doi: 10.15407/ujpe62.03.0271

O.O. VAKHNENKO

Bogolyubov Institute for Theoretical Physics
(14b, Metrologichna Str., Kyïv 03680, Ukraine; e-mail: vakhnenko@bitp.kiev.ua)

PACS 02.30.Ik, 63.20.Ry, 45.05.+x, 81.07.De, 36.20.-r

DISTINCTIVE FEATURES OF THE INTEGRABLE NONLINEAR SCHRÖDINGER SYSTEM ON A RIBBON OF TRIANGULAR LATTICE

The dynamics of an integrable nonlinear Schrödinger system on a triangular-lattice ribbon is shown to be critical against the value of background parameter regulated by the limiting values of concomitant fields. Namely at the critical point, the number of basic field variables is reduced by half and the Poisson structure of the system becomes degenerate. On the other hand, outside the critical point, the form of Poisson structure turns out to be an essentially nonstandard one, and the meaningful procedure of its standardization leads inevitably to the breaking of the mutual symmetry between the standardized basic subsystems. There are two possible realizations of such an asymmetric standardization, each giving rise to a total suppression of field amplitudes in one of the standardized basic subsystems at the critical value of background parameter. In the undercritical region the standardized basic field amplitudes acquire the meaning of probability amplitudes of some nonequivalent intracell bright excitations, whereas in the overcritical region such an interpretation is proven to be incorrect. A proper analysis shows that the overcritical region could be thought as the region of coexistence between the standardized subsystems of bright and dark excitations.

Keywords: integrable nonlinear system, triangular-lattice ribbon, Hamiltonian structure, soliton solution, critical contraction, symmetry breaking.

1. Introduction

In a series of works [1–4], we have proposed [1] and investigated [2–4] an integrable nonlinear Schrödinger-type ladder system, whose network of intersite resonant coupling bonds makes it possible to visualize the spatial arrangement of relevant lattice sites as a ribbon of triangular lattice characterized by two structural elements (sites) in the unit cell. Due to its multicomponent structure consisting of two basic mutually symmetric subsystems and one concomitant subsystem, the primary integrable nonlinear system exhibits a number of important and even unusual properties. Thus, it is capable to incorporate the uniform external magnetic field in terms of Peierls phases, as well as to include the effect of a uniform external field presumably of the electric origin so valuable for a rigorous modeling of Bloch oscillations.

In its original formulation [1], the integrable nonlinear Schrödinger system on a triangular-lattice ribbon (referred to as an integrable nonlinear ladder system with background-controlled intersite resonant coupling) had been found in the framework of the semidiscrete zero-curvature representation

$$\frac{d}{d\tau}L(n|z) = A(n+1|z)L(n|z) - L(n|z)A(n|z) \quad (1.1)$$

with the spectral and evolution operators $L(n|z)$ and $A(n|z)$, being given by certain 4×4 square matrices dependent on the integer space variable n (running from $-\infty$ to $+\infty$), continuous time variable τ, and spectral parameter z.

However, it has been recently shown [2, 3] that the same result can be obtained, by relying upon the spectral and evolution operators given by more simple 2×2 square matrices. Moreover, the general form of such an approach gives rise to the semidiscrete integrable nonlinear system that permits at least two reductions corresponding to two particular models distinguished by the types of their nonlinearities (attractive or repulsive) [3]. Both of reduced nonlinear systems are integrable in the Lax sense [5–7], inasmuch as each of them can be rewritten in terms of an appropriate zero-curvature representation.

Due to the distinct symmetries of their field variables and the qualitatively distinct types of permitted boundary conditions, the system with attractive-type nonlinearities and the system with repulsive-type nonlinearities should exhibit essentially distinct properties and, as a consequence, must possess absolutely distinct types of solutions. Thus, despite being two particular reductions of some general integrable nonlinear lattice system [3], the system with attractive-type nonlinearities and the system with repulsive-type nonlinearities should be treated as the systems of absolutely distinct qualities from the physical point of view. Each of the above-mentioned systems is obliged to be self-sufficient in its own domains of field variables, boundary conditions, and solutions.

In the present article, we consider the most pronounced properties dictated by the so-called natural constraints in combination with the adopted boundary conditions for the field variables as applied to the system with attractive-type nonlinearities.

2. Evolution Equations

Having been written in terms of two pairs of basic field amplitudes $q_+(n)$, $r_+(n)$ and $q_-(n)$, $r_-(n)$ accompanied by one pair of concomitant field amplitudes $\mu(n)$, $\nu(n)$, the evolution equations of the integrable nonlinear Schrödinger system on a triangular-lattice ribbon read [1–4]

$$\begin{aligned}&\mathrm{i}\dot{q}_+(n) + \beta q_-(n-1)[1 + q_+(n)r_+(n)] +\\&+ \alpha q_+(n+1)[q_+(n)r_-(n) - \nu(n)] +\\&+ \alpha\,[q_-(n) + q_+(n)\mu(n)] = 0,\end{aligned} \tag{2.1}$$

$$\begin{aligned}&-\mathrm{i}\dot{r}_+(n) + \alpha r_-(n-1)[1 + r_+(n)q_+(n)] +\\&+ \beta r_+(n+1)[r_+(n)q_-(n) - \mu(n) +\\&+ \beta\,[r_-(n) + r_+(n)\nu(n)] = 0,\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\mathrm{i}\dot{q}_-(n) + \alpha q_+(n+1)[1 + q_-(n)r_-(n)] +\\&+ \beta q_-(n-1)[q_-(n)r_+(n) - \mu(n)] +\\&+ \beta\,[q_+(n) + q_-(n)\nu(n)] = 0,\end{aligned} \tag{2.3}$$

$$\begin{aligned}&-\mathrm{i}\dot{r}_-(n) + \beta r_+(n+1)[1 + r_-(n)q_-(n)] +\\&+ \alpha r_-(n-1)[r_-(n)q_+(n) - \nu(n)] +\\&+ \alpha\,[r_+(n) + r_-(n)\mu(n)] = 0,\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\mathrm{i}\dot{\mu}(n) + \alpha q_+(n+1)[r_+(n) + r_-(n)\mu(n)] +\\&+ \beta\,[q_+(n)r_+(n) - q_-(n)r_-(n)] -\\&- \alpha r_-(n-1)[q_-(n) + q_+(n)\mu(n)] = 0,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&-\mathrm{i}\dot{\nu}(n) + \beta r_+(n+1)[q_+(n) + q_-(n)\nu(n)] +\\&+ \alpha\,[r_+(n)q_+(n) - r_-(n)q_-(n)] -\\&- \beta q_-(n-1)[r_-(n) + r_+(n)\nu(n)] = 0,\end{aligned} \tag{2.6}$$

where the amplitudes within each pair are related by the symmetry of complex conjugation $r_+(n) = q_+^*(n)$, $r_-(n) = q_-^*(n)$, $\nu(n) = \mu^*(n)$, and the overdot denotes the differentiation with respect to the time variable τ. The coupling parameters α and β can be taken as arbitrary complex-valued functions of time restricted by the only property of complex conjugation $\beta^* = \alpha$. The chosen symmetries of field amplitudes and coupling parameters ensure the attractive type of system's nonlinearities. As for the boundary conditions, we assume the basic field amplitudes to be rapidly vanishing at both spatial infinities $|n| \to \infty$ and adopt the concomitant field amplitudes to be supported by an arbitrarily fixed spatially uniform background

$$\lim_{|n|\to\infty} \mu(n) = \mu, \tag{2.7}$$

$$\lim_{|n|\to\infty} \nu(n) = \nu. \tag{2.8}$$

In the general case (*viz* for nonzero values of the limiting quantities μ and ν), two last conditions (2.7) and (2.8) are suitable for treating the suggested semidiscrete nonlinear system (2.1)–(2.6) as a system given on a ribbon of the triangular lattice (see Fig. 1). In so doing, the quantities μ and ν acquire the meaning of additional (background-controlled) coupling parameters.

In order to justify the triangular-lattice ribbon configuration of the underlying space lattice, it is sufficient to consider the linear part of our nonlinear system (2.1)–(2.6) and to observe that the quantities α,

β and $-\alpha\nu$, $-\beta\mu$ should be understood respectively as the parameters of intersite linear and composite intersite linear couplings between the basic fields.

