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CAN QUANTUM GEOMETRODYNAMICS COMPLEMENT GENERAL RELATIVITY?

The properties of the universe as a whole are considered on the grounds of classical and quantum theories. For the maximally symmetric geometry, it is shown that the main equation of the quantum geometrodynamics is reduced to the non-linear Hamilton–Jacobi equation. In the semiclassical approximation, this non-linear equation is linearized and reduces to the Friedmann equation with the additional quantum source of gravity in the form of the stiff Zel'dovich matter. The semiclassical wave functions of the universe, in which different types of matter-energies dominate, are obtained. The cases of the domination of radiation, barotropic fluid, and new quantum matter-energy are discussed. The probability of the transition from the quantum state, where radiation dominates, into the state, in which a barotropic fluid in the form of a dust is dominant, is calculated.

Keywords: universe, general relativity, quantum geometrodynamics, cosmology.

1. Introduction

The answer to the question given in the title to this paper can be provided after the comparative description of the universe in classical and quantum theories. In the quantum theory, the main object of the theory is the state vector (wave function) Ψ. In the general case, the state vector Ψ is a complex-valued function defined in some configuration space Ω, and, without loss of generality, it can always be written as $\Psi_\alpha = \mathcal{A}_\alpha\, e^{i\varphi_\alpha}$, where $\mathcal{A}_\alpha$ and φ_α are real functions of the generalized variables in Ω, and α is a set of quantum numbers, which characterize the state of the system with the state vector Ψ_α. In the region of Ω, where the phase φ_α varies by a large amount on small scales, the system under investigation can be considered as an almost classical system, in a sense that its wave properties are inessential and can be ignored when calculating the parameters of the system [1]. Nevertheless, an almost classical system still has the wave properties. They can give a probabilistic character to parameters of the system, which have no analogs in classical theory. So, the overlap integral, $\langle \Psi_{\alpha'} | \Psi_\alpha \rangle = \int d\Omega\, \mathcal{A}^*_{\alpha'} \mathcal{A}_\alpha\, e^{i(\varphi_\alpha - \varphi_{\alpha'})}$ at $\alpha' \neq \alpha$, can be nonzero because of the contribution of a subregion in Ω, where the difference $\varphi_\alpha - \varphi_{\alpha'}$ of two large phases is small. As a result, the spontaneous transition (or transition under the action of an instantaneous perturbation) $\alpha \to \alpha'$ with the change of a physical state of the system becomes possible. Classical and quantum descriptions of physical properties of the same system appear here as complementary without contradiction to each other.

As is well-known, the universe is subject to classical laws of general relativity on large spacetime scales, whereas it should be described from the quantum-theoretical perspective on small scales comparable with the Planck one. The questions whether the universe preserves the wave properties during its subsequent evolution and whether these properties can be discovered are undoubtedly interesting.

In the present paper, as a working model of spacetime geometry, we choose the maximally symmetric geometry described by the Robertson–Walker metric. In Section 2, the quantum constraint equations imposed on a state vector of the universe are given. These equations are formulated in the representation of the generalized field variables such as the cosmic scale factor and the uniform scalar field. The scalar field is described by some Hermitian Hamiltonian. Its mean values with respect to proper state vectors determine the proper energy of matter in the form of a barotropic fluid contained in the comoving volume (Section 3). In Section 4, it is shown that the main equation of the theory can be rewritten as a nonlinear Hamilton–Jacobi equation. Its nonlinear part is caused by a new source of the gravitational field, which has a purely quantum dynamical nature, and is additional to ordinary matter sources. In Section 5, the classical description of the universe evolving in time according to power and exponential laws are given in comparison with the quantum description of the same universe in the semiclassical approximation. The corresponding wave functions of the universe, in which different types of matter-energies dominate, are obtained. As examples, the cases of the domination of radiation or a barotropic fluid are discussed. The case of domination of a new quantum matter-energy is special. It is shown that its energy density is negative, while its equation of state coincides with the equation of state of the stiff Zel'dovich matter. Such an energy density dominates in the sub-Planck region. Here, the wave function is constant, and the semiclassical equation of motion has an allowed trajectory in imaginary time. The fact that the wave function is non-vanishing near the initial singularity point means that, in this region, there is some source, which provides the origin of the universe with a finite nucleation rate [2, 3] (cf. Refs. [4–6]). In Discussion, the transition probability of the universe from the state, where radiation dominates, into a state, in which a barotropic fluid in the form of a dust is dominant, is calculated.

Throughout the paper, unless otherwise specified, the modified Planck system of units is used. As a result, all quantities in the equations become dimensionless. The length $l_P = \sqrt{2G\hbar/(3\pi c^3)}$ is taken as a unit of length, and the $\rho_P = 3c^4/(8\pi G l_P^2)$ is used as a unit of energy density and pressure. The mass-energy is measured in units of the Planck mass, $m_P c^2 = \hbar c/l_P$. The proper time τ is taken in units of l_P. The time parameter (conformal time) T is expressed in radians. The scalar field is taken in $\phi_P = \sqrt{3c^4/(8\pi G)}$. Here, G is Newton's gravitational constant.

