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**EFFECTIVE MASS OF  $^4\text{He}$  ATOM  
IN SUPERFLUID AND NORMAL PHASES**PACS 05.30.Jp, 68.35.Rh

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*The formula for the temperature dependence of the effective mass of a  $^4\text{He}$  atom in the superfluid and normal phases is obtained. This expression for the effective mass allows one to eliminate infra-red divergences, being applicable at all temperatures, except for a narrow fluctuation region  $0.97 \lesssim T/T_c \leq 1$ . In the high and low temperature limits, as well as in the interactionless limit, the obtained expression reproduces the well known results. The temperature dependence of the heat capacity and the phase transition temperature  $T_c \approx 2.18$  K are calculated, by using the formula obtained for the effective mass. In the framework of the approach proposed in this work, the small critical index  $\eta$  is determined in the random phase approximation. The obtained value corresponds to the well known result.*

*Keywords:* liquid  $^4\text{He}$ , effective mass, critical temperature, critical indices.

**1. Introduction**

The idea that the transition of liquid  $^4\text{He}$  into the superfluid state is a manifestation of the Bose–Einstein condensation was put forward for the first time by F. London [1]. It was the “proximity” of the Bose condensation temperature in an ideal gas with helium parameters to the transition temperature in real  $^4\text{He}$  that suggested him this idea. Although this interpretation of the phase transition is not free from difficulties [2], it correctly describes, in general, modern experiments with cooled gases [3, 4].

The problems in the theory of liquid  $^4\text{He}$ , which remain unresolved till now, include the calculation of corresponding thermodynamic functions in the whole temperature interval and the calculation of the transition temperature into the superfluid state, which would agree with the experimental value. At the qualitative level, a reduction of the critical temperature was substantiated by R. Feynman [5], who introduced the concept of effective particle mass. For intuitive

reasons, he came to a conclusion that, owing to the interaction between particles, the effective mass has to exceed the atomic one. This conclusion is also valid for two-dimensional systems [6].

However, there exists a competing mechanism. The repulsion at short distances effectively increases the system density and, consequently, should increase the Bose condensation temperature. This conclusion is confirmed by the results of theoretical calculations [7–11] and Monte-Carlo simulation carried out for the model of weakly non-ideal Bose gas [12, 13]. In order to put the experimental results obtained in the  $^4\text{He}$ -Vycor system in correspondence with the results of theoretical calculations, the effect of atomic mass renormalization and a shift associated with the repulsive part of the interparticle interaction have to be taken into account simultaneously [14].

In the literature, the value of effective mass at low temperatures was mainly analyzed [15–19]. In works [20, 21], the corresponding temperature dependence was obtained within the variational approach. The properties of helium in the normal phase were studied in works [22, 23], where the effective mass of particles

was used as a fitting parameter to put the calculated structural functions in agreement with experimental curves.

A method of calculation of the effective mass of  ${}^4\text{He}$  atoms in the liquid phase was demonstrated in work [24]. It allows infra-red divergences typical of the phase transition theory to be eliminated. A shortcoming of the proposed approach is the poorly substantiated extrapolation of the “seed” effective mass onto a wide temperature interval (proceeding from the corresponding expression obtained for the zero temperature) and an incorrect behavior of the obtained effective mass in the critical region.

Another approach to the calculation of the effective mass was proposed in work [25]. In its framework, the temperature dependence of the heat capacity was determined. The result obtained turned out in much better agreement with experimental data than the results of calculations on the basis of a “bare” mass. However, the expression for the effective mass, which was obtained in the framework of this approach, does not exclude the mentioned infra-red divergences, because the ideology of the effective mass calculations was not oriented to this purpose.

This work is aimed at finding such an expression for the effective mass, which would eliminate infra-red divergences and reproduce a correct behavior in a vicinity of the critical point (excluding, maybe, a narrow fluctuation region). At the same time, it should be better substantiated theoretically in a wide temperature region. Another task consisted in obtaining the temperature dependence of the heat capacity with the use of the new effective mass and in comparing it with the previous results.

## 2. General Formulas

While calculating the heat capacity for a many-boson system, let us use the expression for its internal energy in the pair correlation approximation [24, 25]. The expression can be obtained by averaging the Hamiltonian with the density matrix found in work [26]:

$$E = N \frac{mc^2}{2} + \sum_{\mathbf{q} \neq 0} \frac{\bar{\varepsilon}_q}{\bar{z}_0^{-1} e^{\beta \bar{\varepsilon}_q} - 1} + \frac{1}{2} \frac{\bar{m}}{m} \sum_{\mathbf{q} \neq 0} \frac{\lambda_q}{1 + \lambda_q \bar{S}_0(q)} \frac{\partial \bar{S}_0(q)}{\partial \beta} +$$

$$+ \frac{1}{4} \sum_{\mathbf{q} \neq 0} \varepsilon_q (\lambda_q^2 + \alpha_q^2 - 1) S(q) + \frac{1}{2} \sum_{\mathbf{q} \neq 0} \varepsilon_q \left[ \frac{\alpha_q}{\text{sh}(\beta E_q)} - \frac{1}{\text{sh}(\beta \varepsilon_q)} \right] + \frac{1}{16} \sum_{\mathbf{q} \neq 0} \varepsilon_q \left( 1 - \frac{1}{\alpha_q^2} \right) \left( \alpha_q - \frac{1}{\alpha_q} - 4\alpha_q^2 \right), \quad (1)$$

