1}{2}\mathrm{diag}(3, 1, -1, -3)$. It is easy to verify that the commutation relations $\left[s^j, s^\ell\right] = i\varepsilon^{j\ell n}s^n$ of the SU(2)-algebra are valid. The Casimir operator for this representation of the SU(2)-algebra is given by $\mathbf{s}^2 = \frac{15}{4}\mathrm{I}_4 = \frac{3}{2}\left(\frac{3}{2} + 1\right)\mathrm{I}_4$, where I_4 is a 4×4 unit matrix.

The general solution of the Schrödinger–Foldy equation (48) is given by

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k e^{-ikx}b^{\check{\alpha}}(\mathbf{k})\mathrm{d}_{\check{\alpha}}, \tag{50}$$

$\check{\alpha} = 1, 2, 3, 4$, $\{\mathrm{d}_{\check{\alpha}}\}$ are the orts of the 4-dimensional Cartesian basis (10) and notations (9) are used. Solution (50) is associated with the stationary complete set $\mathbf{p}$, $s^3 = s_z$ of the momentum and spin projection operators of a spin $s = 3/2$ fermion, respectively.

The equations for the spin projection operator $s^3 = \frac{1}{2}\mathrm{diag}(3, 1, -1, -3)$ eigenvalues are given by

$$s^3\mathrm{d}_1 = \frac{3}{2}\mathrm{d}_1,\ s^3\mathrm{d}_2 = \frac{1}{2}\mathrm{d}_2, \\ s^3\mathrm{d}_3 = -\frac{1}{2}\mathrm{d}_3,\ s^3\mathrm{d}_4 = -\frac{3}{2}\mathrm{d}_4. \tag{51}$$

The interpretation of the amplitudes $b^\alpha(\mathbf{k})$ in (50) follows from Eqs. (51) and similar equations for the operator $\mathbf{p}$ eigenvalues. The functions $b^1(\mathbf{k})$, $b^2(\mathbf{k})$, $b^3(\mathbf{k}$, $b^4(\mathbf{k})$ are the quantum-mechanical momentum-spin amplitudes of a fermion with the spin projection eigenvalues $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{3}{2}$, respectively.

The Schrödinger–Foldy equation (48) and the set $\{\mathrm{f}\}$ of its solutions (50) are invariant with respect to the irreducible unitary spin $s = 3/2$ representation (14) of the Poincaré group $\mathcal{P}$. The corresponding 4×4 matrix-differential generators are given by (12) and (13), where the spin 3/2 SU(2) generators $\mathbf{s} = (s^{\ell n})$ are given in (49).

The validity of this assertion is verified by three steps already explained in Section 2 after formula (14). The corresponding Casimir operators have the form $p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_4$, $W = w^\mu w_\mu = m^2 \mathbf{s}^2 = \frac{3}{2}\left(\frac{3}{2}+1\right) m^2 \mathrm{I}_4$.

8. A Brief Scheme of the Relativistic Canonical Quantum Mechanics of the 8-Component Fermionic Spin $s = (3/2, 3/2)$ Particle-Antiparticle Doublet

This model is constructed in the complete analogy with the RCQM of the 4-component spin $s = (1/2, 1/2)$ particle-antiparticle doublet, which is given in Section 7 of [44] in details (see some important addition in Section 5). Moreover, the principles of constructing and describing such particle-antiparticle multiplet within the framework of the RCQM are in a complete analogy with the principles of description and construction of the spin $s = (1, 1)$ particle-antiparticle doublet considered in Section 11 of [44]. The difference is only in the dimensions of the corresponding spaces and matrices. Therefore, the details can be omitted. The model below is useful for the Σ-hyperon description.

The 8-component fermionic spin $s = (3/2, 3/2)$ particle-antiparticle doublet is constructed as a direct sum of two spin $s = 3/2$ singlets. The spin $s = 3/2$ singlet was considered above in Section 7.

The most important fact is that here the link with the Dirac-like equation is similar to that between the spin $s = (1/2, 1/2)$ particle-antiparticle doublet and the standard 4-component Dirac equation, which was demonstrated in Sections 9 and 10 of [44]. Therefore, the spin $s = (3/2, 3/2)$ particle-antiparticle doublet is of special interest.

The Schrödinger–Foldy equation and the space of states are given by

$$(i\partial_0 - \widehat{\omega}) f(x) = 0, \; f = \text{column}\left(f^1, f^2, ..., f^8\right), \quad (52)$$

$\mathrm{S}^{3,8} \subset \mathrm{H}^{3,8} \subset \mathrm{S}^{3,8*}$. The general solution of Eq. (52) for the spin $s = (3/2, 3/2)$ particle-antiparticle doublet is given by

$$f(x) = \begin{vmatrix} f_{\text{part}} \\ f_{\text{antipart}} \end{vmatrix} = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{-ikx} b^{\mathrm{A}}(\mathbf{k}) \mathrm{d}_{\mathrm{A}}, \quad (53)$$

$\mathrm{A} = \overline{1, 8}$, $\{\mathrm{d}_{\mathrm{A}}\}$ are the orts of the 8-component Cartesian basis (10) and the amplitudes $b^{\mathrm{A}}(\mathbf{k})$ correspond to the spin $s = (3/2, 3/2)$ particle-antiparticle doublet.

The generators of the corresponding SU(2)-spin that satisfy the commutation relations (7) of the SU(2) algebra are as follows:

$$\mathbf{s}_8 = \begin{vmatrix} \mathbf{s} & 0 \\ 0 & -C\mathbf{s}C \end{vmatrix}, \quad (54)$$

where $C\mathrm{I}_4$ is the diagonal 4×4 operator of complex conjugation, and the matrices $\mathbf{s}$ for the single spin $s = 3/2$ particle are given in (49). In the explicit form, the SU(2) spin operators (54) are given by formulae (179) in [44]. The Casimir operator has the form of the 8×8 diagonal matrix $\mathbf{s}^2 = \frac{15}{4}\mathrm{I}_8 = \frac{3}{2}\left(\frac{3}{2}+1\right)\mathrm{I}_8$, where I_8 is the 8×8 unit matrix.

The stationary complete set of operators is given by

$$g = \begin{vmatrix} -\mathrm{I}_4 & 0 \\ 0 & \mathrm{I}_4 \end{vmatrix}, \quad p^j = -i\partial_j, \quad (55)$$

$s_8^3 = \frac{1}{2}\mathrm{diag}(3, 1, -1, -3, -3, -1, 1, 3)$, where g is the charge sign operator, $\mathbf{p} = (p^j)$ is the momentum operator, and $s_8^3 = s_z$ is the operator of the spin (54) projection on the axis z.