2. Quantum Constraint Equations

Let us consider the homogeneous, isotropic, and spatially closed quantum cosmological system (universe). The geometry of such a universe is described by the Robertson–Walker metric. This metric has a maximally symmetric three-dimensional subspace of the four-dimensional space-time. Since we consider the spatially closed universe, the geometry of the space-time depends on a single cosmological parameter, namely the cosmic scale factor a, which describes the overall expansion or contraction of the universe [7]. The scale factor is a field variable, which determines gravity in the formalism under consideration. We assume that, from the beginning, the universe is filled with matter in the form of the uniform scalar field ϕ, the state of which is given by some Hermitian Hamiltonian, $H_\phi = H_\phi^\dagger$. This Hamiltonian is defined in a curved space-time. Therefore, in the general case, it depends on a scale factor a as a parameter, $H_\phi = H_\phi(a)$. In addition, it will be accepted that the universe is filled with a perfect fluid in the form of a relativistic matter (further referred as radiation) with the proper energy $M_\gamma = \frac{E}{2a}$ in the comoving volume $\frac{1}{2}a^3$, where E is a real constant proportional to the number of particles of the perfect fluid. The perfect fluid defines a material reference frame [8, 9].

The restrictions in the form of the first-class constraint equations are imposed on the state vector of the quantum universe $\Psi = \langle a, \phi | \Psi(T) \rangle$, where T is a time parameter. These constraints can be reduced to two equations [9–11],

$$\left(-i\partial_T - \frac{2}{3}E\right)\Psi = 0, \tag{1}$$

$$\left(-\partial_a^2 + a^2 - 2aH_\phi - E\right)\Psi = 0, \tag{2}$$

where Eq. (1) describes the time evolution of Ψ, when the number of particles of the perfect fluid conserves, while Eq. (2) determines the quantum states of the universe at some fixed instant of time $T = T_0$, T_0 is an arbitrary constant taken as a time reference point. The coefficient $\frac{2}{3}$ in Eq. (1) is caused by the choice of the parameter T as the time variable. This time variable is connected with the proper time τ by the differential equation $d\tau = adT$. Following the ADM formalism [12, 13], one can extract the so-called lapse function N, that specifies the time reference scale, from the total differential dT: $dT = Nd\eta$, where η is the "arc time" [14, 15].

The quantum constraints (1) and (2) can be rewritten in the form of the time-dependent Schrödinger-type equation

$$-i\partial_T \Psi = \frac{2}{3}\mathcal{H}\Psi, \tag{3}$$

where

$$\mathcal{H} = -\partial_a^2 + a^2 - 2aH_\phi. \tag{4}$$

The minus sign before the partial derivative ∂_T is stipulated by the specific character of the cosmological problem, namely that the classical momentum conjugate to the variable a is defined with the minus sign [16, 17] (see below).

The partial solution of Eqs. (1) and (2) has a form

$$\Psi(T) = e^{i\frac{2}{3}E(T-T_0)}\Psi(T_0), \tag{5}$$

where the vector $\Psi(T_0) \equiv \langle a, \phi|\psi\rangle$ satisfies the stationary equation

$$\mathcal{H}|\psi\rangle = E|\psi\rangle. \tag{6}$$

From the condition

$$0 = \frac{d}{dT}\int D[a,\phi]\,|\Psi|^2 = \\ = -i\frac{2}{3}\int D[a,\phi]\,\Psi^*\left[\mathcal{H}^\dagger - \mathcal{H}\right]\Psi, \tag{7}$$

where $D[a,\phi]$ is the measure of integration with respect to the fields a and ϕ chosen in an appropriate way, it follows that the operator (4) is Hermitian: $\mathcal{H} = \mathcal{H}^\dagger$.

3. Barotropic Fluid

The Hamiltonian of matter H_ϕ can be diagonalized by means of some state vectors $\langle x|u_k\rangle$ in the representation of the generalized field variable $x = x(a,\phi)$ with the measure of integration $D[a,\phi] = da\,dx$ in Eq. (7).

Assuming that the states $|u_k\rangle$ are orthonormalized, $\langle u_k|u_{k'}\rangle = \delta_{kk'}$, we obtain the equation

$$\langle u_k|H_\phi|u_{k'}\rangle = M_k(a)\,\delta_{kk'}, \tag{8}$$

which determines the proper energy $M_k(a) = \frac{1}{2}a^3\rho_m$ of a substance (barotropic fluid) in a discrete and/or continuous kth state in the volume $\frac{1}{2}a^3$ with the energy density ρ_m and the pressure

$$p_m = w_m(a)\rho_m, \tag{9}$$

where

$$w_m(a) = -\frac{1}{3}\frac{d\ln M_k(a)}{d\ln a} \tag{10}$$

is the equation of state parameter.