where  $\bar{m}$  is the effective mass;  $\bar{\varepsilon}_q = \hbar^2 q^2 / 2\bar{m}$ ,  $\bar{z}_0$ , and  $\bar{S}_0(q)$  are the renormalized one-particle spectrum, activity, and structure factor, respectively, of the ideal Bose gas;  $E_q = \alpha_q \varepsilon_q$  is the spectrum of elementary Bogolyubov excitations;  $\alpha_q = \sqrt{1 + 2N\nu_q / (V\varepsilon_q)}$  is the Bogolyubov factor,  $\nu_q = \int e^{-i\mathbf{q}\mathbf{R}} \Phi(R) d\mathbf{R}$  is the Fourier transform of the pairwise interparticle interaction potential  $\Phi(R)$ ,

$$S(q) = \frac{\bar{S}_0(q)}{1 + \lambda_q \bar{S}_0(q)} \quad (2)$$

is the structure factor of a Bose liquid in the pair correlation approximation; and

$$\lambda_q = \alpha_q \text{th} \left[ \frac{\beta}{2} E_q \right] - \text{th} \left[ \frac{\beta}{2} \varepsilon_q \right]. \quad (3)$$

The distribution of Bose particles with the new spectrum looks like

$$\bar{n}_p = \frac{1}{\bar{z}_0^{-1} e^{\beta \bar{\varepsilon}_p} - 1}, \quad (4)$$

whereas the renormalized one-particle spectrum  $\bar{\varepsilon}_p$  is chosen in the form

$$\bar{\varepsilon}_p = \varepsilon_p + \Delta_p - \Delta_0, \quad (5)$$

where  $\Delta_p$  is a correction to the spectrum, which is to be determined. The value of  $\Delta_0$  depends only on the temperature and is actually responsible for the activity renormalization. After eliminating infra-red divergences, the expression for  $\Delta_p$  looks like [24]

$$\Delta_p = \frac{1}{N\beta} \sum_{\mathbf{q} \neq 0} \frac{\lambda_q}{1 + \lambda_q \bar{S}_0(q)} \bar{n}_{|\mathbf{p}+\mathbf{q}|}. \quad (6)$$

Expression (5) for the renormalized one-particle spectrum can also be written in the form

$$\bar{\varepsilon}_p = \frac{\hbar^2 p^2}{2\bar{m}(p)}, \quad (7)$$

where the quantity  $\bar{m}(p)$  is regarded as the total effective mass of a particle, which depends on the absolute value of wave vector  $\mathbf{p}$ . This form for the spectrum  $\bar{\varepsilon}_p$  was proposed in work [27] in order to exclude infra-red divergences. It will be recalled that the effective mass  $\bar{m}$  is formed by many-particle correlations, starting from four-particle ones, and, generally speaking, it depends on the momentum  $p$ . It is clear that we are interested in the behavior of  $\bar{m}(p)$  as  $p \rightarrow 0$ . As the total effective mass, we will understand the quantity  $\bar{m} = \bar{m}(0)$ . In this connection, let us consider the difference  $\Delta_p - \Delta_0$  as  $p \rightarrow 0$  in more details.

At small  $p$ -values, the renormalized spectrum (5) can be written in the form [24]

$$\bar{\varepsilon}_p = \frac{\hbar^2 p^2}{2\bar{m}}, \quad (8)$$

where

$$\begin{aligned} \frac{m^*}{\bar{m}} = 1 + \frac{1}{2\pi^2 \rho} \int_0^\infty \frac{q^2 \lambda_q}{1 + \lambda_q \bar{S}_0(q)} \times \\ \times \bar{n}_q (1 + \bar{n}_q) \left[ \frac{2}{3} \beta \varepsilon_q (1 + 2\bar{n}_q) - 1 \right] dq. \end{aligned} \quad (9)$$

In our theory, we use the following expression for the temperature dependence of the ‘‘seed’’ effective mass  $m^*$ , which was obtained in work [25]:

$$\begin{aligned} \frac{m}{m^*} = 1 - \frac{1}{3N} \sum_{\mathbf{q} \neq 0} \frac{(\alpha_q - 1)^2}{\alpha_q (\alpha_q + 1)} - \\ - \frac{2}{3N} \sum_{\mathbf{q} \neq 0} \left\{ \frac{\alpha_q^2 + 3}{\alpha_q^2 - 1} [n(\beta \varepsilon_q) - 1/(\beta \varepsilon_q)] - \right. \\ - \frac{3\alpha_q^2 + 1}{\alpha_q (\alpha_q^2 - 1)} [n(\beta E_q) - 1/(\beta E_q)] + \\ \left. + 2 [1/(\beta \varepsilon_q) - \beta \varepsilon_q n(\beta \varepsilon_q) [1 + n(\beta \varepsilon_q)]] \right\}, \end{aligned} \quad (10)$$

where the notation  $n(x) = 1/(e^x - 1)$  is used.

