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INTERACTION OF SPINLESS PARTICLES WITH YUKAWA RING-SHAPED POTENTIAL

We have obtained the approximate solutions of the Klein–Gordon equation with the Yukawa ring-shaped potential, by using the Nikiforov–Uvarov method for a special case of equal scalar and vector potentials. The energy eigenvalues for bound states and the corresponding wave functions are also obtained in a proper approximation. We have also shown that the results can be used to evaluate the energy eigenvalues of the Yukawa, angle-dependent, and Coulomb potentials. The numerical results are discussed and presented in the table and in the figure, which suggest their applicability to other systems. With the adjusted potential parameters given in the table, it is shown that the interaction of spinless (Klein–Gordon) particles with the Yukawa ring-shaped potential gives positive energy eigenvalues for the various quantum states.

K e y w o r d s: spinless particles, Yukawa potential, angle-dependent potential, approximation scheme, Nikiforov–Uvarov method.

1. Introduction

The subject of the non-central potentials has been studied in various fields of nuclear physics and quantum chemistry which concern the interactions between deformed pair of nuclei and ring-shaped molecules like benzene [1–4]. There has been continuous interest in the solutions of Schrödinger, Klein–Gordon, and Dirac equations for some non-central potentials [5].

These equations are solved by means of different methods for exactly solvable potentials such as Supersymmetry Quantum Mechanics (SUSYQM) [6–11], time-dependent perturbation [12], asymptotic iteration method (AIM) [13–16], factorization method [17, 18], functional analysis [19], Nikiforov–Uvarov (NU) method [20–28], and others [29–31]. Yaşuk *et al.* [8] presented an alternative simple method for the exact solution of the Klein–Gordon equation (KGE) in the presence of noncentral equal scalar and vector potentials, by using the Nikiforov–Uvarov method [45].

A spherically harmonic oscillatory ring-shaped potential had been proposed, and the exact complete solutions of the Schrödiger equation (SE) with it were presented via the Nikiforov–Uvarov method by Zhang *et al.* [33]. Bayrak *et al.* [34] and also, Chen *et al.* [35] presented exact solutions of the SE with the Makarov potential by using the asymptotic iteration method and the partial wave method, respectively. Kandirmaz *et al.* also used the path integral method to investigate the coherent states for a particle in the noncentral Hartmann potential [30]. Hamzavi *et al.* studied the Dirac equation with the Hartmann potential [16].

The Yukawa potential or static screened Coulomb potential [36] is given by

$$V(r) = -V_0 \frac{e^{-\alpha r}}{r}, \tag{1}$$

where $V_0 = \alpha' Z$, $\alpha' = (137.037)^{-1}$ is the fine structure constant and Z is the atomic number, and α is the screening parameter. This potential is often used to compute the bound-state normalization and the energy level of neutral atoms [37] which have been studied over the years.

The novel ring-shaped potential was introduced by Berkdemir [26]:

$$V_\theta(\theta) = \frac{\gamma + \beta \sin^2\theta + \eta \sin^4\theta}{\sin^2\theta \cos^2\theta}. \tag{2}$$

Here, β, η and γ are arbitrary constants.

The non-central potential has attracted much attention recently. Antia *et al.* [38] has obtained the approximate analytical solutions of the relativistic KGE with scalar and vector shifted Hulthen plus angle-dependent potentials. Again, Antia *et al.*, [41] have obtained solutions of the non-relativistic SE with Hulthen–Yukawa plus angle-dependent potential within the framework of the Nikiforov–Uvarov (NU) method.

Our choice of this combined potential is based on the motivation derived from the applications of Yukawa and ring-shaped potentials. The Yukawa potential is one of the short-range potentials and has a lot of applications in physics. It has Coulombic behavior for small r and is exponentially damped for large r. It plays the important role in high-energy and particle physics, atomic physics, chemical physics, gravitational plasma physics, and solid state physics [42–43]. This potential could be applied to various branches of nuclear physics and quantum chemistry to describe nucleon-nucleon interactions, the meson-meson interaction, and interactions between the deformed pair of nuclei and ring-shaped molecules like benzene. One of us [44] investigated a solution of the non-relativistic Schrödinger equation with the Yukawa angle-dependent potential and applied it to study diatomic molecules. Being motivated by this success, we will attempt to study the relativistic spinless particles (Klein–Gordon particles) interacting with the Yukawa ring-shaped potential.

This paper aims at obtaining the solutions of the KGE with the Yukawa ring-shaped potential for a special case of equal scalar and vector potentials using the NU method.