Since the form of the Hamiltonian H_ϕ is not specified, then, generally speaking, the proper energy $M_k(a)$ can describe ordinary matter-energy, dark matter, and dark energy. The properties of dark matter and dark energy are summarized in Refs. [7, 18]. In order to demonstrate the possible behavior of M_k as a function of a, let us consider the model of the uniform scalar field with the potential $V(\phi) = \lambda_\alpha\phi^\alpha$, where λ_α is the coupling constant, and α takes arbitrary non-negative values, $\alpha \geq 0$. Then we find [10]

$$M_k(a) = \epsilon_k\left(\frac{\lambda_\alpha}{2}\right)^{\frac{2}{2+\alpha}} a^{\frac{3(2-\alpha)}{2+\alpha}}, \tag{11}$$

where ϵ_k is an eigenvalue of the equation

$$(-\partial_x^2 + x^\alpha - \epsilon_k)|u_k\rangle = 0,$$

and $x = \left(\frac{\lambda_\alpha a^6}{2}\right)^{\frac{1}{2+\alpha}}\phi$ is the rescaled matter scalar field. The equation of state parameter in such a model does not depend on a and has a simple form $w_m(a) = \frac{\alpha-2}{\alpha+2}$. It describes the barotropic fluid in all possible states. In the case of the model ϕ^0, the field ϕ averaged over its quantum states reproduces vacuum (dark energy) in the kth state with the density $\rho_m = \lambda_0\epsilon_k$ and the function $\langle x|u_k\rangle$ in the form of a plane wave e^{ikx} with the wave vector $k = \pm\sqrt{\epsilon_k - 1}$. The model ϕ^1 describes the strings in the kth state with the energy density $\rho_m = \left(\frac{\lambda_1}{2}\right)^{2/3}\frac{2\epsilon_k}{a^2}$, where $\epsilon_k \lesseqgtr 0$ and $|u_k\rangle$ is the Airy function. In the model ϕ^2, the scalar field, after averaging over quantum states, turns into a dust with the total mass $M_k = \sqrt{2\lambda_2}(k + \frac{1}{2})$, where k is the number of dust particles (including dark matter in the corresponding model), and the density $\rho_m = \frac{2M_k}{a^3}$. The model ϕ^4 leads to the relativistic matter with the energy density $\rho_m = \left(\frac{\lambda_4}{2}\right)^{1/3}\frac{2\epsilon_k}{a^4}$, where $\epsilon_k < \infty$, and

$|u_k\rangle$ has the asymptotics in the form of a cylindrical function. In the case $\alpha = \infty$, the field ϕ averaged over the states $|u_k\rangle$ reduces to the stiff Zel'dovich matter with the density $\rho_m = \frac{2\epsilon_k}{a^6}$. For the further discussion, see Ref. [10].

4. Non-linear Hamilton-Jacobi Equation

Assuming that the set of vectors $|u_k\rangle$ is complete, the solution of Eq. (6) can be represented in the form of a superposition of states of the universe with the substance in the kth state in any form described above. We have

$$|\psi\rangle = \sum_k |u_k\rangle\langle u_k|\psi\rangle, \tag{12}$$

where the wave function $f(a) \equiv \langle u_k|\psi\rangle$ satisfies the equation

$$\left[-\partial_a^2 + a^2 - 2aM(a)\right] f = Ef. \tag{13}$$

The index k is omitted here and below, since, in what follows, we consider the universe with the proper energy of the substance in a specific kth state, $M_k(a) \equiv M(a)$. Because the operator (4) is Hermitian, it follows that the operator on the left-hand side of Eq. (13) is Hermitian as well. This equation determines the wave function corresponding to the particular eigenvalue E. Depending on the form of $M(a)$, the constant E can take the values lying in a discrete or continuous spectrum of the states of the proper energy of radiation $M_\gamma = \frac{1}{2}a^3\rho_\gamma$ with the energy density $\rho_\gamma = \frac{E}{a^4}$ and the pressure $p_\gamma = \frac{1}{3}\rho_\gamma$. Thus, the value of constant E is determined through the quantum numbers enumerating the states of the substance and radiation.

We look for the solution of Eq. (13) in the form of a wave propagating along the a direction

$$f(a) = Ae^{iS(a)}, \tag{14}$$

where A is the normalizing constant, and the phase $S(a)$ is a complex function

$$S(a) = S_R(a) + iS_I(a) \tag{15}$$

(with S_R and S_I real). In the general case, the solution of Eq. (13) is the superposition of the wave function $f(a)$ and its complex conjugate $f^*(a)$. Substituting Eq. (14) into Eq. (13), we find that $S_R(a)$ satisfies the non-linear equation

$$(\partial_a S_R)^2 + a^2 - 2aM(a) - E = Q(a), \tag{16}$$

where the function

$$Q(a) = \frac{3}{4}\left(\frac{\partial_a^2 S_R}{\partial_a S_R}\right)^2 - \frac{1}{2}\frac{\partial_a^3 S_R}{\partial_a S_R} \tag{17}$$

describes the new source of the gravitational field with the energy density

$$\rho_Q = \frac{Q(a)}{a^4}, \tag{18}$$

which is additional to the ordinary matter (substance and radiation). This source has the quantum dynamical nature. It emerges as a result of the expansion (or contraction) of the universe as a whole. The equation of state of a quantum source of matter-energy has a form

$$p_Q = w_Q(a)\rho_Q, \tag{19}$$

where p_Q is the pressure, and

$$w_Q(a) = \frac{1}{3}\left(1 - \frac{d\ln Q(a)}{d\ln a}\right) \tag{20}$$

is the equation of state parameter. The first term in Eq. (20) takes the correction for relativity into account. The additional source of matter-energy has a proper energy $M_Q = \frac{1}{2}a^3\rho_Q = \frac{Q}{2a}$ contained in the volume $\frac{1}{2}a^3$.