The equations for eigenvalues of the operators g, $s_8^3 = s_z$ have the form

$$g\mathrm{d}_1 = -\mathrm{d}_1,\; g\mathrm{d}_2 = -\mathrm{d}_2,\; g\mathrm{d}_3 = -\mathrm{d}_3,\; g\mathrm{d}_4 = -\mathrm{d}_4,$$
$$g\mathrm{d}_5 = +\mathrm{d}_5,\; g\mathrm{d}_6 = +\mathrm{d}_6,\; g\mathrm{d}_7 = +\mathrm{d}_7,\; g\mathrm{d}_8 = +\mathrm{d}_8, \quad (56)$$

$$s_8^3\mathrm{d}_1 = \frac{3}{2}\mathrm{d}_1,\; s_8^3\mathrm{d}_2 = \frac{1}{2}\mathrm{d}_2,\; s_8^3\mathrm{d}_3 = -\frac{1}{2}\mathrm{d}_3,$$
$$s_8^3\mathrm{d}_4 = -\frac{3}{2}\mathrm{d}_4,\; s_8^3\mathrm{d}_5 = -\frac{3}{2}\mathrm{d}_5,\; s_8^3\mathrm{d}_6 = -\frac{1}{2}\mathrm{d}_6, \quad (57)$$
$$s_8^3\mathrm{d}_7 = \frac{1}{2}\mathrm{d}_7,\; s_8^3\mathrm{d}_8 = \frac{3}{2}\mathrm{d}_8.$$

The equations for eigenvalues of the momentum operator $\mathbf{p}$ can be found in [44].

Therefore, the functions $b^1(\mathbf{k})$, $b^2(\mathbf{k})$, $b^3(\mathbf{k})$, $b^4(\mathbf{k})$ in solution (53) are the momentum-spin amplitudes of a massive fermion with the spin $s = 3/2$ and the spin projection $(3/2, 1/2, -1/2, -3/2)$, respectively; $b^5(\mathbf{k})$, $b^6(\mathbf{k})$, $b^7(\mathbf{k})$, $b^8(\mathbf{k})$ are the momentum-spin amplitudes of the antiparticle (antifermion) with the spin $s = 3/2$ and the spin projection $(-3/2, -1/2, 1/2, 3/2)$, respectively.

In addition to the bosonic $\mathcal{P}$ invariance, which was considered in Section 13 in [44], the Schrödinger–Foldy equation (52) (and the set {f} of its solutions

(53)) is invariant with respect to the reducible unitary fermionic representation (14) of the Poincaré group $\mathcal{P}$, whose Hermitian 8×8 matrix-differential generators are given by (12) and (13), where the spin $s=(3/2,3/2)$ SU(2) generators $\mathbf{s}=(s^{\ell n})$ are given in (54).

The proof is similar to that given in Section 2 after formula (14). The Casimir operators of this reducible fermionic spin $s=(3/2,3/2)$ representation of the group $\mathcal{P}$ have the form $p^2=\widehat{p}^{\mu}\widehat{p}_{\mu}=m^2\mathrm{I}_8$, $W==w^{\mu}w_{\mu}=m^2\mathbf{s}_8^2=\frac{3}{2}\left(\frac{3}{2}+1\right)m^2\mathrm{I}_8$.

9. Covariant Field Equation for the 8-Component Spin $s=(3/2,3/2)$ Fermionic Particle-Antiparticle Doublet

The model is constructed in a complete analogy with the consideration in Section 22 of [44].

The start of this derivation is given in Section 8 above, where the RCQM of the 8-component fermionic spin $s=(3/2,3/2)$ particle-antiparticle doublet is considered. The second step is the transition from the Schrödinger–Foldy equation (52) to the canonical field equation. This step, as shown in Section 6, is possible only for the anti-Hermitian form of the operators. Nevertheless, the resulting operators can be chosen in the standard Hermitian form and do not contain the operator C of complex conjugation. The last step of the transition from the canonical field equation to the covariant local field equation is fulfilled in analogy with the FW transformation [2] (transition operator (35) for $N=4$).

Thus, the canonical field equation for the 8-component spin $s=(3/2,3/2)$ fermionic particle-antiparticle doublet (8-component analogy of the FW equation) is found from the Schrödinger–Foldy equation (52) on the basis of the transformation v_8:

$$v_8=\begin{vmatrix}\mathrm{I}_4 & 0\\ 0 & C\mathrm{I}_4\end{vmatrix},\quad v_8^{-1}=v_8^{\dagger}=v_8,\quad v_8v_8=\mathrm{I}_8;\quad \phi=v_8f,$$
$$f=v_8\phi;\quad v_8\hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}}v_8=\hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}},$$
$$v_8\hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}}v_8=\hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}},$$

and is given by

$$(i\partial_0-\Gamma_8^0\widehat{\omega})\phi(x)=0,\quad \phi=\mathrm{column}(\phi^1,\phi^2,...,\phi^8),\tag{58}$$

where

$$\Gamma_8^0=\begin{vmatrix}\mathrm{I}_4 & 0\\ 0 & -\mathrm{I}_4\end{vmatrix}.\tag{59}$$

Above in the definition of the operator v_8, $\hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}}$ is an arbitrary operator from the RCQM of the 8-component particle-antiparticle doublet in the anti-Hermitian form, e.g., the operator $(\partial_0+i\widehat{\omega})$ of the equation of motion, the operator of spin (54), *etc.*, $\hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}}$ is an arbitrary operator from the canonical field theory of the 8-component particle-antiparticle doublet in the anti-Hermitian form, and $C\mathrm{I}_4$ is the 4×4 operator of complex conjugation.

The general solution of Eq. (58) in the case of spin $s=(3/2,3/2)$ fermionic particle-antiparticle doublet is found with the help of the above-given transformation v_8 from the general solution (53) of the Schrödinger–Foldy equation (52) and is given by

$$\phi(x)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\,\times$$
$$\times\left[e^{-ikx}b^{\mathrm{A}}(\mathbf{k})\mathrm{d}_{\mathrm{A}}+e^{ikx}b^{*\mathrm{B}}(\mathbf{k})\mathrm{d}_{\mathrm{B}}\right],\tag{60}$$

where $\mathrm{A}=\overline{1,4}$, $\mathrm{B}=\overline{5,8}$, the orts of the 8-component Cartesian basis are given in (10) with $N=8$, and the quantum-mechanical interpretation of the amplitudes $(b^{\mathrm{A}}(\mathbf{k}),\,b^{*\mathrm{B}}(\mathbf{k}))$ is given according to (56) and (57).

The SU(2) spin operators, which satisfy the commutation relations (7) and commute with the operator $(i\partial_0-\Gamma_8^0\widehat{\omega})$ of the equation of motion (58), are derived from the corresponding RCQM operators (54) on the basis of the transformations v_8. These canonical field spin operators are given by

$$\mathbf{s}_8=\begin{vmatrix}\mathbf{s} & 0\\ 0 & \mathbf{s}\end{vmatrix},\quad \mathbf{s}_8^2=\frac{3}{2}\left(\frac{3}{2}+1\right)\mathrm{I}_8,\tag{61}$$

where the 4×4 operators $\mathbf{s}$ are given in (49). In the explicit form, the SU(2) spin operators (61) are given by formulae (279) in [44].