It can be shown [2, 3] that the local densities

$$\rho_-(n) = \ln[\mu(n) - q_-(n)r_+(n)], \quad (2.9)$$

$$\rho_0(n) = \ln[1+\mu(n)\nu(n) + q_+(n)r_+(n) + q_-(n)r_-(n)], \quad (2.10)$$

$$\rho_+(n) = \ln[\nu(n) - q_+(n)r_-(n)] \quad (2.11)$$

entering the three lowest local conservation laws of the system under study (2.1)–(2.6) are mutually dependent due to the property

$$\dot{\rho}_-(n) = \dot{\rho}_0(n) = \dot{\rho}_+(n). \quad (2.12)$$

On the one hand, the chain of these equalities (2.12) forces the limiting values μ and ν of concomitant fields $\mu(n)$ and $\nu(n)$ to be time-independent. On the other hand, it should be treated as the differential version of two following natural constraints

$$\exp[\rho_-(n) - \rho_0(n)] = \frac{\mu}{1+\mu\nu}, \quad (2.13)$$

$$\exp[\rho_+(n) - \rho_0(n)] = \frac{\nu}{1+\mu\nu}, \quad (2.14)$$

where the main background parameter $\mu\nu$ can acquire only the nonnegative values by virtue of its definition. The natural constraints (2.13) and (2.14) imply that the concomitant fields $\mu(n)$ and $\nu(n)$ are actually dependent on the basic fields $q_+(n)$, $r_+(n)$ and $q_-(n)$, $r_-(n)$. Namely this observation prescribes us to call the fields $\mu(n)$, $\nu(n)$ as the concomitant ones.

It is necessary to stress that the phases of complex-valued coupling parameters can be interpreted as the magnetic fluxes threading the plane of a lattice, *i.e.* as the Peierls phases [8–10]. Moreover, the coupling parameters are capable to incorporate the impact of an external linear potential on the dynamics of the primary system (2.1)–(2.6) via the appropriate modification of their time dependences [4].

3. Poisson Structure and Hamiltonian Formulation of the Primary System

In view of the existence of the natural constraints (2.13) and (2.14), it is reasonable to introduce the

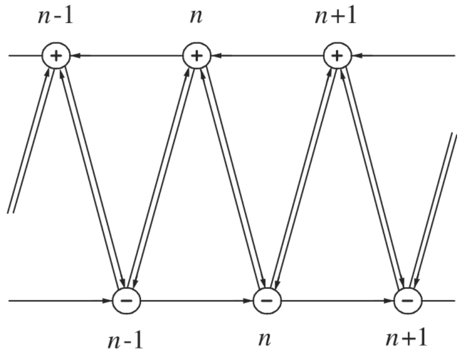

Fig. 1. Fragment of a triangular-lattice ribbon associated with the integrable nonlinear system under consideration (2.1)–(2.6). Each arrow directed to a particular site indicates the linear or composite linear coupling between this site and the site, where the arrow begins. The quantities $q_+(n)$ and $r_+(n)$ determine two complex conjugate nearly amplitudes of probability to find an upper site within the n-th unit cell being excited. The quantities $q_-(n)$ and $r_-(n)$ determine two complex conjugate nearly amplitudes of probability to find a lower site within the n-th unit cell being excited

Poisson bracket $\{\Phi, \Psi\}$ between two arbitrary functions Φ and Ψ exclusively in terms of the basic (*i.e.*, truly independent) functions $q_+(n)$, $r_+(n)$ and $q_-(n)$, $r_-(n)$.

In so doing, it is convenient to operate with the unified field variables $\mathrm{y}_\lambda(n)$ (with $\lambda = 1, 2, 3, 4$) linked to the basic ones $q_+(n)$, $r_+(n)$ and $q_-(n)$, $r_-(n)$ by the relations [3]

$$\mathrm{y}_1(n) = q_-(n), \quad (3.1)$$

$$\mathrm{y}_2(n) = q_+(n), \quad (3.2)$$

$$\mathrm{y}_3(n) = r_-(n), \quad (3.3)$$

$$\mathrm{y}_4(n) = r_+(n). \quad (3.4)$$

Then, according to the general rule [11–13], the Poisson bracket $\{\Phi, \Psi\}$ related to the inspected system (2.1)–(2.6) is determined by the expression

$$\{\Phi, \Psi\} = -\sum_{\lambda=1}^{4}\sum_{\varkappa=1}^{4}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\partial\Phi}{\partial \mathrm{y}_\lambda(n)}\mathrm{J}_{\lambda\varkappa}(n|m)\frac{\partial\Psi}{\partial \mathrm{y}_\varkappa(m)}, \quad (3.5)$$

where the matrix elements $\mathrm{J}_{\lambda\varkappa}(n|m)$ of a skew-symmetric ($\mathrm{J}_{\varkappa\lambda}(m|n) = -\mathrm{J}_{\lambda\varkappa}(n|m)$) structure matrix [12] or symplectic operator [13] were found to be specified by the following formulas [3]:

$$\mathrm{J}_{13}(n|m) = -\mathrm{i}[1 + q_-(n)r_-(n)]\delta_{nm}, \quad (3.6)$$

$$\mathrm{J}_{14}(n|m) = -\mathrm{i}[q_-(n)r_+(n) - \mu(n)]\delta_{nm}, \tag{3.7}$$

$$\mathrm{J}_{23}(n|m) = -\mathrm{i}[q_+(n)r_-(n) - \nu(n)]\delta_{nm}, \tag{3.8}$$

$$\mathrm{J}_{24}(n|m) = -\mathrm{i}[1 + q_+(n)r_+(n)]\delta_{nm}, \tag{3.9}$$

and

$$\mathrm{J}_{31}(n|m) = +\mathrm{i}[1 + r_-(n)q_-(n)]\delta_{nm}, \tag{3.10}$$

$$\mathrm{J}_{32}(n|m) = +\mathrm{i}[r_-(n)q_+(n) - \nu(n)]\delta_{nm}, \tag{3.11}$$

$$\mathrm{J}_{41}(n|m) = +\mathrm{i}[r_+(n)q_-(n) - \mu(n)]\delta_{nm}, \tag{3.12}$$

$$\mathrm{J}_{42}(n|m) = +\mathrm{i}[1 + r_+(n)q_+(n)]\delta_{nm}. \tag{3.13}$$

All other matrix elements are equal to zero identically. Thus, the form of a Poisson structure written in terms of the basic field variables $q_+(n)$, $r_+(n)$ and $q_-(n)$, $r_-(n)$ related to the primary nonlinear integrable system (2.1)–(2.6) is seen to be essentially nonstandard. The list of all Poisson brackets between field variables acquires the form [2, 3]

$$\{q_+(m), r_+(n)\} = +\mathrm{i}\,[1 + q_+(n)r_+(n)]\delta_{nm}, \tag{3.14}$$

$$\{q_+(m), r_-(n)\} = +\mathrm{i}\,[q_+(n)r_-(n) - \nu(n)]\delta_{nm}, \tag{3.15}$$

$$\{q_-(m), r_-(n)\} = +\mathrm{i}\,[1 + q_-(n)r_-(n)]\delta_{nm}, \tag{3.16}$$

$$\{q_-(m), r_+(n)\} = +\mathrm{i}\,[q_-(n)r_+(n) - \mu(n)]\delta_{nm}, \tag{3.17}$$

$$\{q_+(m), q_+(n)\} = 0 = \{r_+(m), r_+(n)\}, \tag{3.18}$$

$$\{q_+(m), q_-(n)\} = 0 = \{r_+(m), r_-(n)\}, \tag{3.19}$$

$$\{q_-(m), q_-(n)\} = 0 = \{r_-(m), r_-(n)\}, \tag{3.20}$$

$$\{\mu(m), \nu(n)\} = \mathrm{i}\,[q_+(n)r_+(n) - q_-(n)r_-(n)]\delta_{nm}, \tag{3.21}$$