The derivative of the imaginary part of the phase $S(a)$ is

$$\partial_a S_I = \frac{1}{2}\frac{\partial_a^2 S_R}{\partial_a S_R}. \tag{21}$$

In order to clarify the physical meaning of the quantum correction in Eq. (16), we rewrite the source function $Q(a)$ as

$$Q(a) = (\partial_a S_I)^2 - \partial_a^2 S_I, \tag{22}$$

where Eq. (21) was used. From whence, it follows that if the imaginary part S_I of phase (15) is a slowly varying function of a, then one can set $Q(a) \approx 0$. In this case, Equation (16) becomes the Hamilton–Jacobi equation for the classical action S_{cl} and the wave function (14) takes the form

$$f(a) = \text{const}\, e^{iS_{cl}}. \tag{23}$$

It describes the quantum universe in the semiclassical approximation, when $S_R = S_{cl}$. In classical mechanics, the momentum is equal to the first derivative of the action with respect to the generalized coordinate. In general relativity, Hamilton's equations of motion lead to [17]

$$\partial_a S_{cl} = -\frac{da}{dT} = -a\dot{a}, \tag{24}$$

where we denote $\dot{a} = \frac{da}{d\tau}$. Equations (23) and (24) describe the same universe, but from the different point of view, namely as a quantum system in the approximation $S_R = S_{cl}$ or as a classical object obeying the laws of general relativity.

To determine the physical meaning of the derivative $\partial_a S_R$, we calculate the probability flux density for the universe to be a hypersurface with radius a in a four-dimensional space. From Eq. (16), it follows that the probability flux density is described by the expression

$$J_a = \frac{1}{2i}\left(f^* \partial_a f - f \partial_a f^*\right). \tag{25}$$

Taking Eq. (14) into account, we find

$$J_a = |f|^2\, \partial_a S_R, \tag{26}$$

where $|f|^2 = |A|^2 e^{-2S_I}$ is the probability density for the universe to have the scale factor a. Equation (26) shows that the wave function (14) describes the expansion of the universe as a whole with the generalized momentum $\partial_a S_R$. The generalized action S_R is a solution of the non-linear Hamilton–Jacobi equation (16).

One can make sure that the energy density (18) is the quantum correction to the energy density of the substance and radiation by rewriting Eq. (16) in dimensional physical units

$$(\partial_a S_R)^2 + \left(\frac{3\pi c^3}{2G}\right)^2 a^2 \times \\ \times \left[1 - \frac{8\pi G}{3c^4} a^2 \left(\rho_m + \rho_\gamma + \rho_Q\right)\right] = 0, \tag{27}$$

where

$$\rho_m = \frac{M(a)}{2\pi^2 a^3}, \quad \rho_\gamma = \frac{E}{a^4}, \quad \rho_Q = \frac{\hbar^2}{6\pi^3}\frac{G}{c^2}\frac{Q(a)}{a^4} \tag{28}$$

are the energy densities of the substance, radiation, and quantum addition measured in GeV/cm^3. Here, a is taken in cm, $M(a)$ in GeV, E in GeV cm ($\hbar c$), S_R in GeV s ($\hbar$), whereas Q is in cm^{-2} and it has the same form as in Eq. (17). From Eq. (27), it follows that the quantum correction is proportional to $\hbar^2$. In the formal limit $\hbar \to 0$, Eq. (27) turns into the Hamilton–Jacobi equation of general relativity. A more rigorous approach requires the change to dimensionless variables, which do not contain dimensional fundamental constants G, c, and $\hbar$. The impact of the quantum correction $Q(a)$ on the dynamics of the universe as a whole is determined by how quickly the amplitude $A\, e^{-S_I(a)}$ of the wave function (14) changes with a. In Eq. (16), this impact depends on how its terms behave themselves, as a increases (decreases). In the models, in which Eq. (13) can be integrated exactly, Eq. (16) also admits a solution in an analytical form [9, 10].

5. Classical-Quantum Correspondence

Let us assume that the universe evolves in time τ with a power-law scale factor

$$a = \beta\, \tau^\alpha, \tag{29}$$

where α and β are some arbitrary constants [1]. Then the generalized momentum in the semiclassical approximation is equal to

$$\partial_a S_R = -\alpha\, \beta^{\frac{1}{\alpha}} a^{\frac{2\alpha-1}{\alpha}}, \tag{30}$$

and the quantum source function (17) takes the form

$$Q(a) = \frac{\gamma_\alpha}{a^2}, \tag{31}$$

where the numerator

$$\gamma_\alpha = \frac{(2\alpha-1)(4\alpha-1)}{4\alpha^2} \tag{32}$$

does not depend on a. The universal dependence of the source function (31) on the scale factor allows one to find the equation of state for quantum matter-energy for any values of the parameters α and β. So, it follows from Eq. (20) and (31) that the equation of state parameter (20) is $w_Q = 1$, and the energy density decreases, as a increases, according to the law

$$\rho_Q = \frac{\gamma_\alpha}{a^6}. \tag{33}$$

[1] Here, the constant α differs from the parameter α in Eq. (11). We use the same letter to emphasize that, in both cases, different types of matter-energy correspond to different values of α.