It is easy to see that the critical-point divergence on the right-hand side of equality (9) originates from the integrand at small  $q$ -values. This singularity is logarithmic, as will be shown later. Such a divergence is typical of critical phenomena. Our task consists in isolating this singularity and finding a correct expression for the effective mass. For this purpose, let us consider the following equality, which follows from

work [24]:

$$\frac{m^*}{\bar{m}} = 1 + \lim_{p \rightarrow 0} \frac{\Delta_p - \Delta_0}{\varepsilon_p}, \quad (11)$$

where

$$\Delta_p - \Delta_0 = \frac{1}{N\beta} \sum_{\mathbf{q} \neq 0} \frac{\lambda_q}{1 + \lambda_q \bar{S}_0(q)} \{ \bar{n}_{|\mathbf{q}+\mathbf{p}|} - \bar{n}_q \}. \quad (12)$$

In expression (12), we isolate the quantity

$$\begin{aligned} \Delta_\infty = \frac{1}{N\beta} \sum_{\mathbf{q} \neq 0} \frac{\lambda_q}{1 + \lambda_q \bar{S}_0(q)} \left\{ \frac{1}{\bar{z}_0^{-1} - 1 + \bar{z}_0^{-1} \beta \bar{\varepsilon}_{|\mathbf{q}+\mathbf{p}|}} - \right. \\ \left. - \frac{1}{\bar{z}_0^{-1} - 1 + \bar{z}_0^{-1} \beta \bar{\varepsilon}_q} \right\}, \end{aligned} \quad (13)$$

which contains the indicated non-analyticity in whole and, simultaneously, is much more convenient for the analysis. The next step consists in finding a series expansion for  $\Delta_\infty$  in the interval of small  $p$ -values and confining the series to terms proportional to  $p^2$ , because the higher-order terms give no contribution to the effective mass owing to equality (11).

On the right-hand side of equality (13), we change from summation to integration:

$$\begin{aligned} \Delta_\infty = \frac{p_0^2 \bar{z}_0}{4\pi^2 \beta \rho} \int_0^\infty \frac{\lambda_q dq}{1 + \lambda_q \bar{S}_0(q)} \times \\ \times \left\{ \frac{q}{2p} \ln \left| \frac{P_0^2 + (q+p)^2}{P_0^2 + (q-p)^2} \right| - \frac{2q^2}{P_0^2 + q^2} \right\}, \end{aligned} \quad (14)$$

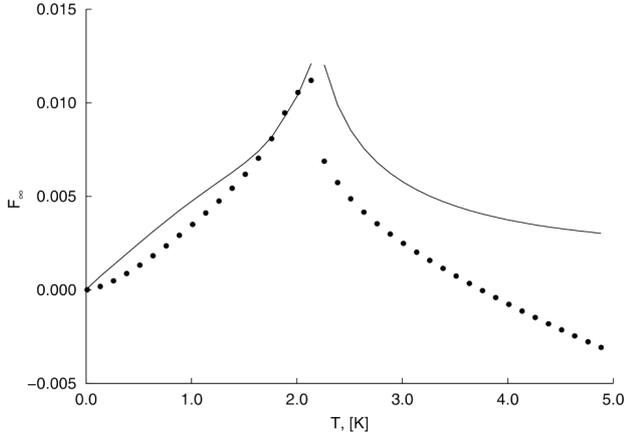
where  $p_0^2 = 2\bar{m}/(\beta \hbar^2)$  and  $P_0 = p_0 \sqrt{(1 - \bar{z}_0)}$ . In the subcritical region, the activity  $\bar{z}_0 = 1$ . Therefore,  $P_0 = 0$  here, and the expressions obtained above become a little simpler.

Let us change the variables:  $q/p = x$  and  $dq = p dx$ . Then

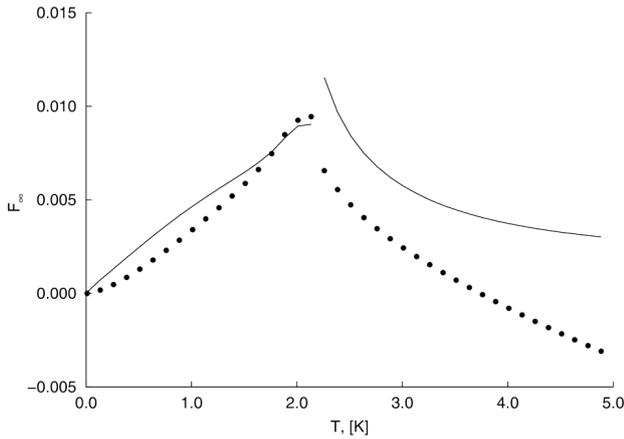
$$\begin{aligned} \Delta_\infty = \frac{p_0^2 \bar{z}_0 p}{4\pi^2 \beta \rho} \int_0^\infty \frac{\lambda_{px}}{1 + \lambda_{px} \bar{S}_0(px)} \times \\ \times \left\{ \frac{x}{2} \ln \left| \frac{P_0^2/p^2 + (x+1)^2}{P_0^2/p^2 + (x-1)^2} \right| - \frac{2x^2}{P_0^2/p^2 + x^2} \right\} dx. \end{aligned} \quad (15)$$

The function  $\lambda_{px}/(1 + \lambda_{px} \bar{S}_0(px))$  is finite and tends to zero, as  $x \rightarrow \infty$  (at a fixed  $p$ ). On the other hand, the function

$$\frac{x}{2} \ln \left| \frac{P_0^2/p^2 + (x+1)^2}{P_0^2/p^2 + (x-1)^2} \right| - \frac{2x^2}{P_0^2/p^2 + x^2}$$



**Fig. 1.** Temperature dependence of the quantity  $F_\infty = \lim_{p \rightarrow 0} \Delta_\infty(p)/\varepsilon_p$  at  $p = 0.01$ . Points correspond to the exact expression, and the solid curve to the accepted approximation



**Fig. 2.** The same as in Fig. 1, but for  $p = 0.1$

falls down to zero in the interval  $x > 1$ . Moreover,

$$\int_0^\infty \left\{ \frac{x}{2} \ln \left| \frac{P_0^2/p^2 + (x+1)^2}{P_0^2/p^2 + (x-1)^2} \right| - \frac{2x^2}{P_0^2/p^2 + x^2} \right\} dx = 0$$

for any  $P_0$ - and  $p$ -values. Those facts allow us to assert (especially if the matter concerns the critical region) that only the interval, where the  $px$ -values in the integrand are small, makes a non-disappearing contribution to the quantity  $\Delta_\infty/p^2$  as  $p \rightarrow 0$ .