The organization of this paper is as follows: In Section 2, we review the Nikiforov–Uvarov method. In Section 3, we present the solutions of the KGE with the Yukawa ring-shaped potential. In Section 4, we discuss our results and make conclusions in Section 5.

2. Review of the Nikiforov–Uvarov Method

Nikiforov and Uvarov [45] have presented a method to obtain the exact solution of the second-order differential equations such as the Schrödinger, Klein–Gordon, and Dirac equations.

As for the SE

$$\psi''(x) + (E - V(x))\,\psi(x) = 0 \tag{3}$$

of the hypergeometric type, it be solved by applying the appropriate transformation, $s = s(x)$,

$$\psi''(s) + \frac{\bar{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{4}$$

where $\sigma(s)$ and $\bar{\sigma}(s)$ must be polynomials of the at most second degree, and $\bar{\tau}(s)$ is a first-degree polynomial. $\psi(s)$ is a function of the hypergeometric type. In order to find the solution of Eq. (4), we set the wave functions as

$$\psi(s) = \phi(s)\,\chi(s). \tag{5}$$

Substituting Eq. (5) into Eq. (4), Eq. (4) is reduced to the hypergeometric-type equation:

$$\sigma(s)\,\chi''(s) + \tau(s)\,\chi'(s) + \chi(s) = 0, \tag{6}$$

where the wave function $\phi(s)$ is defined as the logarithmic derivative. We have

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}, \tag{7}$$

where $\pi(s)$ is a polynomial of the at most first order, and $\sigma(s)$ is a polynomial of the at most second order.

Likewise, the hypergeometric type function $\chi(s)$ in Eq. (6) for a fixed n is given by the Rodrigues relation as

$$\chi_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n}\left[\sigma^n(s)\,\rho(s)\right], \tag{8}$$

where B_n is the normalization constant, and the weight function $\rho(s)$ must satisfy the condition

$$\frac{d}{ds}\left(\sigma(s)\,\rho(s)\right) = \tau(s)\,\rho(s) \tag{9}$$

with

$$\tau(s) = \bar{\tau}(s) + 2\pi(s). \tag{10}$$

Therefore, the function $\pi(s)$ and the parameters required for the NU method are defined as follows:

$$\pi(s) = \frac{\sigma' - \bar{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \bar{\tau}}{2}\right)^2 - \bar{\sigma} + k\sigma}, \tag{11}$$

$$\lambda = k + \pi'(s). \tag{12}$$

Based on the NU method, the k-value of the expression under the square root must be a square of polynomials and this is possible, if and only if its discriminant is zero. With this, the new equation for eigenvalues becomes

$$\lambda = \lambda_n = -\frac{nd\tau}{ds} - \frac{n(n-1)}{2}\frac{d^2\sigma}{ds^2}, \quad n = 0, 1, 2, \ldots. \tag{13}$$

By comparing Eq. (12) and Eq. (13), we obtain the energy eigenvalues.

The parametric generalization of the NU method that is valid for any non-central potential is given by the generalized hypergeometric-type equation [46]:

$$\psi''(s) + \frac{c_1 - c_2 s}{s(1 - c_3 s)}\psi'(s) + \frac{1}{s^2(1 - c_3)^2} \times \\ \times [-\xi_1 s^2 + \xi_2 s - \xi_3]\psi(s) = 0. \tag{14}$$

Equation (6) is solved by comparing it with Eq. (4). We get the following polynomials:

$$\bar{\tau}(s) = (c_1 - c_2 s), \ \sigma(s) = s(1 - c_3 s), \\ \bar{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3. \tag{15}$$

Now, substituting (15) into (11), we find

$$\pi(s) = c_4 + c_5 s \pm \left[(c_6 - c_3 k_\pm)s^2 + \\ + (c_7 + k_\pm)s + c_8\right]^{1/2}, \tag{16}$$

where

$$c_4 = \frac{1}{2}(1 - c_1), \ c_5 = \frac{1}{2}(c_2 - 2c_3), \ c_6 = c_5^2 + \xi_1, \\ c_7 = 2c_4 c_5 - \xi_2, \ c_8 = c_4^2 + \xi_3. \tag{17}$$

The resulting value of k in relation (16) is obtained from the condition that the function under the square root is a square of a polynomials, which yields

$$k_\pm = -(c_7 + 2c_3 c_8) \pm 2\sqrt{c_8 c_9}, \tag{18}$$

where

$$c_9 = c_3 c_7 + c_2^2 c_8 + c_6. \tag{19}$$

The new $\pi(s)$ for each k becomes

$$\pi(s) = c_4 + c_5 s - [(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}], \tag{20}$$

where

$$k_- = -(c_7 + 2c_3 c_8) - a\sqrt{c_8 c_9}. \tag{21}$$

Using (10), we obtain

$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s - \\ - 2[(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}]. \tag{22}$$