As was mentioned in Section 3, the stiff Zel'dovich matter has such a density. The energy density of this quantum matter can be negative ($\frac{1}{4} < \alpha < \frac{1}{2}$), positive ($\alpha > \frac{1}{2}$, and $\alpha < \frac{1}{4}$), or vanish ($\alpha = \frac{1}{2}$, and $\alpha = \frac{1}{4}$).

From the point of view of quantum theory, the expansion of the universe with the scale factor (29) is described by the semiclassical wave function

$$f_\alpha(a) = A_\alpha\, a^{-\frac{2\alpha-1}{2\alpha}} \exp\left\{-i\frac{\alpha^2 \beta^{\frac{1}{\alpha}}}{3\alpha - 1}\, a^{\frac{3\alpha-1}{\alpha}}\right\} \tag{34}$$

at $\alpha \neq \frac{1}{3}$, and

$$f_{\frac{1}{3}}(a) = A_{\frac{1}{3}}\, a^{\frac{1}{2} - i\frac{\beta^3}{3}}. \tag{35}$$

The complex conjugate function $f^*_\alpha(a)$ corresponds to the wave propagating toward the initial cosmological singularity point, $a = 0$, and describes the contracting universe.

In the limit of infinitely large values of α, we have $\lim_{\alpha\to\infty} \gamma_\alpha = 2$, and the quantum correction in Eq. (16) equals

$$Q(a) = \frac{2}{a^2}. \tag{36}$$

The same additional term is produced by the universe expanding exponentially. Really, setting

$$a = a(0)\, e^{\sqrt{\rho_v}\tau}, \tag{37}$$

where ρ_v is some constant (e.g., $\rho_v = \frac{\Lambda}{3}$, where Λ is the cosmological constant), we calculate the momentum of the universe

$$\partial_a S_R = -\sqrt{\rho_v}\, a^2. \tag{38}$$

Substituting (38) into Eq. (17), we obtain the quantum correction in the form (36). The exponentially expanding universe is described by the semiclassical wave function

$$f_\infty(a) = A_\infty\, \frac{1}{a}\, e^{-i\frac{\sqrt{\rho_v}}{3}\, a^3}. \tag{39}$$

At infinity, this function vanishes, but it diverges at the point $a = 0$. However, this point is inaccessible, since $\tau \geq 0$ in Eq. (37). Function (39) can be normalized to a constant. The probability flux density (26) for the wave function (39) is

$$J_\infty = -\sqrt{\rho_v}\, |A_\infty|^2. \tag{40}$$

Approximation (31) linearizes Eq. (16), and it can be considered as the Hamilton–Jacobi equation for the generalized action S_R with the additional source of the gravitational field with the energy density (33). Using Eq. (30), this equation can be easily reduced to the Friedmann equation for the Hubble expansion rate $H = \frac{\dot{a}}{a}$. In the semiclassical approximation, $\partial_a S_R = -a\dot{a}$, we have

$$H^2 = \frac{2M(a)}{a^3} + \frac{E}{a^4} + \frac{\gamma_\alpha}{a^6} - \frac{1}{a^2}. \tag{41}$$

In addition to the energy density of the substance ($\sim M(a)\, a^{-3}$) and radiation ($\sim a^{-4}$), this equation contains the energy density of the quantum source ($\sim a^{-6}$).

Let us find out what restriction on the solutions of Eqs. (16) and (41) is imposed by Eq. (29). Since it is assumed that the universe evolves according to law (29) with a given α, this means, according to the standard model, that the approximation of a single component domination in the total energy density of matter energy ρ is used. It has the form [7, 19, 20]

$$\rho \sim \frac{1}{a^{2/\alpha}} \quad \text{for } a \sim \tau^\alpha. \tag{42}$$

The substitution of Eqs. (30) and (31) into Eq. (16) gives the condition

$$\alpha^2\, \beta^{\frac{2}{\alpha}}\, a^{\frac{2}{\alpha}(2\alpha-1)} + a^2 - 2\, a\, M(a) - E = \gamma_\alpha\, a^{-2}. \tag{43}$$

Let $\alpha = \frac{1}{2}$. Then $\gamma_{\frac{1}{2}} = 0$, and Eq. (43) is equivalent to

$$\frac{2M(a)}{a^3} + \frac{E}{a^4} - \frac{1}{a^2} = \frac{\beta^4}{4}\frac{1}{a^4}. \tag{44}$$

Then Eq. (41) yields

$$H^2 = \left(\frac{\beta^2}{2a^2}\right)^2 \quad \text{or} \quad H = \frac{1}{2\tau}, \tag{45}$$

if Eq. (29) is used. This equation describes the spatially flat universe, in which radiation dominates. Hence, model (29) does not contradict Eq. (41). Such a universe expands with constant momentum

$$\partial_a S_R = -\frac{\beta^2}{2}, \tag{46}$$

and, according to Eq. (17), an additional quantum source of energy is not generated, $Q(a) = 0$. The quantum properties of such a universe are described by the wave function

$$f_{\frac{1}{2}}(a) = A_{\frac{1}{2}}\, e^{-i\sqrt{E}a} = A_{\frac{1}{2}}\, e^{-iHa^3}. \tag{47}$$

This function is a solution to Eq. (13), in which the terms a^2 and $2aM(a)$ are omitted, i.e. the quantum universe is spatially flat and contains nothing but radiation. Comparing Eq. (47) with Eq. (34) at $\alpha = \frac{1}{2}$, we find the constant β of Eq. (29):

$$\beta = \left(2\sqrt{E}\right)^{1/2}. \tag{48}$$

This expression coincides with the one that can be obtained directly from the solution of Eq. (41) in the approximation specified above.