$$\{\mu(m), \mu(n)\} = 0 = \{\nu(m), \nu(n)\}, \tag{3.22}$$

$$\{\mu(m), r_-(n)\} = +\mathrm{i}\,[r_+(n) + r_-(n)\mu(n)]\delta_{nm}, \tag{3.23}$$

$$\{\mu(m), q_+(n)\} = -\mathrm{i}\,[q_-(n) + q_+(n)\mu(n)]\delta_{nm} \tag{3.24}$$

$$\{\nu(m), q_-(n)\} = -\mathrm{i}\,[q_+(n) + q_-(n)\nu(n)]\delta_{nm}, \tag{3.25}$$

$$\{\nu(m), r_+(n)\} = +\mathrm{i}\,[r_-(n) + r_+(n)\nu(n)]\delta_{nm}, \tag{3.26}$$

$$\{\mu(m), r_+(n)\} = 0 = \{\nu(m), q_+(n)\}, \tag{3.27}$$

$$\{\mu(m), q_-(n)\} = 0 = \{\nu(m), r_-(n)\}. \tag{3.28}$$

Now, it is not difficult to verify that the nonlinear Schrödinger system on a triangular-lattice ribbon (2.1)–(2.6) permits the concise Hamiltonian representation given by the equations [2, 3]

$$\dot{q}_+(n) = \{H, q_+(n)\}, \tag{3.29}$$

$$\dot{r}_+(n) = \{H, r_+(n)\}, \tag{3.30}$$

$$\dot{q}_-(n) = \{H, q_-(n)\}, \tag{3.31}$$

$$\dot{r}_-(n) = \{H, r_-(n)\}, \tag{3.32}$$

$$\dot{\mu}(n) = \{H, \mu(n)\}, \tag{3.33}$$

$$\dot{\nu}(n) = \{H, \nu(n)\}, \tag{3.34}$$

where the Hamiltonian function H is determined by the expression [2, 3]

$$H = -\sum_{m=-\infty}^{\infty} \alpha\,[\mu(m) + q_+(m)r_-(m-1) - \mu] - \\ -\sum_{m=-\infty}^{\infty} \beta\,[\nu(m) + r_+(m)q_-(m-1) - \nu]. \tag{3.35}$$

Though the Hamiltonian function itself (3.35) does not manifest any nonlinear interaction due to be given by the quadratic form with respect to the field variables, the nonlinear interactions still appear in the primary dynamic system (2.1)–(2.6) thanks to the highly nonstandard form of the relevant Poisson brackets (3.14)–(3.28). The question arises whether it is possible to standardize the form of a Poisson structure and to carry over all nonlinear interactions directly into the standardized Hamiltonian function. Under some (terminologically veiled but plausible) conditions, the positive statement on this problem proclaims the Darboux theorem [11–13]. However, it does not give any reasonable prescription how to perform such a standardization. The first rational hint in resolving the puzzle of standardization has been prompted to us by the fact of the pronounced criticality of the primary (unstandardized) nonlinear system (2.1)–(2.6) against the governing background parameter $\mu\nu$. We consider the theme of criticality of the system in the next section.

4. Background-Controlled Contraction of Field Variables and Degeneration of Poisson Structure

In order to reveal the criticality of the primary nonlinear system (2.1)–(2.6) against the governing background parameter $\mu\nu$, let us rewrite two natural constraints (2.13) and (2.14) by means of three formulas

$$\mu(n) - q_-(n)r_+(n) = \mu\,\exp[+\rho(n)], \tag{4.1}$$

$$1 + \mu(n)\nu(n) + q_+(n)r_+(n) + q_-(n)r_-(n) = \\ = (1 + \mu\nu)\,\exp[+\rho(n)], \tag{4.2}$$

$$\nu(n) - q_+(n)r_-(n) = \nu \exp[+\rho(n)], \tag{4.3}$$

where the common real quantity $\rho(n)$ could be thought as the total density of excitations on both chains of a ladder lattice.

Then, combining the above relations (4.1)–(4.3) according to scheme [(4.2)][(4.2)]–4[(4.1)][(4.3)] and making some minor rearrangements, we come to the expression [4]

$$[1 - \mu(n)\nu(n)]^2 + [q_+(n)r_+(n) - q_-(n)r_-(n)]^2 + \\ + 2[q_+(n) + \nu(n)q_-(n)][r_+(n) + \mu(n)r_-(n)] + \\ + 2[q_-(n) + \mu(n)q_+(n)][r_-(n) + \nu(n)r_+(n)] = \\ = (1 - \mu\nu)^2 \exp[+2\rho(n)], \tag{4.4}$$

which is seen to be essentially critical against the value of background parameter $\mu\nu$. Precisely, at $\mu\nu = 1$ its right-hand side vanishes identically, and we are obliged to equalize each term on the left-hand side part to zero in view of the nonnegativity of each such term evident from the inherent symmetries $r_+^*(n) = q_+(n)$, $r_-^*(n) = q_-(n)$ and $\nu^*(n) = \mu(n)$ of field amplitudes.

These demands valid only at the critical point $\mu\nu = 1$ are tantamount to the extra set of constraints

$$\mu(n)\nu(n) = 1, \tag{4.5}$$

$$q_+(n) + \nu(n)q_-(n) = 0 = r_+(n) + \mu(n)r_-(n), \tag{4.6}$$

$$q_-(n) + \mu(n)q_+(n) = 0 = r_-(n) + \nu(n)r_+(n), \tag{4.7}$$

that contract the primary multicomponent nonlinear dynamic system (2.1)–(2.6) given on a ribbon of triangular lattice with two sites in the unit cell into the two-component nonlinear dynamic system

$$+ \mathrm{i}\dot{q}(n) + [\alpha q(n+1) + \beta q(n-1)][1 + q(n)r(n)] = 0, \tag{4.8}$$

$$- \mathrm{i}\dot{r}(n) + [\beta r(n+1) + \alpha r(n-1)][1 + r(n)q(n)] = 0 \tag{4.9}$$

given on a purely one-dimensional lattice with one site in the unit cell. Here, the contracted field variables $q(n)$ and $r(n)$ are defined according to the parametrization formulas

$$q_+(n) = q(n) \exp[+\mathrm{i}(2\delta - \pi)(n - 1/2)], \tag{4.10}$$

$$r_+(n) = r(n) \exp[-\mathrm{i}(2\delta - \pi)(n - 1/2)], \tag{4.11}$$

$$q_-(n) = q(n) \exp[+\mathrm{i}(2\delta - \pi)(n + 1/2)], \tag{4.12}$$

$$r_-(n) = r(n) \exp[-\mathrm{i}(2\delta - \pi)(n + 1/2)], \tag{4.13}$$

$$\mu(n) = \exp[+2\mathrm{i}\delta], \tag{4.14}$$

$$\nu(n) = \exp[-2\mathrm{i}\delta], \tag{4.15}$$

where the real phase parameter δ is assumed to be time-independent. Thus, at the critical point $\mu\nu = 1$, the primary nonlinear integrable system (2.1)–(2.6) shrinks into the simpler system (4.8), (4.9) that can be referred to as a generalization of the integrable Ablowitz–Ladik system [14, 15] to the case of time-dependent coupling parameters α and β [16]. As a result, the number of independent field variables is reduced by half, while the concomitant field variables are trivialized to the mere constants.

However, either in the undercritical region $\mu\nu < 1$ or in the overcritical region $\mu\nu > 1$, the system remains being multicomponent and cannot be reduced to a simpler one by any transformation. This statement is in lines with the fact that the Poisson structure of the primary integrable system (2.1)–(2.6) degenerates, as it will be seen only at the critical point $\mu\nu = 1$.