Now, let us proceed to expanding the quantities  $\lambda_q$  and  $\bar{S}_0(q)$  in a vicinity of the zero wave vector,  $q = 0$ . For  $\lambda_q$ , the result is obtained easily:

$$\lambda_q = \beta \rho \nu_0 + o(q), \tag{16}$$

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where  $\rho$  is the density of the Bose system. On the contrary, the form of the series expansion for the structure factor of the ideal Bose gas depends on the temperature interval. It looks like

$$S_0(q) = \frac{4\bar{m}(1-(T/T_c)^{3/2})}{\beta \hbar^2 q^2} + \frac{\bar{m}^2}{2\rho \hbar^4 \beta^2} \frac{1}{q} + 1 + o(q) \tag{17}$$

in the subcritical region ( $T < T_c$ ) and

$$S_0(q) = \frac{\bar{m}^2}{\pi \rho \hbar^4 \beta^2} \frac{1}{q} \operatorname{arctg} \left( \frac{q}{2P_0} \right) + o(q) \tag{18}$$

in the supercritical one. This result follows immediately from the expression for the structure factor of the ideal Bose gas [26]

$$\begin{aligned} \bar{S}_0(q) &= 1 + \frac{\bar{m}}{4\pi^2 \rho \beta \hbar^2} \frac{1}{q} \int_0^\infty \frac{p}{\bar{z}_0^{-1} e^{\beta \frac{\hbar^2 q^2}{2\bar{m}}} - 1} \times \\ &\times \ln \left| \frac{1 - \bar{z}_0 e^{-\beta \frac{\hbar^2 (p+q)^2}{2\bar{m}}}}{1 - \bar{z}_0 e^{-\beta \frac{\hbar^2 (p-q)^2}{2\bar{m}}}} \right| dp. \end{aligned}$$

In order to analyze expression (15) analytically, let us apply such approximations for the quantities  $\lambda_q$  and  $\bar{S}_0(q)$ , which contain only the expansion terms presented above. One can check the adequacy of those approximations with the help of the numerical analysis, the results of which are exhibited in Figs. 1 and 2.

The further analysis of expression (15) will be carried out separately in the sub- and supercritical temperature intervals and at the very critical point, because the approaches, which should be applied in each of those cases, are different.

### 3. Calculations in the Subcritical Temperature Interval

In the subcritical temperature interval ( $T < T_c$ ), expression (15) reads

$$\Delta_\infty = \frac{p_0^2 \varkappa p^3}{4\pi^2 \beta \rho} \int_0^\infty \frac{x^2 \left( x \ln \left| \frac{x+1}{x-1} \right| - 2 \right) dx}{(1 + \varkappa)x^2 p^2 + \gamma x p + 2n_0 \varkappa} + o(p^2),$$

where

$$\varkappa = \beta \rho \nu_0; \quad \gamma = \frac{\bar{m}^2 \nu_0}{2\hbar^4 \beta}; \quad n_0 = 1 - \left( \frac{T}{T_c} \right)^{3/2}. \tag{19}$$

Applying the formula

$$x \ln \left| \frac{x+1}{x-1} \right| - 2 = \int_{-1}^1 \frac{x^2 da}{x^2 - a^2} - \int_{-1}^1 da = \int_{-1}^1 \frac{a^2 da}{x^2 - a^2}$$

and changing the order of integration, we obtain

$$\begin{aligned} \Delta_\infty &= \frac{2\bar{m}\varkappa p^3}{(2\pi\hbar\beta)^2\rho} \int_{-1}^1 da \times \\ &\times \int_0^\infty \frac{x^2 a^2 dx}{(x^2 - a^2)[(1 + \varkappa)x^2 p^2 + \gamma xp + 2n_0\varkappa]} + o(p^2). \end{aligned} \quad (20)$$

The denominator of the integrand should be factorized, and the whole integrand should be expanded in simple fractions. Then the elementary integration over the variable  $x$  gives

$$\begin{aligned} \Delta_\infty &= -\frac{p_0^2 \varkappa p}{4\pi^2 \beta \rho (1 + \varkappa)} \int_{-1}^1 a^2 da \left\{ \frac{p^2 a \ln |a|}{2(ap - x_1)(ap - x_2)} - \right. \\ &- \frac{p^2 a \ln |a|}{2(ap + x_1)(ap + x_2)} + \frac{px_2^2 \ln |x_2/p|}{(a^2 p^2 - x_2^2)(x_1 - x_2)} - \\ &\left. - \frac{px_1^2 \ln |x_1/p|}{(a^2 p^2 - x_1^2)(x_1 - x_2)} \right\} + o(p^2), \end{aligned} \quad (21)$$

where  $x_1/p$  and  $x_2/p$  are roots of the quadratic equation

$$(1 + \varkappa)x^2 p^2 + \gamma xp + 2n_0\varkappa = 0, \quad (22)$$

and

$$x_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 8n_0(1 + \varkappa)\varkappa}}{2(1 + \varkappa)}. \quad (23)$$