The wave function of the continuous state $f_{\frac{1}{2}}(a) \equiv f_{\frac{1}{2}}(a;E)$ (47) can be normalized to a delta-function

$$\langle f_{\frac{1}{2}}(E)|f_{\frac{1}{2}}(E')\rangle = \delta(E-E'). \tag{49}$$

This condition determines the normalizing constant $A_{\frac{1}{2}}$ (up to an inessential phase factor),

$$|A_{\frac{1}{2}}|^2 = \frac{1}{2\sqrt{E}}. \tag{50}$$

According to Eqs. (46) and (48), the probability flux density (26) for the universe with the wave function (47) and the amplitude from Eq. (50) does not depend on a and E and equals

$$J_{\frac{1}{2}} = -\frac{1}{2}. \tag{51}$$

In this case, the conservation law is fulfilled, $\partial_a J_{\frac{1}{2}} = 0$. The minus sign in Eq. (51) shows that the matter flux is directed away from the observer, i.e. matter objects (galaxies) move away from the observer, by demonstrating the effect of expansion of the universe.

Using the wave packet (proper differential)

$$\bar{f}_{\frac{1}{2}}(a;E) = \int\limits_{E-\delta}^{E+\delta} dE' f_{\frac{1}{2}}(a;E') \tag{52}$$

with the width $2\delta \ll 1$, $\delta > 0$, the wave function $f_{\frac{1}{2}}(a;E)$ can be normalized to 1 as follows:

$$\langle f_{\frac{1}{2}}(E)|\bar{f}_{\frac{1}{2}}(E)\rangle = 1. \tag{53}$$

From Eqs. (49) and (52), it follows that integral (53) does not depend on δ. The smaller δ, the greater the accuracy, with which the wave packet (52) reproduces the wave function (47). The wave function (47) with amplitude (50) can be interpreted as a part of the de Broglie wave propagating along a with the momentum $\partial_a S_R = -\sqrt{E}$.

In the case $\alpha = \frac{2}{3}$, we have $\gamma_{\frac{2}{3}} = \frac{5}{16}$, and the quantum source function (17) is

$$Q(a) = \frac{5}{16}\frac{1}{a^2}. \tag{54}$$

The quantum source is characterized by the positive energy density ρ_Q (18). The condition (43) is written as

$$\frac{2M(a)}{a^3} + \frac{E}{a^4} + \frac{5}{16}\frac{1}{a^6} - \frac{1}{a^2} = \frac{4}{9}\left(\frac{\beta}{a}\right)^3, \tag{55}$$

and Eq. (41) gives the Hubble expansion rate

$$H^2 = \frac{4}{9}\left(\frac{\beta}{a}\right)^3 \text{ or } H = \frac{2}{3\tau}. \tag{56}$$

The latter equation describes a spatially flat universe, where the non-relativistic matter with the mass $M(a) \equiv M = const$ dominates. In this case, the constant β is determined by Eqs. (56) and (41), in which the domination of matter is taken into account:

$$\beta = \left(\frac{3}{2}\sqrt{2M}\right)^{2/3}. \tag{57}$$

The universe expands as a whole with the momentum

$$\partial_a S_R = -\sqrt{2M}\, a^{1/2} = -Ha^2. \tag{58}$$

The wave function has a form

$$f_{\frac{2}{3}}(a) = \frac{A_{\frac{2}{3}}}{a^{1/4}} \exp\left\{-i\frac{2}{3}\sqrt{2M}\, a^{3/2}\right\} = \frac{A_{\frac{2}{3}}}{a^{1/4}} e^{-i\frac{2}{3}Ha^3}. \tag{59}$$

This function is the asymptotics of the Airy function at $2aM \gg 1$. The Airy function is a solution of Eq. (13) with $M(a) = M = const$ for the spatially flat universe (the term a^2 in Eq. (13) should be omitted). In the domain $a \gg \frac{E}{2M}$, the Airy function describes the continuous state with respect to E and can be normalized to a delta-function $\delta(E-E')$ [1]. Asymptotics (59) describes the state with $E=0$. Therefore, it is convenient to normalize it by the condition

$$\langle f_{\frac{2}{3}}(M)|f_{\frac{2}{3}}(M')\rangle = \delta(M-M'), \tag{60}$$

where we denote $f_{\frac{2}{3}}(a) \equiv f_{\frac{2}{3}}(a;M)$. In order to calculate the parameters of the universe in state (59), the wave function can be normalized to 1,

$$\langle f_{\frac{2}{3}}(M)|\bar{f}_{\frac{2}{3}}(M)\rangle = 1, \tag{61}$$

instead of (60), where

$$\bar{f}_{\frac{2}{3}}(a;M) = \int\limits_{M-\delta}^{M+\delta} dM' f_{\frac{2}{3}}(a;M') \tag{62}$$