Indeed, the basic properties of the Poisson structure are dictated by the determinant of the structure matrix $\mathrm{J}_{\lambda\varkappa}(n|m)$ [11]. Since the structure matrix $\mathrm{J}_{\lambda\varkappa}(n|m)$ is diagonal in the spatial indices n and m, it is sufficient to deal solely with the determinant $\mathscr{D}(n)$ of the local structure matrix, i.e., with the determinant of the 4×4 square matrix $\mathrm{J}_{\lambda\varkappa}(n|n)$ marked by λ and $\varkappa$ as the only running indices. According to relations (3.6)–(3.13) specifying the elements of the structure matrix, the explicit expression for the local determinant $\mathscr{D}(n)$ is given by the formula

$$\mathscr{D}(n) = \big\{[1 + q_+(n)r_+(n)][1 + q_-(n)r_-(n)] - \\ - [\mu(n) - q_-(n)r_+(n)][\nu(n) - q_+(n)r_-(n)]\big\}^2. \tag{4.16}$$

This expression shows clearly that, at the critical value $\mu\nu = 1$, the determinant $\mathscr{D}(n)$ of the local structure matrix and, hence, the determinant $\prod_{m=-\infty}^{\infty} \mathscr{D}(m)$ of the whole structure matrix turns to zero identically by virtue of the criticality constraints (4.5)–(4.7) reducing the number of true field variables exclusively at $\mu\nu = 1$. According to the general terminology [11], the Poisson bracket (3.5) considered at the critical value of background parameter $\mu\nu = 1$ should be treated as a degenerate one.

5. Symmetry Broken Standardizations of Field Variables

At the zero background parameter $\mu\nu = 0$, the problem of standardization becomes tantamount to that proposed in our previous works [3, 17, 18]. Therefore, its solution is given by the formulas

$$q_+(n) = Q_+(n)\sqrt{\frac{\exp[Q_+(n)R_+(n)] - 1}{Q_+(n)R_+(n)}}, \quad (5.1)$$

$$r_+(n) = R_+(n)\sqrt{\frac{\exp[Q_+(n)R_+(n)] - 1}{Q_+(n)R_+(n)}} \quad (5.2)$$

and

$$q_-(n) = Q_-(n)\sqrt{\frac{\exp[Q_-(n)R_-(n)] - 1}{Q_-(n)R_-(n)}}, \quad (5.3)$$

$$r_-(n) = R_-(n)\sqrt{\frac{\exp[Q_-(n)R_-(n)] - 1}{Q_-(n)R_-(n)}} \quad (5.4)$$

with the concomitant fields $\mu(n)$ and $\nu(n)$ to be excluded by the expressions

$$\mu(n) = q_-(n)r_+(n), \quad (5.5)$$

$$\nu(n) = q_+(n)r_-(n). \quad (5.6)$$

In so doing, the list of fundamental Poisson brackets related to the standardized fields $Q_+(n)$, $R_+(n)$ and $Q_-(n)$, $R_-(n)$ reads

$$\{Q_+(m), R_+(n)\} = +\mathrm{i}\,\delta_{nm}, \quad (5.7)$$

$$\{Q_+(m), Q_+(n)\} = 0 = \{R_+(m), R_+(n)\}, \quad (5.8)$$

$$\{Q_+(m), R_-(n)\} = 0 = \{Q_+(m), Q_-(n)\}, \quad (5.9)$$

$$\{Q_-(m), R_+(n)\} = 0 = \{R_-(m), R_+(n)\}, \quad (5.10)$$

$$\{Q_-(m), Q_-(n)\} = 0 = \{R_-(m), R_-(n)\}, \quad (5.11)$$

$$\{Q_-(m), R_-(n)\} = +\mathrm{i}\,\delta_{nm}. \quad (5.12)$$

Thus, at the zero background parameter, we come to two symmetric interacting subsystems of bright excitations located on the opposite edges of a zigzag-like lattice, where the quantities $Q_+(n)R_+(n)$ and $Q_-(n)R_-(n)$ acquire the meaning of the excitation densities on the plus and minus labeled edges, respectively.

At a nonzero value of background parameter $\mu\nu \neq \neq 0$, our numerous attempts to find the standardization of the system in a symmetric form have not produced any reasonable result, until, by relying upon the criticality of the system, we came to the conclusion that the standardized system must be an essentially asymmetric one with respect to two pairs of standardized field variables, and the degree of such an asymmetry must be regulated by the value of background parameter $\mu\nu$ or, more precisely, by the limiting values μ and ν of concomitant fields $\mu(n)$ and $\nu(n)$.

On the preparatory stage of the standardization procedure, it was necessary to introduce the so-called intermediate field variables $u_+(n)$, $v_+(n)$ and $u_-(n)$, $v_-(n)$ serving to exclude the concomitant field variables $\mu(n)$ and $\nu(n)$ in the most natural way. Then, considering the expression for the total excitation density $\rho(n)$ written in terms of either of two sets $q_+(n)$, $r_+(n)$, $u_-(n)$, $v_-(n)$ or $q_-(n)$, $r_-(n)$, $u_+(n)$, $v_+(n)$ of mixed primary-intermediate field variables, we revealed its perfect separation into two parts determined by the primary and intermediate field variables, respectively. This observation in combination with the recipes developed in our previous works [19, 20] for the standardization of the simpler Ablowitz–Ladik system provided us with a strong background for the standardization of the integrable nonlinear Schrödinger system on a triangular-lattice ribbon (2.1)–(2.6).

We do not reproduce here all rather cumbersome calculations substantiating the symmetry-broken standardization, but only summarize the main results. As the matter of fact, there are two physically equivalent possibilities in a practical realization of the asymmetric standardization that we call as minus- and plus-asymmetric standardizations.

Thus, the minus-asymmetric standardization is defined by the transformation formulas

$$Q_+(n) = \sqrt{\frac{q_+(n)}{r_+(n)}\ \ln[1 + q_+(n)r_+(n)]}, \quad (5.13)$$

$$R_+(n) = \sqrt{\frac{r_+(n)}{q_+(n)}\ \ln[1 + q_+(n)r_+(n)]} \quad (5.14)$$

and

$$F_-(n) = \sqrt{\frac{q_-(n) + \mu(n)q_+(n)}{r_-(n) + \nu(n)r_+(n)}} \times \times \sqrt{\ln\frac{1+\mu(n)\nu(n)+q_+(n)r_+(n) + q_-(n)r_-(n)}{(1 + \mu\nu)[1 + q_+(n)r_+(n)]}}, \quad (5.15)$$

$$G_-(n) = \sqrt{\frac{r_-(n) + \nu(n)r_+(n)}{q_-(n) + \mu(n)q_+(n)}} \times \times \sqrt{\ln\frac{1+\mu(n)\nu(n)+q_+(n)r_+(n) + q_-(n)r_-(n)}{(1 + \mu\nu)[1 + q_+(n)r_+(n)]}}, \quad (5.16)$$

which are valid outside the critical point, (*i.e.*, at $\mu\nu \neq 1$). The list of fundamental Poisson brackets related to two sets $Q_+(n)$, $R_+(n)$ and $F_-(n)$, $G_-(n)$ of standardized fields was found to be

$$\{Q_+(m), R_+(n)\} = +\mathrm{i}\,\delta_{nm}, \tag{5.17}$$

$$\{Q_+(m), Q_+(n)\} = 0 = \{R_+(m), R_+(n)\}, \tag{5.18}$$

$$\{Q_+(m), G_-(n)\} = 0 = \{Q_+(m), F_-(n)\}, \tag{5.19}$$

$$\{F_-(m), R_+(n)\} = 0 = \{G_-(m), R_+(n)\}, \tag{5.20}$$

$$\{F_-(m), F_-(n)\} = 0 = \{G_-(m), G_-(n)\}, \tag{5.21}$$

$$\{F_-(m), G_-(n)\} = +\mathrm{i}\,\delta_{nm}. \tag{5.22}$$

It is remarkable that the formulas defining the inverse transformation

$$q_+(n) = Q_+(n)\sqrt{\frac{\exp[Q_+(n)R_+(n)] - 1}{Q_+(n)R_+(n)}}, \tag{5.23}$$

$$r_+(n) = R_+(n)\sqrt{\frac{\exp[Q_+(n)R_+(n)] - 1}{Q_+(n)R_+(n)}} \tag{5.24}$$

and

$$q_-(n) = \sqrt{\frac{F_-(n)}{G_-(n)}\{1 - \mu\nu\exp[+F_-(n)G_-(n)]\}} \times \\ \times \sqrt{\{\exp[+F_-(n)G_-(n)] - 1\}} - \\ - \mu\exp[+F_-(n)G_-(n)] \times \\ \times \sqrt{\frac{Q_+(n)}{R_+(n)}\{\exp[+Q_+(n)R_+(n)] - 1\}}, \tag{5.25}$$