The stationary complete set of operators is given by the operators $g,\mathbf{p}$, $s_8^3=s_z$ of the charge sign, momentum, and spin projection, respectively (see Section 8 for details). The equations for eigenvectors and eigenvalues of the spin projection operator $s_8^3=s_z$ from (61) have the form

$$\begin{aligned}&s_8^3\mathrm{d}_1=\frac{3}{2}\mathrm{d}_1,\ s_8^3\mathrm{d}_2=\frac{1}{2}\mathrm{d}_2,\ s_8^3\mathrm{d}_3=-\frac{1}{2}\mathrm{d}_3,\\ &s_8^3\mathrm{d}_4=-\frac{3}{2}\mathrm{d}_4, s_8^3\mathrm{d}_5=\frac{3}{2}\mathrm{d}_5,\ s_8^3\mathrm{d}_6=\frac{1}{2}\mathrm{d}_6,\\ &s_8^3\mathrm{d}_7=-\frac{1}{2}\mathrm{d}_7,\ s_8^3\mathrm{d}_8=-\frac{3}{2}\mathrm{d}_8.\end{aligned}\tag{62}$$

Therefore, the functions $b^1(\mathbf{k})$, $b^2(\mathbf{k})$, $b^3(\mathbf{k})$, and $b^4(\mathbf{k})$ in solution (60) are the momentum-spin amplitudes of a massive fermion with the spin $s = 3/2$ and the spin projection $(3/2, 1/2, -1/2, -3/2)$, respectively; $b^5(\mathbf{k})$, $b^6(\mathbf{k})$, $b^7(\mathbf{k})$, and $b^8(\mathbf{k})$ are the momentum-spin amplitudes of the antiparticle (antifermion) with the spin $s = 3/2$ and the spin projection $(3/2, 1/2, -1/2, -3/2)$, respectively.

Note that the direct quantum-mechanical interpretation of the amplitudes in solution (60) should be given in the framework of the RCQM. Such interpretation is already given in Section 8 after Eqs. (57).

The generators of the reducible unitary fermionic spin $s = (3/2, 3/2)$ doublet representation of the Poincaré group $\mathcal{P}$, with respect to which the canonical field equation (58) and the set $\{\phi\}$ of its solutions (60) are invariant, are derived from the RCQM set of generators (12) and (13) with spin (54) on the basis of the transformations v_8. These Hermitian 8×8 matrix-differential generators are given by

$$\widehat{p}^0 = \Gamma_8^0\widehat{\omega} \equiv \Gamma_8^0\sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell,$$
$$\widehat{j}^{\ell n} = x^\ell\widehat{p}^n - x^n\widehat{p}^\ell + s_8^{\ell n} \equiv \widehat{m}^{\ell n} + s_8^{\ell n}, \qquad (63)$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0\widehat{p}^\ell - $$
$$-\frac{1}{2}\Gamma_8^0\left\{x^\ell, \widehat{\omega}\right\} + \Gamma_8^0\frac{(\mathbf{s}_8\times\mathbf{p})^\ell}{\widehat{\omega}+m}, \qquad (64)$$

where the explicit form of SU(2) spin $s = (3/2, 3/2)$ is given in (61).

It is easy to prove by the direct verification that generators (63) and (64) commute with the operator $(i\partial_0 - \Gamma_8^0\widehat{\omega})$ of the canonical field equation (58) and satisfy the commutation relations (11) of the Lie algebra of the Poincaré group $\mathcal{P}$. The Casimir operators for representation (63), (64) with SU(2) spin (61) are given by $p^2 = \widehat{p}^\mu\widehat{p}_\mu = m^2 \mathrm{I}_8$, $W = w^\mu w_\mu = m^2\mathbf{s}_8^2 = \frac{3}{2}\left(\frac{3}{2}+1\right)\mathrm{I}_8$.

Thus, due to the eigenvalues in Eqs. (62), forms of solution (60) with positive and negative frequencies, and the above-given Bargmann–Wigner analysis of the Casimir operators, one can come to a conclusion that Eq. (58) describes the 8-component canonical field (the fermionic particle-antiparticle doublet) with the spins $s = (3/2, 3/2)$ and $m > 0$.

The operator of the transition to the covariant local field theory representation (the 8×8 analogy of the 4×4 FW transformation operator [2]) is given by: $V_8^{\mp} = \frac{\mp\boldsymbol{\Gamma}_8\cdot\mathbf{p}+\widehat{\omega}+m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}}$, $V_8^- = (V_8^+)^\dagger$, $V_8^- V_8^+ = V_8^+ V_8^- = \mathrm{I}_8$, $\psi = V_8^-\phi$, $\phi = V_8^+\psi$; $\hat{q}_{\mathrm{D}} = V_8^-\hat{q}_{\mathrm{CF}}V_8^+$, $\hat{q}_{\mathrm{CF}} = V_8^+\hat{q}_{\mathrm{D}}V_8^-$, where $\hat{q}_{\mathrm{D}}$ is an arbitrary operator (both in the Hermitian and anti-Hermitian form) in the covariant local field theory representation. The inverse transformation is valid as well. Thus, on the basis of the transformation V_8^-, the 8-component Dirac-like equation is found from the canonical field equation (58) in the form

$$\left[i\partial_0 - \Gamma_8^0(\boldsymbol{\Gamma}_8\cdot\mathbf{p} + m)\right]\psi(x) = 0. \qquad (65)$$

In the formulae for the operator transformation V_8^- and in Eq. (65), the Γ_8^μ matrices are given by

$$\Gamma_8^0 = \begin{vmatrix} \mathrm{I}_4 & 0 \\ 0 & -\mathrm{I}_4 \end{vmatrix}, \quad \Gamma_8^j = \begin{vmatrix} 0 & \Sigma^j \\ -\Sigma^j & 0 \end{vmatrix}, \qquad (66)$$

where Σ^j are the 4×4 Pauli matrices

$$\Sigma^j = \begin{vmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{vmatrix}, \qquad (67)$$

and σ^j are the standard 2×2 Pauli matrices. The matrices Σ^j satisfy the similar commutation relations as the standard 2×2 Pauli matrices and have other similar properties. The matrices Γ_8^μ (66) satisfy the anticommutation relations of the Clifford–Dirac algebra in the form (37) with $N = 4$.

Note that Eq. (65) is not the ordinary direct sum of two Dirac equations. Therefore, it is not the complex Dirac–Kähler equation [56]. Moreover, it is not the standard Dirac–Kähler equation [57].

The solution of Eq. (65) is derived from solution (60) of this equation in the canonical representation (58) on the basis of the transformation V_8^- and is given by

$$\psi(x) = V_8^-\phi(x) = \frac{1}{(2\pi)^{3/2}}\int d^3k \times$$
$$\times\left[e^{-ikx}b^{\mathrm{A}}(\mathbf{k})\mathrm{v}_{\mathrm{A}}^-(\mathbf{k}) + e^{ikx}b^{*\mathrm{B}}(\mathbf{k})\mathrm{v}_{\mathrm{B}}^+(\mathbf{k})\right], \qquad (68)$$

where $\mathrm{A} = \overline{1,4}$, $\mathrm{B} = \overline{5,8}$, and the 8-component spinors $(\mathrm{v}_{\mathrm{A}}^-(\mathbf{k}), \mathrm{v}_{\mathrm{B}}^+(\mathbf{k}))$ are given by (257) in [44].