After the corresponding transformations and the integration over the variable  $a$ , we obtain

$$\begin{aligned} \Delta_\infty &= \frac{p_0^2 \varkappa}{4\pi^2 \beta \rho (1 + \varkappa)} \frac{1}{x_1 - x_2} \left\{ 2(x_2^2 - x_1^2) + \right. \\ &+ \frac{x_2^3}{p^2} \left( \text{dilog} \left[ 1 + \frac{p}{x_2} \right] - \text{dilog} \left[ 1 - \frac{p}{x_2} \right] \right) - \\ &- \frac{x_1^3}{p^2} \left( \text{dilog} \left[ 1 + \frac{p}{x_1} \right] - \text{dilog} \left[ 1 - \frac{p}{x_1} \right] \right) + \\ &+ 2x_2^2 \ln |x_2/p| \left( 1 - \frac{x_2}{p} \text{arcth} \left[ \frac{p}{x_2} \right] \right) - \\ &\left. - 2x_1^2 \ln |x_1/p| \left( 1 - \frac{x_1}{p} \text{arcth} \left[ \frac{p}{x_1} \right] \right) \right\} + o(p^2), \end{aligned} \quad (24)$$

where

$$\text{dilog}[x] = \int_1^x \ln(y)/(1-y) dy.$$

Expanding the obtained expression in  $p$ , we find that the quantities proportional to  $p^2$  originate exclusively from the last two terms in the braces. As a result, we obtain

$$\Delta_\infty = \frac{p_0^2 \varkappa}{3\pi^2 \beta \rho (1 + \varkappa)} \frac{\ln |x_1/x_2|}{(x_1 - x_2)} p^2 + o(p^2). \quad (25)$$

When approaching the critical point, one of the roots, say  $x_2$ , tends to zero, and we obtain a logarithmic divergence for the quantity  $\Delta_\infty$  in a vicinity of the critical point. What is the effective mass in this case? Returning to the analysis of the expressions for  $x_1$  and  $x_2$ , we may conclude that there exists a temperature  $T_F$ , at which  $x_1$  and  $x_2$  are real-valued quantities. In this case, the function  $\text{arc tanh}(p/x_2)$  is no more finite in the temperature interval between  $T_F$  and  $T_c$  and diverges when approaching the critical point. Until the quantity  $x_2$  remains complex, the  $\text{arc tanh}$  function can be expressed in terms of the trigonometric arctangent, which is finite. The temperature  $T_F$  can be easily found by putting the discriminant of the quadratic equation (22) equal to zero. Its numerical solution gives  $T_F \approx 2.13$  K if the critical temperature  $T_c \approx 2.18$  K. One can see that this is a very narrow interval, which can be interpreted as a fluctuation one, i.e. when the fluctuations of the Bose condensate becomes comparable with its amount. It can also be considered as a region similar to the Ginzburg region, where the perturbation calculation method fails. In any case, in the framework of our approach, we cannot draw any proper conclusion about the effective mass in this narrow interval. Other methods, e.g., the renorm-group approach, are required to analyze this region. The numerical analysis testifies that the contribution  $\Delta_\infty$  to the effective mass is very insignificant at temperatures below  $T_F$ . The analytical form of this contribution to the right-hand side of Eq. (11) is as follows:

$$\frac{p_0^4 \varkappa}{6\pi^2 \rho (1 + \varkappa)} \frac{\ln |x_1/x_2|}{(x_1 - x_2)}. \quad (26)$$

#### 4. Calculations at Critical Point

In order to elucidate the divergence character of the quantity  $\Delta_\infty/p^2$ , regarded as a function of  $p$ , at

the critical point, let us make calculations in this case. Let us return to formula (20) and put  $T = T_c$  in it, which means that  $n_0 = 0$ . Then,

$$\Delta_\infty = \frac{p_0^2 \varkappa p^2}{2\pi^2 \beta \rho} \int_0^1 da \int_0^\infty \frac{xa^2 dx}{(x^2 - a^2)[(1 + \varkappa)xp + \gamma]} + o(p^2). \quad (27)$$

Again, let us factorize the denominator of the integrand, expand the resulting integrand in simple fractions, and integrate over the variable  $x$ . As a result, we obtain

$$\Delta_\infty = \frac{p_0^2 \varkappa p^2}{2\pi^2 \beta \rho (1 + \varkappa)} \int_0^1 \frac{a^2 x_0 \ln |a/x_0|}{a^2 p^2 - x_0^2} da + o(p^2),$$

where  $x_0/p$  is a root of the equation

$$(1 + \varkappa)xp + \gamma = 0,$$

and  $x_0 = -\gamma/(1 + \varkappa)$ . Changing the variables,  $a/|x_0| = \xi$ , in the integral above, we have

$$\begin{aligned} \Delta_\infty &= \frac{p_0^2 \varkappa x_0^2 p}{2\pi^2 \beta \rho (1 + \varkappa)} \int_0^{1/|x_0|} \frac{\xi^2 \ln \xi}{\xi^2 - 1} + o(p^2) = \\ &= -\frac{p_0^2 \varkappa x_0^2 p}{2\pi^2 \beta \rho (1 + \varkappa)} \int_0^{1/|x_0|} \xi^2 \ln \xi d\xi + o(p^2), \end{aligned} \quad (28)$$

since  $1/|x_0| \sim p$  ( $p \rightarrow 0$ ). As a result, we obtain

$$\Delta_\infty = \frac{p_0^2 \varkappa p^2}{18\pi^2 \beta \rho (1 + \varkappa) |x_0|} \left(1 - 3 \ln \left| \frac{p}{x_0} \right| \right) + o(p^2).$$