is the wave packet with the width $2\delta \ll 1$, and $\delta > 0$. The amplitude $A_{\frac{2}{3}}$ in Eq. (59) can be found under the assumption that the probability flux density is conserved in the expanding universe and equals to the probability flux density in the radiation-dominated era (51). As a result, we obtain

$$|A_{\frac{2}{3}}|^2 = \frac{1}{2\sqrt{2M}}. \tag{63}$$

The case $\alpha = \frac{1}{3}$ in the quantum description appears to be special. The wave function has the form (35). Parameter (32) takes the value $\gamma_{\frac{1}{3}} = -\frac{1}{4}$, and the quantum source function (17) is

$$Q(a) = -\frac{1}{4a^2}, \tag{64}$$

so that the energy density (18) is negative

$$\rho_Q = -\frac{1}{4a^6}. \tag{65}$$

Then condition (43) takes the form

$$\frac{2M(a)}{a^3} + \frac{E}{a^4} - \frac{1}{4a^6} - \frac{1}{a^2} = \frac{1}{9}\left(\frac{\beta}{a}\right)^6. \tag{66}$$

Equation (41) together with Eq. (66) define the Hubble expansion rate

$$H^2 = \frac{1}{9}\left(\frac{\beta}{a}\right)^6 \quad \text{or} \quad H = \frac{1}{3\tau}. \tag{67}$$

This equation describes a spatially flat universe, in which the stiff Zel'dovich matter with the energy density (65) dominates. From Eq. (41), it follows that, for such a universe,

$$H^2 = -\frac{1}{4a^6}, \tag{68}$$

because the energy densities of a substance and radiation and the curvature term should be neglected. This corresponds to the domain of values $a < 1$ (sub-Planck scales). Equations (67) and (68) impose a restriction on the allowed values of the parameter β

$$\frac{\beta^6}{9} = -\frac{1}{4}. \tag{69}$$

If this condition is satisfied, then it means that, in the sub-Planck region, where the energy density (65) dominates, the wave function (35) is either constant

$$f_{\frac{1}{3}}(a) = A_{\frac{1}{3}} \quad \text{at} \quad \frac{1}{2} = i\frac{\beta^3}{3}, \tag{70}$$

or increases linearly with a,

$$f_{\frac{1}{3}}(a) = A_{\frac{1}{3}}\, a \quad \text{at} \quad \frac{1}{2} = -i\frac{\beta^3}{3}. \tag{71}$$

In the case of Eq. (70), we have $f_{\frac{1}{3}}(0) = A_{\frac{1}{3}} = \text{const}$, i.e. there is a source at the point $a=0$. This may indicate that the universe can originate from the initial cosmological singularity point with a finite nucleation rate $\Gamma \sim |f_{\frac{1}{3}}(0)|^2$. In a more rigorous consideration, it appears that such an origin occurs not from the point $a=0$, but from the whole domain of values of $a \leq \frac{1}{2\sqrt{E}}$, where the wave function has the form (70) [2, 3]. In this domain, there exists the classical trajectory in the imaginary time $t = -i\tau +$ const, which is a solution of Eq. (41), where the energy density of a substance and the curvature term are neglected. Near $a \sim 0$, one has

$$a = \left(-\frac{3}{2}i\tau\right)^{1/3} \equiv \beta\, \tau^{1/3}. \tag{72}$$

From this solution, the condition on β follows unambiguously, which chooses solution (70).

We come to the conclusion that the quantum-mechanical description of the universe in the semi-classical approximation admits the possibility of the origin of the universe from a sub-Planck domain. As was shown in Refs. [2, 3], the origin of the universe is accompanied by a change in the space-time topology, so that the geometry conformal to a unit four-sphere in a five-dimensional Euclidean flat space changes into the geometry conformal to a unit four-hyperboloid embedded in a five-dimensional Lorentz-signatured flat space. On the boundary, where these two subregions adjoin each other, there is a jump with the metric signature change [5, 6].

6. Discussion

Where and how can the wave properties of the almost classical universe become apparent? The state vector $|\psi\rangle$ (12), which does not depend on the time explicitly, is the superposition of the wave functions $f_k(a)$ of all possible kth states of matter in the universe. The wave functions $f_k \equiv f$ are determined by Eq. (13) for different proper energies of matter $M_k(a) \equiv M(a)$. Therefore, in the general case, the overlap integral $\langle f_{k'}|f_k\rangle$ with $k' \neq k$ will be nonzero. In model (29), the type of matter is defined by the parameter α. Let us calculate the probability of the transition of the universe from the state, where radiation dominates ($\alpha = \frac{1}{2}$), into a state, in which a barotropic fluid in the form of a dust is dominant ($\alpha = \frac{2}{3}$). This probability is determined by the expression

$$w\,(\text{rad.} \to \text{dust}) = \frac{|\langle f_{\frac{2}{3}}|f_{\frac{1}{2}}\rangle|^2}{\langle f_{\frac{1}{2}}|\bar{f}_{\frac{1}{2}}\rangle\langle f_{\frac{2}{3}}|\bar{f}_{\frac{2}{3}}\rangle}. \tag{73}$$