$$r_-(n) = \sqrt{\frac{G_-(n)}{F_-(n)}\{1 - \mu\nu\exp[+F_-(n)G_-(n)]\}} \times \\ \times \sqrt{\{\exp[+F_-(n)G_-(n)] - 1\}} - \\ - \nu\exp[+F_-(n)G_-(n)] \times \\ \times \sqrt{\frac{R_+(n)}{Q_+(n)}\{\exp[+Q_+(n)R_+(n)] - 1\}}, \tag{5.26}$$

$$\mu(n) = \mu\exp[+F_-(n)G_-(n)] + \sqrt{\frac{R_+(n)F_-(n)}{Q_+(n)G_-(n)}} \times \\ \times \sqrt{\{1 - \mu\nu\exp[+F_-(n)G_-(n)]\}} \times \\ \times \sqrt{\{\exp[+F_-(n)G_-(n)] - 1\}} \times \\ \times \sqrt{\{\exp[+Q_+(n)R_+(n)] - 1\}}, \tag{5.27}$$

$$\nu(n) = \nu\exp[+F_-(n)G_-(n)] + \sqrt{\frac{Q_+(n)G_-(n)}{R_+(n)F_-(n)}} \times \\ \times \sqrt{\{1 - \mu\nu\exp[+F_-(n)G_-(n)]\}} \times \\ \times \sqrt{\{\exp[+F_-(n)G_-(n)] - 1\}} \times \\ \times \sqrt{\{\exp[+Q_+(n)R_+(n)] - 1\}} \tag{5.28}$$

inserted into the Hamiltonian function (3.35) allow us to exclude the concomitant fields $\mu(n)$ and $\nu(n)$ from the further consideration.

It can be shown that the quantity $Q_+(n)R_+(n)$ acquires the real nonnegative values at all admissible values $\mu\nu \geq 0$ of background parameter $\mu\nu$. Hence, it can be treated as the number of bright Q_+R_+ excitations within the n-th unit cell. In contrast, the quantity $F_-(n)G_-(n)$, though being always a real-valued one, remains nonnegative only in the undercritical region $\mu\nu < 1$. Hence, it can be treated as the number of bright F_-G_- excitations within the n-th unit cell only at $\mu\nu < 1$. Moreover, the number of F_-G_- excitations in this region turns out to be bounded from above by the restriction $\mu\nu\exp[+F_-(n)G_-(n)] \leq 1$. When the background parameter $\mu\nu$ tends to its critical value $\mu\nu = 1$, the density of F_-G_- excitations tends to zero on the whole lattice. In the overcritical region $\mu\nu > 1$, the quantity $F_-(n)G_-(n)$ acquires the nonpositive values bounded below by the restriction $1 \leq \mu\nu\exp[+F_-(n)G_-(n)]$.

Similarly, the plus-asymmetric standardization is defined by the transformation formulas

$$Q_-(n) = \sqrt{\frac{q_-(n)}{r_-(n)}\;\ln[1 + q_-(n)r_-(n)]}, \tag{5.29}$$

$$R_-(n) = \sqrt{\frac{r_-(n)}{q_-(n)}\;\ln[1 + q_-(n)r_-(n)]}, \tag{5.30}$$

$$F_+(n) = \sqrt{\frac{q_+(n) + \nu(n)q_-(n)}{r_+(n) + \mu(n)r_-(n)}} \times \\ \times \sqrt{\ln\frac{1+\mu(n)\nu(n)+q_+(n)r_+(n) + q_-(n)r_-(n)}{(1+\mu\nu)[1 + q_-(n)r_-(n)]}}, \tag{5.31}$$

and

$$G_+(n) = \sqrt{\frac{r_+(n) + \mu(n)r_-(n)}{q_+(n) + \nu(n)q_-(n)}} \times \\ \times \sqrt{\ln\frac{1+\mu(n)\nu(n)+q_+(n)r_+(n) + q_-(n)r_-(n)}{(1+\mu\nu)[1 + q_-(n)r_-(n)]}} \tag{5.32}$$

valid outside the critical point, (*i.e.*, at $\mu\nu \neq 1$). The list of Poisson brackets related to two sets $Q_-(n)$,

$R_-(n)$ and $F_+(n)$, $G_+(n)$ of standardized fields was found to be

$$\{Q_-(m), R_-(n)\} = +\mathrm{i}\,\delta_{nm}, \tag{5.33}$$

$$\{Q_-(m), Q_-(n)\} = 0 = \{R_-(m), R_-(n)\}, \tag{5.34}$$

$$\{Q_-(m), G_+(n)\} = 0 = \{Q_-(m), F_+(n)\}, \tag{5.35}$$

$$\{F_+(m), R_-(n)\} = 0 = \{G_+(m), R_-(n)\}, \tag{5.36}$$

$$\{F_+(m), F_+(n)\} = 0 = \{G_+(m), G_+(n)\}, \tag{5.37}$$

$$\{F_+(m), G_+(n)\} = +\mathrm{i}\,\delta_{nm}. \tag{5.38}$$

The formulas defining the inverse transformation read

$$q_-(n) = Q_-(n)\sqrt{\frac{\exp[Q_-(n)R_-(n)] - 1}{Q_-(n)R_-(n)}}, \tag{5.39}$$

$$r_-(n) = R_-(n)\sqrt{\frac{\exp[Q_-(n)R_-(n)] - 1}{Q_-(n)R_-(n)}} \tag{5.40}$$

and

$$q_+(n) = \sqrt{\frac{F_+(n)}{G_+(n)}\{1 - \mu\nu\exp[+F_+(n)G_+(n)]\}} \times$$
$$\times \sqrt{\{\exp[+F_+(n)G_+(n)] - 1\}} -$$
$$- \nu\exp[+F_+(n)G_+(n)] \times$$
$$\times \sqrt{\frac{Q_-(n)}{R_-(n)}\{\exp[+Q_-(n)R_-(n)] - 1\}}, \tag{5.41}$$

$$r_+(n) = \sqrt{\frac{G_+(n)}{F_+(n)}\{1 - \mu\nu\exp[+F_+(n)G_+(n)]\}} \times$$
$$\times \sqrt{\{\exp[+F_+(n)G_+(n)] - 1\}} -$$
$$- \mu\exp[+F_+(n)G_+(n)] \times$$
$$\times \sqrt{\frac{R_-(n)}{Q_-(n)}\{\exp[+Q_-(n)R_-(n)] - 1\}}, \tag{5.42}$$

$$\mu(n) = \mu\exp[+F_+(n)G_+(n)] + \sqrt{\frac{Q_-(n)G_+(n)}{R_-(n)F_+(n)}} \times$$
$$\times \sqrt{\{1 - \mu\nu\exp[+F_+(n)G_+(n)]\}} \times$$
$$\times \sqrt{\{\exp[+F_+(n)G_+(n)] - 1\}} \times$$
$$\times \sqrt{\{\exp[+Q_-(n)R_-(n)] - 1\}}, \tag{5.43}$$

$$\nu(n) = \nu\exp[+F_+(n)G_+(n)] + \sqrt{\frac{R_-(n)F_+(n)}{Q_-(n)G_+(n)}} \times$$
$$\times \sqrt{\{1 - \mu\nu\exp[+F_+(n)G_+(n)]\}} \times$$
$$\times \sqrt{\{\exp[+F_+(n)\,G_+(n)] - 1\}} \times$$
$$\times \sqrt{\{\exp[+Q_-(n)R_-(n)] - 1\}}. \tag{5.44}$$

Thus, in the case of plus-asymmetric standardization, we are also capable to exclude the concomitant fields $\mu(n)$ and $\nu(n)$ from the Hamiltonian function (3.35).