The spinors $(\mathrm{v}_{\mathrm{A}}^-(\mathbf{k}), \mathrm{v}_{\mathrm{B}}^+(\mathbf{k}))$ are derived from the orts $\{\mathrm{d}_\alpha\}$ of the Cartesian basis (10) for the case of $N = 8$ with the help of the transformation V_8^-. The spinors $(\mathrm{v}_{\mathrm{A}}^-(\mathbf{k}), \mathrm{v}_{\mathrm{B}}^+(\mathbf{k}))$ satisfy the relations of orthonormalization and completeness similar to the corresponding relations for the standard 4-component Dirac spinors (see, e.g., [55]).

The direct quantum-mechanical interpretation of the amplitudes in solution (68) should be given in the framework of the RCQM. Such interpretation is already given in Section 8 after Eqs. (57).

In the covariant local field theory, the operators of the SU(2) spin, which satisfy the corresponding commutation relations $[s^j_{8\mathrm{D}}, s^\ell_{8\mathrm{D}}] = i\varepsilon^{j\ell n} s^n_{8\mathrm{D}}$ and commute with the operator $[i\partial_0 - \Gamma^0_8(\mathbf{\Gamma}_8 \cdot \mathbf{p} + m)]$ of Eq. (65), are derived from the pure matrix operators (61) with the help of the transition operator V^-_8: $\mathbf{s}_{8\mathrm{D}} = V^-_8 \mathbf{s}_8 V^+_8$. The explicit form of these $s = (3/2, 3/2)$ SU(2) generators was given already by formulae (284)–(287) in [44].

The equations for eigenvectors and eigenvalues of the operator $s^3_{8\mathrm{D}}$ (286) in [44] follow from Eqs. (62) and the transformation V^-_8. In addition, the action of the operator $s^3_{8\mathrm{D}}$ (formula (286) in [44]) on the spinors $(\mathrm{v}^-_\mathrm{A}(\mathbf{k}), \mathrm{v}^+_\mathrm{B}(\mathbf{k}))$ (257) in [44] also leads to the result

$$\begin{aligned} &s^3_{8\mathrm{D}}\mathrm{v}^-_1(\mathbf{k}) = \frac{3}{2}\mathrm{v}^-_1(\mathbf{k}),\ s^3_{8\mathrm{D}}\mathrm{v}^-_2(\mathbf{k}) = \frac{1}{2}\mathrm{v}^-_2(\mathbf{k}),\\ &s^3_{8\mathrm{D}}\mathrm{v}^-_3(\mathbf{k}) = -\frac{1}{2}\mathrm{v}^-_3(\mathbf{k}),\ s^3_{8\mathrm{D}}\mathrm{v}^-_4(\mathbf{k}) = -\frac{3}{2}\mathrm{v}^-_4(\mathbf{k}),\\ &s^3_{8\mathrm{D}}\mathrm{v}^+_5(\mathbf{k}) = \frac{3}{2}\mathrm{v}^+_5(\mathbf{k}),\ s^3_{8\mathrm{D}}\mathrm{v}^+_6(\mathbf{k}) = \frac{1}{2}\mathrm{v}^+_6(\mathbf{k}),\\ &s^3_{8\mathrm{D}}\mathrm{v}^+_7(\mathbf{k}) = -\frac{1}{2}\mathrm{v}^+_7(\mathbf{k}),\ s^3_{8\mathrm{D}}\mathrm{v}^+_8(\mathbf{k}) = -\frac{3}{2}\mathrm{v}^+_8(\mathbf{k}). \end{aligned} \tag{69}$$

In order to verify Eqs. (69), the identity $(\tilde{\omega}+m)^2 + (\mathbf{k})^2 = 2\tilde{\omega}(\tilde{\omega} + m)$ is used. In the case $\mathrm{v}^+_\mathrm{B}(\mathbf{k})$, the substitution $\mathbf{k} \to -\mathbf{k}$ is made in the expression $s^3_{8\mathrm{D}}(\mathbf{k})$ (formula (286) of [44]).

Equations (69) determine the interpretation of the amplitudes in solution (68). This interpretation is similar to the given above after Eqs. (62). Nevertheless, the direct quantum-mechanical interpretation of amplitudes should be made in the framework of the RCQM (see Section 8).

The explicit form of the $\mathcal{P}$-generators of the fermionic representation of the Poincaré group $\mathcal{P}$, with respect to which the covariant equation (65) and the set $\{\psi\}$ of its solutions (68) are invariant, is derived from generators (63) and (64) with the SU(2) spin (61) on the basis of the transformation V^-_8. The corresponding generators are given by

$$\widehat{p}^0 = \Gamma^0_8(\mathbf{\Gamma}_8 \cdot \mathbf{p} + m), \quad \widehat{p}^\ell = -i\partial_\ell, \tag{70}$$

$$\begin{aligned} &\widehat{j}^{\ell n} = x^\ell_\mathrm{D}\widehat{p}^n - x^n_\mathrm{D}\widehat{p}^\ell + s^{\ell n}_{8\mathrm{D}} \equiv \widehat{m}^{\ell n} + s^{\ell n}_{8\mathrm{D}},\\ &\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0\widehat{p}^\ell - \frac{1}{2}\left\{x^\ell_\mathrm{D}, \widehat{p}^0\right\} + \frac{\widehat{p}^0(\mathbf{s}_{8\mathrm{D}} \times \mathbf{p})^\ell}{\widehat{\omega}(\widehat{\omega} + m)}, \end{aligned} \tag{71}$$

where the spin matrices $\mathbf{s}_{8\mathrm{D}} = (s^{\ell n}_{8\mathrm{D}})$ are given by (284)–(286) in [44], and the operator $\mathbf{x}_\mathrm{D}$ has the form

$$\mathbf{x}_\mathrm{D} = \mathbf{x} + \frac{i\mathbf{\Gamma}_8}{2\widehat{\omega}} - \frac{\mathbf{s}^\Gamma_8 \times \mathbf{p}}{\widehat{\omega}(\widehat{\omega} + m)} - \frac{i\mathbf{p}(\mathbf{\Gamma}_8 \cdot \mathbf{p})}{2\widehat{\omega}^2(\widehat{\omega} + m)}, \tag{72}$$

with the spin matrices $\mathbf{s}_8 = \frac{i}{2}(\Gamma^2_8\Gamma^3_8, \Gamma^3_8\Gamma^1_8, \Gamma^1_8\Gamma^2_8)$.

It is easy to verify that generators (71) and (72) with the SU(2) spin (formulae (284)–(286) from [44]) commute with the operator $\left[i\partial_0 - \Gamma^0_8(\mathbf{\Gamma}_8 \cdot \mathbf{p} + m)\right]$ of Eq. (65), satisfy the commutation relations (11) of the Lie algebra of the Poincaré group, and the corresponding Casimir operators are given by $p^2 = \widehat{p}^\mu\widehat{p}_\mu = m^2\mathrm{I}_8$, $W = w^\mu w_\mu = m^2\mathbf{s}^2_{8\mathrm{D}} = \frac{3}{2}\left(\frac{3}{2} + 1\right)m^2\mathrm{I}_8$.