Hence, we showed that the quantity  $\Delta_\infty/p^2$  diverges at the critical point as  $\ln |p|$  ( $p \rightarrow 0$ ). Such a singularity is typical of critical phenomena. It can be interpreted as a consequence of the expansion of the one-particle spectrum of a Bose liquid in a vicinity of the critical point:

$$\begin{aligned} \frac{\hbar^2 \tilde{p}^2}{2\tilde{m}} \left( \frac{p}{\tilde{p}} \right)^{2-\eta} &= \frac{\hbar^2 \tilde{p}^2}{2\tilde{m}} \left( \frac{p}{\tilde{p}} \right)^2 e^{-\eta \ln(p/\tilde{p})} = \\ &= \frac{\hbar^2 p^2}{2\tilde{m}} \left( 1 - \eta \ln \left( \frac{p}{\tilde{p}} \right) \right) + o(\eta), \end{aligned} \quad (29)$$

where  $\eta$  is the small critical index, and  $\tilde{p}$  a characteristic scale of the wave vector in a vicinity of the critical point. Taking into account that only the quantity

$\Delta_\infty$  gives a non-zero contribution to the one-particle spectrum of a Bose liquid at the critical point, we obtain the following equation for the determination of  $\eta$  and  $\tilde{p}$ :

$$\begin{aligned} \frac{p_0^2 \varkappa p^2}{18\pi^2 \beta \rho (1 + \varkappa) |x_0|} \left( 1 - 3 \ln \left| \frac{p}{x_0} \right| \right) = \\ = \frac{p^2}{p_0^2 \tilde{\beta}} \left( 1 - \eta \ln \left( \frac{p}{\tilde{p}} \right) \right). \end{aligned} \quad (30)$$

From whence, we have

$$\begin{aligned} \eta &= \frac{4}{3\pi^2} \approx 0.135, \\ \tilde{p} &= |x_0| \exp \left( \frac{\eta - 3}{3\eta} \right) \approx 1.68 \cdot 10^{-3} \text{ \AA}^{-1}. \end{aligned} \quad (31)$$

The result for the small critical index  $\eta$  was obtained for the first time in works [28,29]. The cited authors used a method of expansion in reciprocal powers of the order parameter dimensionality. The random-phase approximation reproduces only the first term of this expansion. Therefore, it is no wonder that the result obtained for the small critical index differs from the result of Monte-Carlo simulations [30].

## 5. Calculations at Above-Critical Temperatures

At temperatures higher than the critical one, quantity (15) acquires the form

$$\begin{aligned} \Delta_\infty &= \frac{p_0^2 \tilde{z}_0 \varkappa}{4\pi^2 \beta \rho} \int_0^\infty \frac{qdq}{q + \tilde{\gamma} \operatorname{arctg} \left( \frac{q}{2P_0} \right)} \times \\ &\times \left\{ \frac{q}{2p} \ln \left| \frac{P_0^2 + (q+p)^2}{P_0^2 + (q-p)^2} \right| - \frac{2q^2}{P_0^2 + q^2} \right\}, \end{aligned} \quad (32)$$

where  $\tilde{\gamma} = 2\gamma/\pi$ . Differentiating it with respect to  $p$ , integrating the result again over  $p$ , and changing the order of integration, we obtain

$$\begin{aligned} \Delta_\infty &= \frac{p_0^2 \tilde{z}_0 \varkappa}{4\pi^2 \beta \rho} \frac{1}{p} \int_0^p dp \int_0^\infty \frac{qdq}{q + \tilde{\gamma} \operatorname{arctg} \left( \frac{q}{2P_0} \right)} \times \\ &\times \frac{2qp^2(q^2 - p^2 - 3P_0^2)}{[P_0^2 + (p+q)^2][P_0^2 + (p-q)^2][P_0^2 + q^2]}. \end{aligned} \quad (33)$$

In order to calculate this integral, we symmetrize the limits of integration over  $q$ , make an analytical continuation of the integrand into the upper half-plane

of the complex  $q$ -variable, and close the contour of integration by a semicircle of radius  $R$ . In the limit  $R \rightarrow \infty$ , the integral along the semicircle  $R$  equals zero, because the power of the integrand's denominator is larger by two than the power of the numerator. As a result, our integral is equal to a sum of residues at the analytical continuation of the integrand into the upper half-plane times  $2\pi i$ . Only three singular points of the integrand fall within this half-plane:  $q = p + iP_0$ ,  $q = -p + iP_0$ , and  $q = iP_0$ . (Note, by the way, that the multiplier in the denominator with the arctan function does not equal to zero over the whole complex  $q$ -plane.) As a result of calculations, we obtain

$$\begin{aligned} \Delta_\infty = & \frac{p_0^2 \bar{z}_0 \varkappa \pi i}{4\pi^2 \beta \rho} \int_0^p dp \left( \frac{2P_0^2}{iP_0 + \tilde{\gamma} \arctg(i/2)} + \right. \\ & + \frac{(-p + iP_0)^2}{-p + iP_0 + \tilde{\gamma} \arctg\left(-\frac{p}{2P_0} + \frac{i}{2}\right)} + \\ & \left. + \frac{(p + iP_0)^2}{p + iP_0 + \tilde{\gamma} \arctg\left(\frac{p}{2P_0} + \frac{i}{2}\right)} \right). \end{aligned} \quad (34)$$