In the complete analogy with the case of minus-asymmetric standardization, it can be shown that, in the case of plus-asymmetric standardization, the quantity $Q_-(n)R_-(n)$ acquires the real nonnegative values at all values of background parameter $\mu\nu$. Hence, it can be treated as the number of bright Q_-R_- excitations within the n-th unit cell. In contrast, the quantity $F_+(n)G_+(n)$, though being always the real-valued one, remains nonnegative only in the undercritical region $\mu\nu < 1$. Hence, it can be treated as the number of bright F_+G_+ excitations within the n-th unit cell only at $\mu\nu < 1$. Moreover, the number of F_+G_+ excitations in this region turns out to be bounded above by the restriction $\mu\nu\exp[+F_+(n)G_+(n)] \leq 1$. When the background parameter $\mu\nu$ tends to its critical value $\mu\nu = 1$, the density of F_+G_+ excitations tends to zero on the whole lattice. In the overcritical region $\mu\nu > 1$, the quantity $F_+(n)G_+(n)$ acquires the nonpositive values bounded below by the restriction $1 \leq \mu\nu\exp[+F_+(n)G_+(n)]$.

6. Standardized Systems in Terms of a One-Soliton Solution

In this section, we illustrate some general results of the previous section concerning the asymmetric standardizations (5.13)–(5.16) and (5.29)–(5.32) of the primary integrable nonlinear system (2.1)–(2.6) by the example of a one-soliton solution.

Following our recent works [2, 3], the formulas for a one-soliton solution of the unstandardized system (2.1)–(2.6) given on an infinite ribbon of triangular lattice and characterized by the attractive-type nonlinearities read

$$q_+(n) = +\sinh(2\gamma)\frac{\exp[+2\mathrm{i}(\varkappa_+ + \varkappa_-)(n - \xi - y) + \mathrm{i}\chi]}{\cosh[2(\gamma_+ + \gamma_-)(n - s - x)]}, \tag{6.1}$$

$$q_-(n) = -\sinh(2\gamma)\frac{\exp[+2\mathrm{i}(\varkappa_+ + \varkappa_-)(n + \xi - y) + \mathrm{i}\chi]}{\cosh[2(\gamma_+ + \gamma_-)(n + s - x)]}, \tag{6.2}$$

$$\mu(n) = \mu -$$
$$- \frac{\exp(+2\mathrm{i}\varkappa)\sinh(2\gamma)\sinh[2(\gamma_+ + \gamma_- - \gamma)]}{\cosh[2(\gamma_+ + \gamma_-)(n - s - x)]\cosh[2(\gamma_+ + \gamma_-)(n + s - x)]}, \tag{6.3}$$

and

$$r_+(n) = +\sinh(2\gamma)\frac{\exp[-2\mathrm{i}(\varkappa_+ + \varkappa_-)(n-\xi-y)-\mathrm{i}\chi]}{\cosh[2(\gamma_+ + \gamma_-)(n-s-x)]}, \tag{6.4}$$

$$r_-(n) = -\sinh(2\gamma)\frac{\exp[-2\mathrm{i}(\varkappa_+ + \varkappa_-)(n+\xi-y)-\mathrm{i}\chi]}{\cosh[2(\gamma_+ + \gamma_-)(n+s-x)]}, \tag{6.5}$$

$$\nu(n) = \nu - \\ -\frac{\exp(-2\mathrm{i}\varkappa)\sinh(2\gamma)\sinh[2(\gamma_+ + \gamma_- - \gamma)]}{\cosh[2(\gamma_+ + \gamma_-)(n-s-x)]\cosh[2(\gamma_+ + \gamma_-)(n+s-x)]}. \tag{6.6}$$

Here, two pairs of the real constant parameters γ_+, $\varkappa_+$ and γ_-, $\varkappa_-$ are defined through two arbitrary real constant spectral parameters γ and $\varkappa$ and two constant boundary values μ and ν of concomitant fields $\mu(n)$ and $\nu(n)$ by the two sets of equations:

$$\exp(+2\gamma_+ + 2\mathrm{i}\varkappa_+) = \exp(+2\gamma + 2\mathrm{i}\varkappa) + \mu, \tag{6.7}$$

$$\exp(+2\gamma_+ - 2\mathrm{i}\varkappa_+) = \exp(+2\gamma - 2\mathrm{i}\varkappa) + \nu, \tag{6.8}$$

and

$$\exp(-2\gamma_- + 2\mathrm{i}\varkappa_-) = \exp(-2\gamma + 2\mathrm{i}\varkappa) + \mu, \tag{6.9}$$

$$\exp(-2\gamma_- - 2\mathrm{i}\varkappa_-) = \exp(-2\gamma - 2\mathrm{i}\varkappa) + \nu, \tag{6.10}$$

respectively. Another two real constant parameters s and ξ relevant to our consideration are determined by the formulas

$$2s = 1 - \frac{\gamma}{\gamma_+ + \gamma_-}, \tag{6.11}$$

$$2\xi = 1 - \frac{\varkappa}{\varkappa_+ + \varkappa_-}\ . \tag{6.12}$$

These two parameters s and ξ are responsible for the coordinate and phase splittings between the plus-labeled (upper-chain) and minus-labeled (lower-chain) basic soliton components $q_+(n)$, $r_+(n)$ and $q_-(n)$, $r_-(n)$ and consequently can serve as indicators of the convergence between them when the background parameter $\mu\nu$ tends to unity.

Relying upon definitions (6.7)–(6.10) of parameters γ_+, $\varkappa_+$ and γ_-, $\varkappa_-$, one can readily obtain the expression

$$\sinh[2(\gamma_+ + \gamma_- - \gamma)] = (1-\mu\nu)\sinh(2\gamma)K(\gamma,\varkappa|\mu,\nu), \tag{6.13}$$

where $K(\gamma,\varkappa|\mu,\nu)$ is certain essentially positive factor. This formula (6.13) clearly indicates that the sign of its left-hand term $\sinh[2(\gamma_+ + \gamma_- - \gamma)]$ is completely determined by the sign of product $(1-\mu\nu)\sinh(2\gamma)$. In particular, due to this property, the concomitant one-soliton components (6.3) and (6.6) when calculated at the critical point $\mu\nu = 1$ are reduced to their limiting constant values μ and ν. The same property of the term $\sinh[2(\gamma_+ + \gamma_- - \gamma)]$ will be shown to determine the main characteristics of standardized one-soliton components.

Indeed, applying the formulas of minus-asymmetric standardization (5.13)–(5.16) to the quantities $Q_+(n)R_+(n)$ and $F_-(n)G_-(n)$ calculated on the multicomponent one-soliton solution (6.1)–(6.6), we obtain

$$Q_+(n)R_+(n) = \ln\Bigg\{1 + \\ + \frac{\sinh(2\gamma)\sinh(2\gamma)}{\cosh[2(\gamma_+ + \gamma_-)(n-x-s)]\cosh[2(\gamma_+ + \gamma_-)(n-x-s)]}\Bigg\}, \tag{6.14}$$

$$F_-(n)G_-(n) = \ln\Bigg\{1 + \\ + \frac{\sinh(2\gamma)\sinh[2(\gamma_+ + \gamma_- - \gamma)]}{\cosh[2(\gamma_+ + \gamma_-)(n-x-3s+1)]\cosh[2(\gamma_+ + \gamma_-)(n-x+s)]}\Bigg\}. \tag{6.15}$$

Thus, in accordance with the general theory, the quantity $Q_+(n)R_+(n)$, when calculated on the one-soliton solution, acquires real nonnegative values at all admissible values of background parameter $\mu\nu$ and can be treated as the number of bright Q_+R_+ excitations within the n-th unit cell. In contrast, the sign of the quantity $F_-(n)G_-(n)$ calculated on the one-soliton solution is seen to be totally manifested by the sign of the parameter $1-\mu\nu$. Hence, the quantity $F_-(n)G_-(n)$ can be treated as the number of bright F_-G_- excitations within the n-th unit cell only at $\mu\nu < 1$. Moreover, in the limiting case of critical point $\mu\nu = 1$, the F_-G_- component of a minus-asymmetric soliton is vanished completely.