As was already explained in details in the previous sections, the conclusion that Eq. (65) describes the local field of a fermionic particle-antiparticle doublet of the spin $s = (3/2, 3/2)$ and mass $m > 0$ (and its solution (68) is the local fermionic field of the above-mentioned spin and nonzero mass) follows from the analysis of Eqs. (69) and the above-given calculation of the Casimir operators p^2, $W = w^\mu w_\mu$. Hence, Eq. (65) describes the spin $s = (3/2, 3/2)$ particle-antiparticle doublet on the same level, on which the standard 4-component Dirac equation describes the spin $s = (1/2, 1/2)$ particle-antiparticle doublet. Moreover, the external argument in the validity of such interpretation is the link with the corresponding RCQM of spin $s = (3/2, 3/2)$ particle-antiparticle doublet, where the quantum-mechanical interpretation is direct and evident. Therefore, the fermionic spin $s = (3/2, 3/2)$ properties of Eq. (65) are proved.

Contrary to the bosonic spin $s = (1, 0, 1, 0)$ properties of Eq. (65) found in [44] (Section 22), the fermionic spin $s = (1/2, 1/2, 1/2, 1/2)$ properties of this equation are evident. The fact that Eq. (65) describes the multiplet of two fermions with the spin $s = 1/2$ and two antifermions with that spin can be proved much more easier than the above-given consideration. The proof is similar to that given in the standard 4-component Dirac model. The detailed consideration can be found in Sections 7, 9, and 10 of [44]. Therefore, Eq. (65) has more extended property of the Fermi–Bose duality than the standard Dirac equation [47–51]. This equation has the property of the Fermi–Bose triality. The property of the Fermi–Bose triality of the manifestly covariant equation (65) means that this equation describes on equal level (i) the spin $s = (1/2, 1/2, 1/2, 1/2)$ multiplet of two spin $s = (1/2, 1/2)$ fermions and two spin $s = (1/2, 1/2)$

antifermions, (ii) the spin $s = (1, 0, 1, 0)$ multiplet of the vector and scalar bosons together with their antiparticles, (iii) the spin $s = (3/2, 3/2)$ particle-antiparticle doublet.

It is evident that Eq. (65) is new in comparison with the Pauli–Fierz [58], Rarita–Schwinger [59], and Davydov [60] equations for the spin $s = 3/2$ particle. Contrary to the 16-component equations from [58–60], Eq. (65) is 8-component and does not need any additional condition. Formally, Eq. (65) has likely some similar features with the Bargmann–Wigner equation [61] for arbitrary spin, when the spin value is taken 3/2. The transformation $V_8^{\mp} = \frac{\mp\boldsymbol{\Gamma}_8\cdot\mathbf{p}+\widehat{\omega}+m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}}$ looks like the transformation of Pursey [62] in the case of $s = 3/2$. Nevertheless, the difference is clear. The model given here is derived from the first principles of the RCQM (not from the FW-type representation of the canonical field theory). Our consideration is original and new. The link with the corresponding RCQM, the proof of the symmetry properties and relativistic invariance, the well-defined spin operator (284–286) in [44], the features of the Fermi–Bose duality (triality) of Eq. (65), the interaction with an electromagnetic field, and many other characteristics are suggested firstly.

The interaction, quantization, and Lagrange approach in the above-given spin $s = (3/2, 3/2)$ model are completely similar to the Dirac 4-component theory and standard quantum electrodynamics. For example, the Lagrange function of the system of an interacting 8-component spinor and the electromagnetic field (in the terms of the 4-vector potential $A^{\mu}(x)$) is given by

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}\left(\overline{\psi}(x)\Gamma_8^{\mu}\frac{\partial\psi(x)}{\partial x^{\mu}} - \frac{\partial\overline{\psi}(x)}{\partial x^{\mu}}\Gamma_8^{\mu}\psi(x)\right) - - m\overline{\psi}(x)\psi(x) + q\overline{\psi}(x)\Gamma_8^{\mu}\psi(x)A_{\mu}(x), \quad (73)$$

where $\overline{\psi}(x)$ is the independent Lagrange variable, and $\overline{\psi} = \psi^{\dagger}\Gamma_8^0$ in the space of solutions $\{\psi\}$. In Lagrangian (73), $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the electromagnetic field tensor in terms of the potentials, which play the role of variational variables in this Lagrange approach.

Therefore, the covariant local quantum field theory model for the interacting particles with spin $s = 3/2$ and photons can be constructed in a complete analogy to the construction of the modern quantum electrodynamics. This model can be useful for the investigations of the processes with interacting hyperons and photons.

10. Interaction

In general, this paper is about free noninteracting fields and particle states. Note, at first, that the free noninteracting fields and particle states are the physical reality of the same level as the interacting fields and the corresponding particle states. Nevertheless, this means that the interaction between a fields can be easily introduced on the every step of consideration. One test model with interaction is considered in explicit form in Section 9. The interaction cannot be a deficiency in these constructions and can be introduced in many places by the method similar to formula (73).

11. Discussions and Conclusions

In the presented article, our experience [3, 40, 47–51, 63–65] in the time span 2002–2014 in the investigation of the spin $s = 1/2$ and $s = 1$ fields is applied for the first time to the higher spin cases $s = 3/2$ and $s = 2$ (in the form of electronic preprint, it has been presented for the first time in [44]). Thus, our "old" papers are augmented by the list of new results for higher spins and the generalization to arbitrary spins. Moreover, the system of different vertical and horizontal links between the descriptions of particles with arbitrary spin on the levels of relativistic quantum mechanics, canonical field theory (of the FW type), and locally covariant field theory is suggested.

Among the results of this paper, the original method of derivation of the Dirac (and the Dirac-like equations for higher spins) is suggested. In order to determine the place of this derivation among the other known methods given in [3], the different ways of the derivation of the Dirac equation [66–83] have been reviewed (for completeness, we mention additionally works [84–86]). Thus, a review of the different derivations of the Dirac equation demonstrates that the general method presented in Section 6 is original and new. Here, the Dirac equation is derived from the 4-component Schrödinger–Foldy equation (3) of the RCQM. The RCQM model of the spin $s = (1/2, 1/2)$ particle-antiparticle doublet is mentioned in Section 5 and is considered in details in Section 7 of [44]. Hence, the Dirac equation is derived here from the more fundamental model of the same physical reality, which is presented by the RCQM of the spin $s = (1/2, 1/2)$ particle-antiparticle doublet.

The system of vertical and horizontal links between the RCQM and field theory, which is proved above, has different useful applications. One of the fundamental applications is the participation in the discussion about the antiparticle negative mass. We emphasize [3] that the model of the RCQM and the corresponding field theory do not need the appealing to the antiparticle negative mass concept [87–90]. Further, the application of the RCQM can be useful for the analysis of the experimental situation found in [91]. Such analysis is of interest due to the fact that (as it is demonstrated here) the RCQM is the most fundamental model of the fermionic particle-antiparticle doublet. Another interesting application of the RCQM is inspired by work [92], where the quantum electrodynamics is reformulated in the FW representation. The author of work [92] essentially used the result in [88] on the negative mass of antiparticles. Starting from the RCQM, we are able not to appeal to the concept of antiparticle's negative mass.