Using the wave functions (47) and (59), we have

$$|\langle f_{\frac{2}{3}}|f_{\frac{1}{2}}\rangle|^2 = |A_{\frac{2}{3}}|^2\,|A_{\frac{1}{2}}|^2\, I(E, M), \tag{74}$$

where

$$I(E, M) = \left|\int_0^\infty da\, a^{-1/4} \times \exp\left\{i\sqrt{E}\, a^{3/2}\left(\frac{1}{\sqrt{a_c(E, M)}} - \frac{1}{\sqrt{a}}\right)\right\}\right|^2, \tag{75}$$

and

$$a_c(E, M) = \frac{9}{8}\,\frac{E}{M}. \tag{76}$$

The integration in Eq. (75) can be performed analytically. As a result, $I(E, M)$ will have the form of the sum of terms containing generalized hypergeometric functions. However, for our illustrative purposes, it is sufficient to calculate the integral in Eq. (75), by using the method of approximate calculation of overlap integrals of semiclassical wave functions mentioned in Introduction. In the region $a \gg a_c$, the exponential function in the integrand oscillates rapidly, and its contribution into the integral is exponentially small [1]. The main contribution into the integral comes from the region near $a = a_c$, where the exponential function is almost 1. In this approximation, we have

$$I(E, M) = \frac{3}{\sqrt{2}}\left(\frac{E}{M}\right)^{3/2}. \tag{77}$$

Then, with regard for Eqs. (50), (53), (62), (63), and (77), we obtain the following simple expression for probability (73):

$$w\,(\text{rad.} \to \text{dust}) = \frac{3}{8}\,\frac{E}{M^2}. \tag{78}$$

We estimate a_c, I, and w, by using the values $E_0 = 1.86\times10^{118}$ and $M_0 = 0.92\times10^{61}$. They correspond to the modern values of the energy densities of radiation and matter, $\rho_\gamma^0 = 2.61 \times 10^{-10}$ GeV cm^{-3} and $\rho_m^0 = \rho_{\text{crit}} = 0.48 \times 10^{-5}$ GeV cm^{-3}, and the Hubble length $a_0 \equiv \frac{c}{H_0} = 1.37 \times 10^{28}$ cm taken as a rough estimate of the size of the observable universe [2]. We have

$$a_c(E_0, M_0) = 2.27 \times 10^{57} (= 1.69 \times 10^{24}\ \text{cm}), \tag{79}$$

$$I(E_0, M_0) = 1.93 \times 10^{86}, \tag{80}$$

$$w_0 = 0.83 \times 10^{-4}. \tag{81}$$

Then the redshift is $z_c = 2.41\, z_{eq}$, where $z_{eq} = 3360$ is the redshift of matter-radiation equality. Parameter (80) is close to the number of photons $N_\gamma = 4.47 \times 10^{87}$ in the volume $\frac{4}{3}\pi a_0^3 = 1.09 \times 10^{85}$ cm^3. Probability (81) coincides with the matter density contrast $\frac{\Delta\rho_m}{\rho_m} \sim 10^{-4}$ in the era of matter-radiation equality, when perturbations begin to grow mainly at the expense of cold dark matter like WIMPs (see, e.g., Ref. [21]). Such a coincidence is not accidental. Let us consider fluctuations of the matter density, which occur on account of radiation. Then we can write $\Delta\rho_m = (\rho_\gamma + \rho_m) - \rho_m = \rho_\gamma$, and the matter density contrast is equal to $\frac{\Delta\rho_m}{\rho_m} = \frac{E}{2aM}$, where a is the scale factor of the universe with mass M. The quantum

[2] All astrophysical constants and parameters, used here and below, are taken from Ref. [7].

calculations show that the mean value of the scale factor in the state of the universe with mass $M \gg 1$ is $\langle a \rangle = M$ [9]. For the modern values a_0 and M_0 for the observable universe, the following equality holds $a_0 \approx 2M_0$. So that for the matter density contrast, we have

$$\frac{\Delta\rho_m}{\rho_m} = \zeta \frac{E}{M^2}, \quad \text{with } \frac{1}{4} < \zeta < \frac{1}{2}. \tag{82}$$

For $\zeta = \frac{3}{8}$, this agrees with Eq. (78).

This example demonstrates that the parameters of the classical theory, which are calculated in general relativity with the use of the data of astronomical observations, are in quite good agreement with predictions of quantum geometrodynamics.

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ЧИ МОЖЕ КВАНТОВА ГЕОМЕТРОДИНАМІКА ДОПОВНИТИ ЗАГАЛЬНУ ТЕОРІЮ ВІДНОСНОСТІ?

Р е з ю м е

Властивості всесвіту як цілого розглядаються з позицій класичної та квантової теорії. Для максимально симетричної геометрії показано, що основне рівняння квантової геометродинаміки зводиться до нелінійного рівняння Гамільтона–Якобі. У квазікласичному наближенні це нелінійне рівняння лінеаризується та зводиться до рівняння Фрідмана з додатковим квантовим джерелом гравітаційного поля із гранично жорстким рівнянням стану, запропонованим Зельдовичем. Отримано квазікласичні хвильові функції всесвіту, в якому домінують різні типи матерії-енергії. Розглядаються випадки домінування випромінювання, баротропної рідини та нової квантової матерії-енергії. Обчислена імовірність переходу із квантового стану, в якому домінує випромінювання, у стан, в якому домінуючою є баротропна рідина у формі пилу.