On the other hand, applying the formulas of plus-asymmetric standardization (5.29)–(5.32) to the quantities $Q_-(n)R_-(n)$ and $F_+(n)G_+(n)$ calculated on the multicomponent one-soliton solution (6.1)–(6.6), we obtain

$$Q_-(n)R_-(n) = \ln\Bigg\{1 + \\ + \frac{\sinh(2\gamma)\sinh(2\gamma)}{\cosh[2(\gamma_+ + \gamma_-)(n-x+s)]\cosh[2(\gamma_+ + \gamma_-)(n-x+s)]}\Bigg\}, \tag{6.16}$$

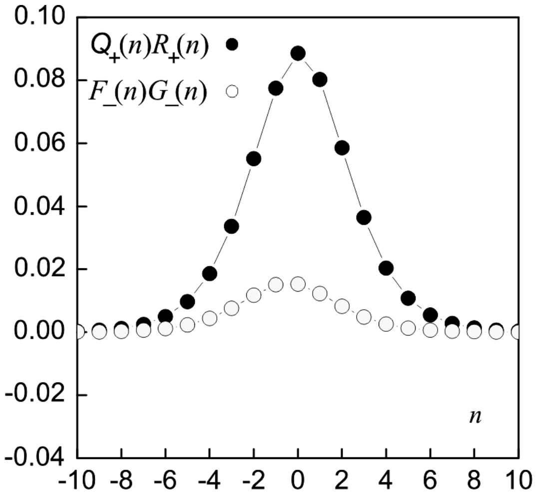

Fig. 2. The distribution of strong $Q_+(n)R_+(n)$ (full circles) and weak $F_-(n)G_-(n)$ (empty circles) one-soliton components over the unit cell number n for the case of minus-asymmetric standardization at $\mu = 0.7 = \nu$, $\gamma = 0.15$, $\varkappa = 0$, $x = 0$ according to formulas (6.14) and (6.15)

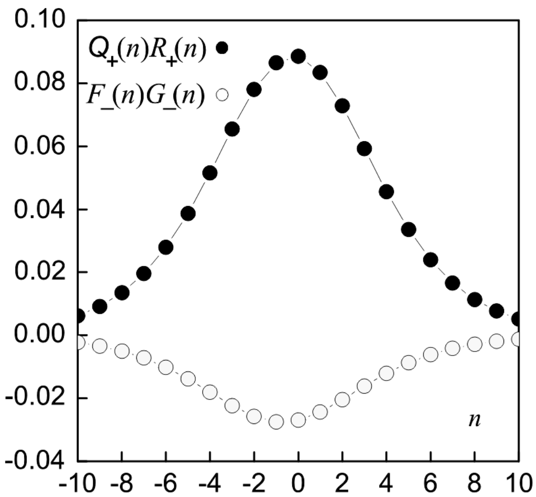

Fig. 3. The distribution of strong $Q_+(n)R_+(n)$ (full circles) and weak $F_-(n)G_-(n)$ (empty circles) one-soliton components over the unit cell number n for the case of minus-asymmetric standardization at $\mu = 1.9 = \nu$, $\gamma = 0.15$, $\varkappa = 0$, $x = 0$ according to formulas (6.14) and (6.15)

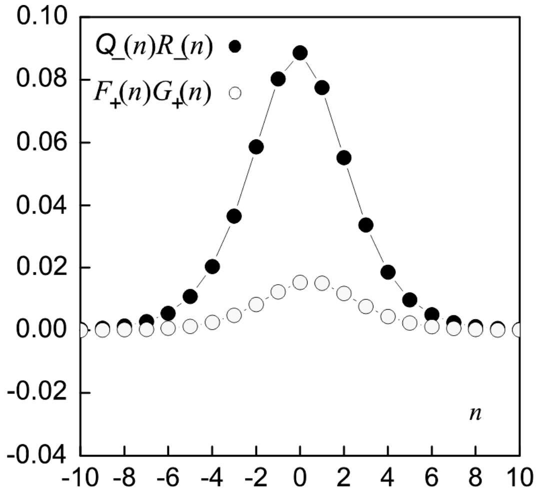

Fig. 4. The distribution of strong $Q_-(n)R_-(n)$ (full circles) and weak $F_+(n)G_+(n)$ (empty circles) one-soliton components over the unit cell number n for the case of plus-asymmetric standardization at $\mu = 0.7 = \nu$, $\gamma = 0.15$, $\varkappa = 0$, $x = 0$ according to formulas (6.16) and (6.17)

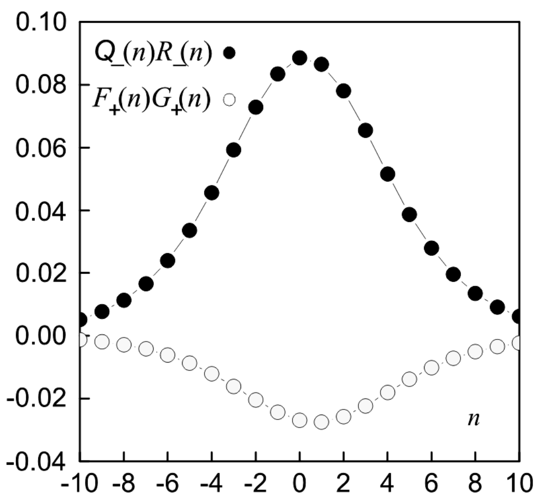

Fig. 5. The distribution of strong $Q_-(n)R_-(n)$ (full circles) and weak $F_+(n)G_+(n)$ (empty circles) one-soliton components over the unit cell number n for the case of plus-asymmetric standardization at $\mu = 1.9 = \nu$, $\gamma = 0.15$, $\varkappa = 0$, $x = 0$ according to formulas (6.16) and (6.17)

$$F_+(n)G_+(n) = \ln\left\{1 + \frac{\sinh(2\gamma)\sinh[2(\gamma_+ + \gamma_- - \gamma)]}{\cosh[2(\gamma_+ + \gamma_-)(n - x + 3s - 1)]\cosh[2(\gamma_+ + \gamma_-)(n - x - s)]}\right\}. \tag{6.17}$$

Thus, in accordance with the general theory, the quantity $Q_-(n)R_-(n)$, when calculated on a one-soliton solution, acquires the real nonnegative values at all admissible values of background parameter $\mu\nu$. Hence, it can be treated as the number of bright Q_-R_- excitations within the n-th unit cell. In contrast, the sign of the quantity $F_+(n)G_+(n)$ calculated for a one-soliton solution is seen to be totally manifested by the sign of the parameter $1 - \mu\nu$. Hence, the quantity $F_+(n)G_+(n)$ can be treated as the number of bright F_+G_+ excitations within the n-th unit cell only at $\mu\nu < 1$. Moreover, in the limiting case of critical point $\mu\nu = 1$, the F_+G_+ component of a plus-asymmetric soliton is vanished completely.

Here, we emphasize that, at the critical point $\mu\nu = 1$, the parameter of coordinate splitting s (6.11) turns to zero identically by virtue of formula (6.13) for the functional parameter $\sinh[2(\gamma_+ + \gamma_- - \gamma)]$. Thus, any contradiction between the minus-asymmetric soliton representation and the plus-asymmetric soliton representation is absent. The very existence of two nonequivalent subsystems in either of two asymmetrically standardized systems makes it possible to describe the criticality of the system in the most natural way by eliminating all excitations in one of the subsystems at the critical point.

Figures 2 and 3 calculated according to formulas (6.14) and (6.15) of the minus-asymmetric standardization illustrate the principal distinction in the interplay between two mutually asymmetric one-soliton components at undercritical $\mu\nu < 1$ (Fig. 2) and overcritical $\mu\nu > 1$ (Fig. 3) values of main background parameter $\mu\nu$.

Figures 4 and 5 calculated according to formulas (6.16) and (6.17) of the plus-asymmetric standardization illustrate the principal distinction in the interplay between two mutually asymmetric one-soliton components at undercritical $\mu\nu < 1$ (Fig. 4) and overcritical $\mu\nu > 1$ (Fig. 5) values of main background parameter $\mu\nu$.

7. Conclusion

In this article, we have summarized the most important features of the integrable nonlinear Schrödinger system on a triangular-lattice ribbon in view of a significant role that the semidiscrete nonlinear Schrödinger-type integrable models play in the description of various phenomena from various branches of physics. The list of respective references on physical applications can be found in our recent works [2–4]. We shall not repeat here all results obtained or cited in the main text of the paper inasmuch as the majority of them have been concisely formulated in the abstract. However, it is necessary to emphasize that, due to the criticality of the system, the properties of standardized nonlinear excitations in the undercritical and overcritical regions of the main background parameter are principally distinct. Namely, in the undercritical region, the standardized system consists of two subsystems of bright excitations, while, in the overcritical region, one of the two subsystems converts into a subsystem of dark nonlinear excitations. The mutual symmetry between the standardized subsystems is proven to be essentially broken at all nonzero values of main background parameter. At the zero value of background parameter, we come to two symmetric interacting subsystems of bright excitations located on the opposite edges of the zigzag-like lattice.