One of the general fundamental conclusions is as follows. It is shown by the corresponding comparison that the customary FW representation cannot give a complete quantum mechanical description of a relativistic particle (or particle multiplet). Compare, e.g., the equations for eigenvectors and eigenvalues of the third component of the spin operator in each quantum-mechanical and canonical field theory model above. It is useful also to compare the general solutions in the RCQM and in field theory (it is enough to consider the field theory in the FW representation). Contrary to the RCQM, the general solution in the FW representation consists of positive and negative frequency parts. As a consequence, contrary to the RCQM, the energy has an indefinite sign in the FW representation. Hence, the complete quantum mechanical description of a relativistic particle (or particle multiplet) can be given only in the framework of the RCQM. Therefore, the customary FW transformation is extended here to the form, which gives the link between the locally covariant field theory and the RCQM. Hence, such extended inverse FW transformation is used here to fulfill the synthesis of covariant particle equations. The start of such synthesis is given here from the RCQM and not from the canonical field theory in the representations of the Foldy–Wouthuysen type.

Comparison of the RCQM and the FW representation visualizes the role of von Neumann's axiomatics [41] in this presentation. Therefore, the relation of von Neumann's axiomatics [41] to the overall contents of the paper is direct and unambiguous. It is shown that, among the above-considered models, only the RCQM of arbitrary spin can be formulated in von Neumann's axiomatics, whereas the canonical and covariant field theories cannot be formulated in its framework.

The new operator links $v_{\mathrm{N}} = \begin{vmatrix} \mathrm{I}_{\mathrm{N}} & 0 \\ 0 & C\mathrm{I}_{\mathrm{N}} \end{vmatrix}$, $\mathrm{N} = 2s+1$, found here between the RCQM of arbitrary spin and the canonical (FW type) field theory enabled ones to translate the result found in these models from one model to another. For example, the results of [28–31] from the RCQM can be translated into the canonical field theory. Contrary, the results of [32] from the canonical field theory can be translated into the RCQM (for free noninteracting cases and in the form of anti-Hermitian operators).

The partial case of the Schrödinger–Foldy equation, when the wave function has only one component, is called the spinless Salpeter equation and is under consideration in many recent works [30, 31, 33–39]. The partial wave packet solutions of this equation are given in [35]. The solutions for free massless or massive particle on a line, massless particle in a linear potential, plane wave solution for a free particle (this solution is given here in (8) for the N-component case), free massless particle in three dimensions have been considered. Further in work [36], other time-dependent wave packet solutions of the free spinless Salpeter equation are given. In view of the relation of such wave packets to the Lévy process, the spinless Salpeter equation (in the one-dimensional space) is called in [36] as the Lévy–Schrödinger equation. The several examples of the characteristic behavior of such wave packets have been shown, in particular, the multimodality arising in their evolutions: a feature at variance with the typical diffusive unimodality of both the corresponding Lévy process densities and usual Schrödinger wave functions. Therefore, the interesting task is to extend such consideration to the equations of the N-component relativistic canonical quantum mechanics considered above and to use the links given here in order to transform the wave packet solutions [35, 36] into the solutions of the equations of the locally covariant field theory.

In this article, the original FW transformation [2] is used and slightly generalized for the many-component cases. The improvement of the FW transformation [2] was the task of many authors from the 1950s till now (see, e.g., the recent works [93–96]). Nevertheless, this transformation for the free case of noninteracting spin 1/2 particle-antiparticle doublet is not changed from 1950 (the year of the first publication) till today. A. Silenko was successful in the FW transformation for single particles with spin 0 [93], spin 1/2 [94], and spin 1 [95, 96] interacting with external electric, magnetic, and other fields. In the case of noninteracting particle, when the external electric, magnetic and other external fields are equal to zero, all results of [93–96] and other works are reduced to the earlier results [2, 97]. Therefore, the choice of the exact FW transformation from 1950 in this paper as the initial one (and the basis for further generalizations for arbitrary spins) is evident and well-defined. In our subsequent works, we will consider interacting fields and will use the results in [93–96] and the other recent results, which generalize the FW formulas in the case of interaction.

A few remarks should be added about the choice of the spin operator. The authors of work [98] considered all spin operators for a Dirac particle satisfying some logical and group-theoretic conditions. The discussion of other spin operators proposed in the literature has been presented as well. As a result, only one satisfactory operator has been chosen. This operator is equivalent to the Newton–Wigner spin operator and the FW mean-spin operator. Contrary to such way, the situation here is evident. Above, the choice of the spin operator for a spin $s = 1/2$ particle-antiparticle doublet is unique. The explicit form for such operator follows directly from the main principles of the RCQM of a spin $s = 1/2$ particle-antiparticle doublet, which are formulated in Section 7 of [44] and here. This operator is given in formulae (24). After that, the links between the RCQM, FW representation, and Dirac model unambiguously give, at first, the FW spin (25) and finally spin (141) in [44], which is the FW mean-spin operator. Therefore, the similar consideration for the higher spin doublets gives unambiguously the well-defined higher spin operators, which are presented in Sections 22–27 of [44]. These new mean-spin operators (259)–(261), (284)–(286), and (359) in [44] for the N-component Dirac-type equations for higher spins 1, 3/2 and 2 are the interesting independent results.

The goal of work [39] is a comprehensive analysis of the intimate relationship between jump-type stochastic processes (e.g. Lévy flights) and nonlocal (due to integro-differential operators involved) quantum dynamics. A special attention is paid to the spinless Salpeter equation and the various wave packets, in particular, to their radial expression in 3D. Furthermore, Foldy's approach [13] is used to encompass free Maxwell theory. The consideration in [44] (see, e.g., Sections 13, 22, and 23) demonstrates another link between the Maxwell equations and the RCQM. In the generalization of the Foldy's synthesis of covariant particle equations given here, the Maxwell equations and their analogy for nonzero mass are related to the RCQM of spin $s = (1, 1)$ and spin $s = (1, 0, 1, 0)$ particle-antiparticle doublets (see another approach in [99, 100]). The electromagnetic field equations that follow from the corresponding relativistic quantum mechanical equations have been found in [44]. The new electrodynamical equations containing the hypothetical antiphoton and massless spinless antiboson have been introduced. The Maxwell-like equations for the boson with spin $s = 1$ and $m > 0$ (W-boson) have been introduced as well. In other words, the Maxwell equations for the field with nonzero mass have been introduced in [44].

The covariant consideration of the arbitrary spin field theory given here and in Sections 21–28 of [44] contains the noncovariant representations of the Poincaré algebra. Nevertheless, it is not the deficiency of the given model. For the generators of the Poincaré group $\mathcal{P}$ of spin $s = (1/2, 1/2)$, the covariant form (146) of [44] is well-known. Only in order to have the uniform consideration, the Poincaré generators for spin $s = (1, 0, 1, 0)$, (3/2,3/2), (2,0,2,0), (2,1,2,1) are given in formulae (265), (266), (331), (332), (361), and (362) of [44] in uniform forms of noncovariant operators in covariant theory. After further transformations of sets of these generators in the direction of finding the covariant forms (like (146) of [44]), some sets of generators can be presented in the manifestly covariant forms. For other sets of generators, the covariant forms are extrinsic. Some sets of generators can be presented only in the forms, which are similar to that given in [47–51], where the prime anti-Hermitian operators and specific eigenvectors – eigenvalues equations (with imaginary eigenvalues) are used (see, e.g., formula (21) in [48]).