Without specifying the subsequent rather simple transformations, we present the final result for  $\Delta_\infty$ :

$$\begin{aligned} \Delta_\infty = & -\frac{2\pi}{p} \int_0^p dp \left\{ \frac{(f_2 + P_0)(p^2 - P_0^2) - 2pP_0(f_1 + p)}{(f_1 + p)^2 + (f_2 + P_0)^2} + \right. \\ & \left. + \frac{P_0^2}{P_0 + \tilde{\gamma} \ln(3)/2} \right\}, \end{aligned} \quad (35)$$

where

$$\begin{aligned} f_1 = & \frac{\tilde{\gamma}}{2} \arctg\left(\frac{4pP_0}{3P_0^2 - p^2}\right), \\ f_2 = & -\frac{\tilde{\gamma}}{2} \ln\left(\frac{\sqrt{(3P_0^2 - p^2)^2 + 16p^2P_0^2}}{9P_0^2 + p^2}\right). \end{aligned} \quad (36)$$

We expand the expression obtained in a series in the small parameter  $p$  and keep only the terms proportional to  $p^2$ . As a result, we obtain

$$\begin{aligned} \Delta_\infty = & -\frac{p_0^2 \bar{z}_0 \varkappa \tilde{\gamma}}{27\pi \beta \rho} \times \\ & \times \frac{(-9\tilde{\gamma} \ln^2(3) + 8P_0 + 28\tilde{\gamma} \ln(3) - 16\tilde{\gamma})}{(2P_0 + \tilde{\gamma} \ln(3))^3} p^2 + o(p^2). \end{aligned} \quad (37)$$

The corresponding contribution to the right-hand side of Eq. (11) is as follows:

$$-\frac{p_0^4 \bar{z}_0 \varkappa \tilde{\gamma}}{27\pi \rho} \frac{(-9\tilde{\gamma} \ln^2(3) + 8P_0 + 28\tilde{\gamma} \ln(3) - 16\tilde{\gamma})}{(2P_0 + \tilde{\gamma} \ln(3))^3}. \quad (38)$$

With the help of the numerical analysis, one can get convinced in the smallness of this quantity. Therefore, its contribution to the effective mass can also be neglected.

## 6. Analytical Expression for Effective Mass

Taking into account that the quantity  $\Delta_\infty$  gives an insignificant contribution to the effective mass, which was demonstrated above, and returning to the calculation scheme described in work [24], we obtain the following expression for the effective mass:

$$\bar{m} = \frac{m^*}{(1 + F(T))}, \quad (39)$$

$$\begin{aligned} \text{where} \\ F(T) = & \lim_{p \rightarrow 0} \frac{1}{N\beta\varepsilon_p} \sum_{\mathbf{q} \neq 0} \frac{\lambda_q}{1 + \lambda_q \bar{S}_0(q)} (e^{\mathbf{P} \nabla_q} - 1) \times \\ & \times \left( \bar{n}_q - \frac{1}{\bar{z}_0^{-1}(\beta\bar{\varepsilon}_q + 1 - \bar{z}_0)} \right), \end{aligned} \quad (40)$$

and  $\nabla_q$  is the gradient operator.

Let us expand the operator  $e^{\mathbf{P} \nabla_q}$  in a series and confine the expansion to first three terms, because they give us the required approximation. Making simple transformations, changing from summation to integration, and taking the meaning of notations  $p_0$  and  $P_0$  into account, we obtain the following expression for the quantity  $F(T)$ :

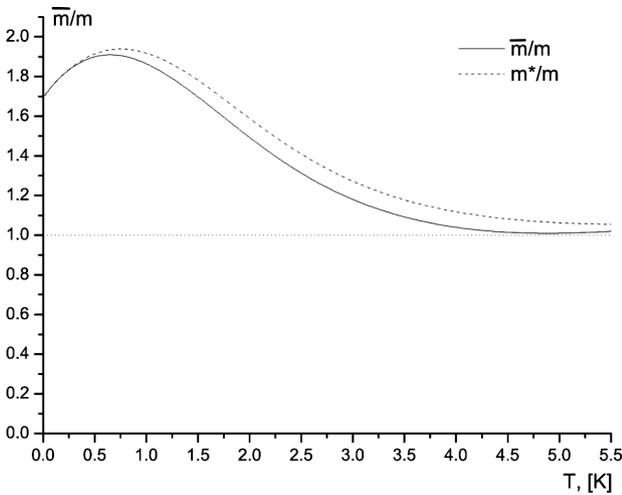
$$\begin{aligned} F(T) = & \frac{1}{2\pi^2 \rho} \int_0^\infty \frac{\lambda_q q^2 dq}{1 + \lambda_q \bar{S}_0(q)} \left( \bar{n}_q (1 + \bar{n}_q) \times \right. \\ & \left. \times \left[ \frac{2}{3} \beta \varepsilon_q (1 + 2\bar{n}_q) - 1 \right] - \frac{\bar{z}_0 (\beta \bar{\varepsilon}_q - 3 + 3\bar{z}_0)}{3 (\beta \bar{\varepsilon}_q + 1 - \bar{z}_0)^3} \right). \end{aligned} \quad (41)$$

A direct inspection easily verifies that the function  $F(T)$  equals zero in the limits of both low and high temperatures. Therefore, in those limits,  $\bar{m} = m^*$ . Using the results of work [25], we obtain that  $\lim_{T \rightarrow 0} \bar{m} \approx 1.7m$  and  $\lim_{T \rightarrow \infty} \bar{m} = m$ .