We have illustrated the consequences of the criticality of the system and the results of symmetry-broken standardizations by the example of a one-soliton solution both analytically and graphically.

Some preliminary results concerning the problem of standardization have been published in our short work [21]. No alternative approaches to the above problem are known. As the matter of fact, since the two alternative sets of asymmetric canonical field variables are already known, one can readily propose a number of canonical transformations to generate one or another new set of canonical field variables. However, any of such new set cannot be treated as an alternative one to the basic asymmetric set originating the very generation procedure.

The work has been supported by the National Academy of Sciences of Ukraine (Division of Physics and Astronomy) within Program No. 0117U000240. Its key results have been reported at the Bogolyubov Conference on Problems of Theoretical Physics (May 24–26, 2016, Kyïv, Ukraine). The author is grateful to Vyacheslav O. Vakhnenko for the preparation of Figures 1–5. The author acknowledges the recommendation given by the anonymous referee to illustrate the standardized soliton solutions by the graphic materials.

1. O.O. Vakhnenko. Integrable nonlinear ladder system with background-controlled intersite resonant coupling. *J. Phys. A: Math. Gen.* **39**, 11013 (2006) [DOI: 10.1088/0305-4470/39/35/005].
2. O.O. Vakhnenko. Integrable nonlinear Schrödinger system on a triangular-lattice ribbon. *J. Phys. Soc. Japan* **84**, 014003 (2015) [DOI: 10.7566/JPSJ.84.014003].
3. O.O. Vakhnenko. Nonlinear integrable model of Frenkel-like excitations on a ribbon of triangular lattice. *J. Math. Phys.* **56**, 033505 (2015) [DOI: 10.1063/1.4914510].

4. O.O. Vakhnenko. Coupling-governed metamorphoses of the integrable nonlinear Schrödinger system on a triangular-lattice ribbon. *Phys. Lett. A* **380**, 2069 (2016) [DOI: 10.1016/j.physleta.2016.04.034].
5. A.C. Newell. *Solitons in Mathematics and Physics* (SIAM Press, 1985) [ISBN: 978-0-89871-196-7].
6. L.A. Takhtadzhyan, L.D. Faddeyev. *Gamil'tonov Podkhod v Teorii Solitonov* (Nauka, 1986); L.A. Takhtadzhyan, L.D. Faddeev. *Hamiltonian Methods in the Theory of Solitons* (Springer, 1987) [ISBN: 978-3-540-69843-2].
7. G.-Z. Tu. On Liouville integrability of zero-curvature equations and the Yang hierarchy. *J. Phys. A: Math. Gen.* **22**, 2375 (1989) [DOI: 10.1088/0305-4470/22/13/031].
8. R. Peierls. Zur theorie des diamagnetismus von leitungselektronen. *Z. Phys.* **80**, 763 (1933) [DOI: 10.1007/BF01342591].
9. O.O. Vakhnenko, M.J. Velgakis. Transverse and longitudinal dynamics of nonlinear intramolecular excitations on multileg ladder lattices. *Phys. Rev. E* **61**, 7110 (2000) [DOI: 10.1103/PhysRevE.61.7110].
10. O.O. Vakhnenko. Enigma of probability amplitudes in Hamiltonian formulation of integrable semidiscrete nonlinear Schrödinger systems. *Phys. Rev. E* **77**, 026604 (2008) [DOI: 10.1103/PhysRevE.77.026604].
11. B.A. Dubrovin, S.P. Novikov, A.F. Fomenko. *Sovremennaya Geometriya. Metody i Prilozheniya* (Nauka, 1986); B.A. Dubrovin, S.P. Novikov, A.F. Fomenko. *Modern Geometry. Methods and Applications* (Springer, 1984) [ISBN: 978-1-4684-9946-9].
12. B.M. Maschke, A.J. Van Der Schaft, P.C. Breedveld. An intrinsic Hamiltonian formulation of network dynamics: Non-standard Poisson structures and gyrators. *J. Franklin Inst.* **329**, 923 (1992) [DOI: 10.1016/S0016-0032(92)90049-M].
13. V.E. Zakharov, E.A. Kuznetsov. Gamil'tonovskiy formalizm dlya nelineynykh voln. *Usp. Fiz. Nauk* **167**, 1137 (1997); V.E. Zakharov, E.A. Kuznetsov. Hamiltonian formalism for nonlinear waves. *Phys.– Uspekhi* **40**, 1087 (1997) [DOI: 10.1070/PU1997v040n11ABEH000304].
14. M.J. Ablowitz, J.F. Ladik. Nonlinear differential-difference equations. *J. Math. Phys.* **16**, 598 (1975) [DOI: 10.1063/1.522558].
15. M.J. Ablowitz, J.F. Ladik. Nonlinear differential–difference equations and Fourier analysis. *J. Math. Phys.* **17**, 1011 (1976) [DOI: 10.1063/1.523009].
16. O.O. Vakhnenko. Solitons in parametrically driven discrete nonlinear Schrödinger systems with the exploding range of intersite interactions. *J. Math. Phys.* **43**, 2587 (2002) [DOI: 10.1063/1.1458059].
17. O.O. Vakhnenko. Solitons on a zigzag-runged ladder lattice. *Phys. Rev. E* **64**, 067601 (2001) [DOI: 10.1103/PhysRevE.64.067601].
18. O.O. Vakhnenko. Inverse scattering transform for the nonlinear Schrödinger system on a zigzag-runged ladder lattice. *J. Math. Phys.* **51**, 103518 (2010) [DOI: 10.1063/1.3481565].
19. O.O. Vakhnenko. Nova povnistyu integrovna dyskretyzatsiya neliniynoho rivnyannya Schrödingera (The new comlpletely integrable discretization of the nonlinear Schrödinger equation). *Ukr. J. Phys.* **40**, 118 (1995).
20. O.O. Vakhnenko, V.O. Vakhnenko. Physically corrected Ablowitz–Ladik model and its application to the Peierls–Nabarro problem. *Phys. Lett. A* **196**, 307 (1995) [DOI: 10.1016/0375-9601(94)00913-A].
21. O.O. Vakhnenko. Asymmetric canonicalization of the integrable nonlinear Schrödinger system on a triangular-lattice ribbon. *Appl. Math. Lett.* **64**, 81 (2017) [DOI: 10.1016/j.aml.2016.07.013].

Received 26.05.16

О.О. Вахненко

ОСОБЛИВІ РИСИ ІНТЕҐРОВНОЇ НЕЛІНІЙНОЇ СИСТЕМИ ШРЬОДІНҐЕРА НА СТЬОЖЦІ ТРИКУТНОЇ ҐРАТКИ

Р е з ю м е

Показано, що динаміка інтеґровної нелінійної системи Шрьодінґера на стьожці трикутної ґратки є критичною відносно величини фонового параметра, регульованого граничними значеннями допоміжних полів. Зокрема, в критичній точці число основних польових змінних скорочується вдвоє, а пуасонівська структура системи стає виродженою. З іншого боку, поза критичною точкою пуасонівська структура системи є суттєво нестандартною, і осмислена процедура її стандартизації неминуче спричиняє порушення взаємної симетрії між стандартизованими основними підсистемами. Існують дві можливі реалізації такої асиметричної стандартизації, кожна з яких призводить до повного подавлення польових амплітуд однієї з основних підсистем при критичному значенні фонового параметра. В докритичній області фонового параметра стандартизовані основні польові амплітуди набувають сенсу амплітуд ймовірності деяких взаємно нееквівалентних внутрішньо коміркових світлих збуджень, в той час як в надкритичній області така інтерпретація стає некоректною. Аналіз показує, що надкритичну область можна трактувати як область співіснування між стандартизованими підсистемами світлих та темних збуджень.