The second reason for the stop on the level of formulae (265), (266), (331), (332), (361), and (362) of [44] is to conserve the important property of the Poincaré generators in the canonical FW type representation. Similarly to the FW-type Poincaré generators, both angular momenta (orbital and spin) in the sets (265), (266), (331), (332), (361), and (362) of [44] commute with the operator of the Dirac-like equation of motion (38). Contrary to it, in the covariant form (146) of [44], only the total angular momentum, which is the sum of orbital and spin angular momenta, commutes with the Diracian.

The main point is as follows. The noncovariance is not the barrier for the relativistic invariance! Not a matter of fact that the noncovariant objects such as the Lebesgue measure d^3x and the noncovariant Poincaré generators are explored, the model of locally covariant field theory of arbitrary spin presented in Section 6 is a relativistic invariant in the following sense. The Dirac-like equation (38) and the set $\{\psi\}$ of its solutions (39) are invariant with respect to the reducible representation of the Poincaré group $\mathcal{P}$, the nonlocal and noncovariant generators of which are given by (41) and (42). Indeed, the direct calculations visualize that generators (41) and (42) commute with the operator of Eq. (38) and satisfy the commutation relations (11) of the Lie algebra of the Poincaré group $\mathcal{P}$.

The partial case of zero mass has been considered briefly in Section 28 of [44].

The 8-component manifestly covariant equation for the spin $s = 3/2$ field found here is the $s = 3/2$ analog of the 4-component Dirac equation for the spin $s = 1/2$ doublet. It is shown that the synthesis of this equation from the relativistic canonical quantum mechanics of the spin $s = 3/2$ particle-antiparticle doublet is completely similar to the synthesis of the Dirac equation from the relativistic canonical quantum mechanics of the spin $s = 1/2$ particle-antiparticle doublet. The difference is only in the value of spin (3/2 and 1/2). On this basis and on the basis of the investigation of solutions and transformation properties with respect to the Poincaré group, this new 8-component equation is suggested and is well defined for the description of spin $s = 3/2$ fermions. Note that the known Rarita–Schwinger (Pauli–Fierz) equation has 16 components and needs the additional condition.

The properties of the Fermi–Bose duality, triality and quadro Fermi–Bose properties of equations found have been discussed briefly.

The results, which are not presented in this small-scale article, can be found in [44] and [101].

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1. P.A.M. Dirac, Proc. R. Soc. Lond. A. **117**, 610 (1928).
2. L.L. Foldy and S.A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
3. V.M. Simulik and I.Yu. Krivsky, Univ. J. Phys. Appl. **2**, 115 (2014).
4. P.A.M. Dirac, Proc. R. Soc. Lond. A. **155**, 447 (1936).
5. H.J. Bhabha, Rev. Mod. Phys. **17**, 200 (1945).
6. D.L. Weaver, C.L. Hammer, and R.H. Good, jr., Phys. Rev. **135**, 241 (1964).
7. P.M. Mathews, Phys. Rev. **143**, 978 (1966).
8. T.J. Nelson and R.H. Good, jr., Rev. Mod. Phys. **40**, 508 (1968).
9. R.F. Guertin, Ann. Phys. (USA) **88**, 504 (1974).
10. R.A. Kraicik and M.M. Nieto, Phys. Rev. D **15**, 433 (1977).
11. R-K. Loide, I. Ots, and R. Saar, J. Phys. A **30**, 4005 (1997).
12. D.M. Gitman and A.L. Shelepin, Int. J. Theor. Phys. **40**, 603 (2001).
13. L.L. Foldy, Phys. Rev. **102**, 568 (1956).
14. E.E. Salpeter, Phys. Rev. **87**, 328 (1952).
15. L.L. Foldy, Phys. Rev. **122**, 275 (1961).
16. R.A. Weder, Annal. de l'Institut H. Poincaré, sec. A **20**, 211 (1974).
17. R.A. Weder, J. Funct. Anal. **20**, 319 (1975).
18. I.W. Herbst, Comm. Math. Phys. **53**, 285 (1977).
19. I.W. Herbst, Comm. Math. Phys. **55**, 316 (1977).
20. P. Castorina, P. Cea, G. Nardulli, and G. Paiano, Phys. Rev. D **29**, 2660 (1984).
21. L.J. Nickisch and L. Durand, Phys. Rev. D. **30**, 660 (1984).
22. A. Martin and S.M. Roy, Phys. Lett. B **233**, 407 (1989).
23. J.C. Raynal, S.M. Roy, V. Singh, A. Martin, and J. Stubble, Phys. Lett. B **320**, 105 (1994).
24. L.P. Fulcher, Phys. Rev. D **50**, 447 (1994).
25. J.W. Norbury, K.M. Maung, and D.E. Kahana, Phys. Rev. A **50**, 3609 (1994).
26. W. Lucha and F.F. Schobert, Phys. Rev. A **54**, 3790 (1996).
27. W. Lucha and F.F. Schobert, Phys. Rev. A **56**, 139 (1997).
28. F. Brau, J. Math. Phys. **39**, 2254 (1998).
29. F. Brau, J. Math. Phys. **40**, 1119 (1999).
30. F. Brau, J. Math. Phys. **46**, 032305 (2005).
31. F. Brau, J. Nonlin. Math. Phys. **12, Suppl. 1**, 86 (2005).
32. T.L. Gill and W.W. Zahary, J. Phys. A. **38**, 2479 (2005).
33. Y. Chargui, L. Chetouani, and A. Trabelsi, J. Phys. A **42**, 355203 (2009).
34. Y. Chargui, A. Trabelsi, and L. Chetouani, Phys. Lett. A **374**, 2243 (2010).