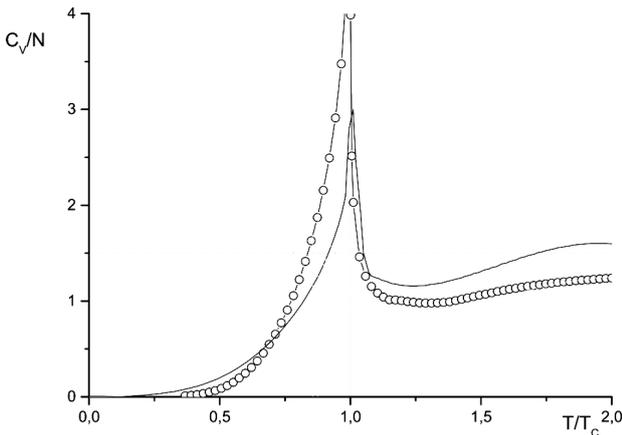
### 7. Numerical Calculation of Effective Mass and Heat Capacity

Let us illustrate the obtained result in the graphic form. For this purpose, we should numerically calculate the ratio  $\bar{m}/m$ . The corresponding calculation is self-consistent, because the expression for  $\bar{m}$  includes the quantities  $\bar{S}_0(q)$ ,  $\bar{\varepsilon}_q$ , and  $\bar{n}_q$ , which depend, in turn, on  $\bar{m}$ . In practice, this situation implies the application of an iteration process, which took 3–4 cycles in our case.

The calculations were carried out for the equilibrium helium density  $\rho = 0.02185 \text{ \AA}^{-3}$ , the parti-



**Fig. 3.** Temperature dependence of the effective mass of  $^4\text{He}$  atom



**Fig. 4.** Temperature dependence of the heat capacity of liquid  $^4\text{He}$ : (solid curve) theoretical result obtained taking the effective mass into account, (circles) experimental data [35–37]

cle mass  $m = 4.0026 \text{ amu}$ , and the sound velocity  $c = 238.2 \text{ m/s}$  in the limit  $T \rightarrow 0$  [32]. The experimentally measured structure factor  $S^{\text{exp}}(q)$  for liquid  $^4\text{He}$  extrapolated to the temperature  $T \rightarrow 0$  [33] rather than the Fourier coefficient for the energy of pairwise interparticle interaction  $\nu_q$  was used as the input information.

In Fig. 3, the temperature dependence of the effective mass of  $^4\text{He}$  atom calculated in the approximation of pair interparticle correlations is exhibited. On its base, using the known formula [34], we also calculated the temperature of the Bose condensation in liquid  $^4\text{He}$ . The obtained value is  $T_c \approx 2.18 \text{ K}$ , which is very close to the experimental value  $T_c = 2.168 \text{ K}$ .

While calculating the heat capacity, we used expression (1) for the internal energy of a many-boson system in the pair correlation approximation. We numerically differentiated it with respect to the temperature. Figure 4 demonstrates the temperature dependence of the heat capacity calculated with regard for the effective mass.

### 8. Conclusions

An expression for the temperature dependence of the effective mass of a  $^4\text{He}$  atom (in both the normal and superfluid phases) is obtained. It allows infrared divergences, which are typical of critical phenomena, to be eliminated. The expression for the effective mass is applicable at all temperatures, except for a narrow fluctuation interval between the temperature  $T_F \approx 2.13 \text{ K}$  and the temperature of phase transition. In the high-temperature limit, as well as when the interparticle interaction is “switched-off”, the effective mass transforms into the “seed” mass of a  $^4\text{He}$  atom. In the low-temperature limit, we obtain a value that coincides with the effective mass of a  $^3\text{He}$  impurity atom in liquid  $^4\text{He}$ , provided that the “seed” mass of a  $^3\text{He}$  atom is substituted by the mass of a  $^4\text{He}$  one [18]. In this context, we note that there is no common opinion concerning the effective mass even at the zero temperature, to say nothing of a wide temperature interval, because the introduction of this quantity into consideration is a phenomenological issue and, to a great extent, depends on the approaches applied for its calculation [15–17].

The behavior of the heat capacity curve theoretically calculated with regard for the effective mass is in much better agreement with the experimental data

than if without it, in particular, in the supercritical region [31]. In addition, in comparison with the “bare” mass, the effective mass obtained in this work gives a better agreement with the experimental data for the heat capacity in the temperature interval of about 0.5 K above the phase transition point [25].

The application of the effective mass made it possible to shift the phase transition point from the value for the ideal Bose gas to the temperature  $T_c \approx 2.18$  K. As was already mentioned, the latter value is very close to the experimental one. The “bare” mass gives rise to  $T_c \approx 1.94$  K in this case [25].

In the framework of the approach proposed in this work, we also succeeded in finding the small critical index  $\eta$  in the random-phase approximation. The obtained value differs rather strongly from the recommended one [30], but simultaneously reproduces the well-known result of this approximation [28].

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ЕФЕКТИВНА МАСА АТОМА  $^4\text{He}$   
В НАДПЛИННІЙ І НОРМАЛЬНІЙ ФАЗАХ

Резюме

Знайдено вираз для температурної залежності ефективної маси атома  $^4\text{He}$  в надплинній і нормальній фазах, який дозволяє усунути інфрачервоні розбіжності і є застосовним при всіх температурах за винятком вузької флуктуаційної області  $0,97 \lesssim T/T_c \leq 1$ . В границі високих і низьких температур, а також в границі виключення взаємодії, отриманий вираз дає відомі результати. На основі ефективної маси розраховано хід кривої теплоємності, а також знайдено температуру фазового переходу  $T_c \approx 2,18$  К. Використовуючи запропонований в роботі підхід, отримано значення малого критичного індексу  $\eta$  в наближенні хаотичних фаз, яке відтворює вже відомий результат цього наближення.