35. K. Kowalski and J. Rembielinski, Phys. Rev. A **84**, 012108 (2011).
36. N.C. Petroni, J. Phys. A **44**, 165305 (2011).
37. C. Semay, Phys. Lett. A **376**, 2217 (2012).
38. Y. Chargui and A. Trabelsi, Phys. Lett. A **377**, 158 (2013).
39. P. Garbaczewski and V. Stephanovich, J. Math. Phys. **54**, 072103 (2013).
40. V.M. Simulik and I.Yu. Krivsky, Ukr. J. Phys. **58**, 1192 (2013).
41. von J. Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932).
42. V.S. Vladimirov, *Methods of the Theory of Generalized Functions* (Taylor and Francis, London, 2002).
43. N.N. Bogoliubov, A.A. Logunov, and I.T. Todorov, *Introduction to Axiomatic Quantum Field Theory* (Springer, Berlin, 1992).
44. V.M. Simulik, arXiv: 1409.2766v2 [quant-ph], Sept. 19, 2014.
45. P. Garbaczewski, Int. J. Theor. Phys. **25**, 1193 (1986).
46. N.H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations* (CRC Press, Florida, 1995), vol. 2.
47. V.M. Simulik and I.Yu. Krivsky, Dopov. Nats. Akad. Nauk Ukrainy, No. 5, 82 (2010).
48. V.M. Simulik and I.Yu. Krivsky, Phys. Lett. A **375**, 2479 (2011).
49. V.M. Simulik, I.Yu. Krivsky, and I.L. Lamer, Cond. Matt. Phys. **15**, 43101(1-10) (2012).
50. V.M. Simulik, I.Yu. Krivsky, and I.L. Lamer, TWMS J. App. Eng. Math. **3**, 46 (2013).
51. V.M. Simulik, I.Yu. Krivsky, and I.L. Lamer, Ukr. J. Phys. **58**, 523 (2013).
52. J.P. Elliott and P.G. Dawber, *Symmetry in Physics* (Macmillan Press, London, 1979), vol. 1.
53. B. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974).
54. B. Thaller, *The Dirac Equation* (Springer, Berlin, 1992).
55. N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields* (Wiley, New York, 1980).
56. I.Yu. Krivsky, R.R. Lompay, and V.M. Simulik, Theor. Math. Phys. **143**, 541 (2005).
57. S.I. Kruglov, Int. J. Theor. Phys. **41**, 653 (2002).
58. M. Fierz and W. Pauli, Proc. R. Soc. Lond. A **173**, 211 (1939).
59. W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).
60. A.S. Davydov, Zh. Eksp. Teor. Fiz. **13**, 313 (1943).
61. V. Bargmann and E.P. Wigner, Proc. Nat. Acad. Sci. U.S. **34**, 211 (1948).
62. D.L. Pursey, Nucl. Phys. **53**, 174 (1964).
63. V.M. Simulik and I.Yu. Krivsky, Rep. Math. Phys. **50**, 315 (2002).
64. V.M. Simulik and I.Yu. Krivsky, Electromag. Phen. **3(9)**, 103 (2003).
65. V.M. Simulik, in *What is the Electron?*, edited by V. Simulik (Apeiron, Montreal, 2005).
66. P.A.M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1958).
67. V.P. Neznamov and A.J. Silenko, J. Math. Phys. **50**, 122302(1-15) (2009).
68. J. Sakurai, *Advanced Quantum Mechanics* (Addison–Wesley, New York, 1967).
69. L. Ryder, *Quantum Field Theory* (Cambridge Univ. Press, Cambridge, 1996).
70. J. Keller, *Theory of the Electron. A Theory of Matter from START* (Kluwer, Dordrecht, 2001).
71. V. Fock and D. Iwanenko, Z. Phys. **54**, 798 (1929).
72. V. Fock, Z. Phys. **57**, 261 (1929).
73. A. Wightman, in *Dispersion Relations and Elementary Particles*, edited by C. De Witt and R. Omnes (Wiley, New York, 1960).
74. H. Sallhofer, Z. Naturforsch. A. **33**, 1378 (1978).
75. H. Sallhofer, Z. Naturforsch. A. **41**, 468 (1986).
76. S.K. Srinivasan and E.C.G. Sudarshan, J. Phys. A **29**, 5181 (1996).
77. L. Lerner, Eur. J. Phys. **17**, 172 (1996).
78. T. Kubo, I. Ohba, and H. Nitta, Phys. Lett. A **286**, 227 (2001).
79. H. Cui, arXiv: quant-ph/0102114. 15 Aug 2001.
80. Y. Ng and H. van Dam, arXiv: hep-th/0211002. 4 Feb 2003.
81. M. Calerier and L. Nottale, Electromag. Phen. **3(9)**, 70 (2003).
82. M. Evans, Found. Phys. Lett. **16**, 369 (2003).
83. M. Evans, Found. Phys. Lett. **17**, 149 (2004).
84. F. Gaioli and E. Alvarez, Am. J. Phys. **63**, 177 (1995).
85. S. Efthimiades, arXiv: quant-ph/0607001v3. 15 Jan 2011.
86. G. D'Ariano and P. Perinotti, arXiv: 1306.1934 [quant-ph], June 8, 2013.
87. H. Bondi, Rev. Mod. Phys. **29**, 423 (1957).
88. E. Recami and G. Zino, Nuovo Cim. A **33**, 205 (1976).
89. G. Landis, J. Propuls. Power **7**, 304 (1990).
90. R. Wayne, Turk. J. Phys. **36**, 165 (2012).
91. W. Kuellin, P. Gaal, K. Reimann, T. Worner, T. Elsaesser, and R. Hey, Phys. Rev. Lett. **104**, 146602(1-4) (2010).
92. V.P. Neznamov, Phys. Part. Nucl. **37**, 86 (2006).
93. A.J. Silenko, Phys. Rev. D **88**, 045004(1–5) (2013).
94. A.J. Silenko, Phys. Rev. A **77**, 012116(1–7) (2008).
95. A.J. Silenko, Phys. Rev. D **87**, 073015 (2013).
96. A.J. Silenko, Phys. Rev. D **89**, 121701(R)(1–6) (2014).
97. K.M. Case, Phys. Rev. **95**, 1323 (1954).
98. P. Caban, J. Rembielinski, and M. Wlodarczyk, Phys. Rev. A **88**, 022119(1–8) (2013).
99. I.Yu. Krivsky, V.M. Simulik, T.M. Zajac, and I.L. Lamer, *Proc. of the 14-th Internat. Conference "Mathematical Methods in Electromagnetic Theory", 28–30 August 2012, Institute of Radiophysics and Electronics, Kharkiv*, p. 201.
100. V.M. Simulik, *Proc. of the 15-th Internat. Conference "Mathematical Methods in Electromagnetic Theory", 26–28 August 2014, Oles Honchar National University, Dnipropetrovsk*, p. 9.

101. V.M. Simulik, arXiv: 1509.04630v1 [quant-ph], Sept. 12, 2015.

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ТА ДІРАКОПОДІБНИХ РІВНЯНЬ ДОВІЛЬНОГО
СПІНУ З ВІДПОВІДНОЇ РЕЛЯТИВІСТСЬКОЇ
КАНОНІЧНОЇ КВАНТОВОЇ МЕХАНІКИ

Р е з ю м е

Одержано нові релятивістські рівняння руху частинок зі спінами $s = 1$, $s = 3/2$, $s = 2$ та ненульовою масою. Наведено опис релятивістської канонічної квантової механіки для частинок довільної маси та спіну. Знайдено зв'язок між релятивістською канонічною квантовою механікою для частинок довільного спіну та коваріантною локальною теорією поля. Розглянуто явно коваріантні польові рівняння довільного спіну, що слідують з квантово-механічних рівнянь. Запропоновано коваріантні локальні польові рівняння для дублету частинка–античастинка спінів $s = (1, 1)$, мультиплету частинка–античастинка спінів $s = (1, 0, 1, 0)$, дублету частинка–античастинка спінів $s = (3/2, 3/2)$, дублету частинка–античастинка спінів $s = (2, 2)$, мультиплету частинка–античастинка спінів $s = (2, 0, 2, 0)$ та мультиплету частинка–античастинка спінів $s = (2, 1, 2, 1)$. Також запропоновано максвеллподібні рівняння для бозону зі спіном $s = 1$ та масою $m > 0$.