The physical condition for the bound-state solution is $\tau' < 0$, and, thus,

$$\tau'(s) = -2c_3 - 2(\sqrt{c_9} + c_3\sqrt{c_8}) < 0. \tag{23}$$

Using Eqs. (12) and (13), we derive the energy equation as

$$(c_2 - c_3)n + c_3 n^2 - (2n + 1)c_5 + \\ + (2n + 1)(\sqrt{c_9} + c_3\sqrt{c_8}) + \\ + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0. \tag{24}$$

The weight function $\rho(s)$ is obtained from Eq. (9) as

$$\rho(s) = s^{c_{10} - 1}(1 - c_3 s)^{\frac{c_{11}}{c_3} - c_{10} - 1}. \tag{25}$$

In view of Eq. (8), we have

$$\chi_n(s) = P_n^{\left(c_{10} - 1, \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1 - 2c_3 s), \tag{26}$$

where

$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, \tag{27}$$

$$c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}), \tag{28}$$

and $P_n^{(\alpha,\beta)}(s)$ are the Jacobi polynomials. Detailed discussions of Jacobi polynomials can be found in the literature [47–48]. The second part of the wave function is obtained from Eq. (7) as

$$\phi(s) = s^{c_{12}}(1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}}, \tag{29}$$

where

$$c_{12} = c_4 + \sqrt{c_8}, \ c_{13} = c_5 - (\sqrt{c_9} + c_3\sqrt{c_8}). \tag{30}$$

Thus, the total wave function becomes

$$\psi(s) = N_n s^{c_{12}}(1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} \times \\ \times P_n^{\left(c_{10} - 1, \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1 - 2c_3 s), \tag{31}$$

where N_n is the normalization constant.

3. Case of Yukawa Ring-Shaped Potential

The Yukawa ring-shaped potential is defined as [36, 49, 50]

$$V(r,\theta) = -\frac{V_0 e^{-\alpha r}}{r} + \frac{\hbar^2}{2\mu}\left(\frac{\gamma + \beta\cos^2\theta + \eta\cos^4\theta}{r^2\cos^2\theta\sin^2\theta}\right), \tag{32}$$

where V_0 is the potential depth, α is the screening parameter, μ is the reduced mass, $\hbar$ is the reduced Planck constant, and γ, β, and η are arbitrary constants.

The potential in (32) can be expressed as

$$V(r,\theta) = -V_r(r) + \frac{\hbar^2}{2\mu}\frac{V_\theta(\theta)}{r^2}. \tag{33}$$

In spherical coordinates, the Klein–Gordon equation for equal scalar and vector potentials can be written as

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - 2(E+M)V(r,\theta) + E^2 - M^2\right]\psi(r,\theta,\phi) = 0, \tag{34}$$

where M is the mass of the particle. The total wave function in Eq. (34) can be defined as

$$\psi(r,\theta,\phi) = \frac{R(r)}{r}Y(\theta,\phi). \tag{35}$$

Substituting Eqs. (32) and (35) into Eq. (34), we have

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{Y}\left[\frac{\partial^2 Y}{\partial\theta^2} + \cot\theta\frac{\partial Y}{\partial\theta}\right] + \frac{1}{Y}\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \frac{2r}{R}(E+M)V_0 e^{-\alpha r} - \frac{2}{Y}(E+M)\frac{\hbar^2}{2\mu}\left[\frac{\gamma + \beta\cos^2\theta + D\cos^4\theta}{\cos^2\theta\sin^2\theta}\right]Y + (E^2 - M^2)r^2 = 0. \tag{36}$$

Simplifying Eq. (36) by separating the variables, the following radial and angular equations are obtained:

$$\frac{d^2R}{dr^2} + \left[\frac{2(E+m)V_0e^{-\alpha r}}{r} + (E^2 - M^2) - \frac{\lambda}{r^2}\right]R(r) = 0, \tag{37}$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} - \frac{\hbar^2}{\mu}(E+M)\times\left[\frac{\gamma + \beta\cos^2\theta + \eta\cos^4\theta}{\cos^2\theta\sin^2\theta}\right]\Theta + \left(\lambda - \frac{m^2}{\sin^2\theta}\right)\Theta = 0, \tag{38}$$

$$\frac{d^2\Phi}{d\phi^2} + m^2\phi = 0, \tag{39}$$

where λ and m^2 are the separation constants, $\lambda = l(l+1)$, where l is the orbital quantum number, and m is the magnetic quantum number. Equation (39) is the azimutal part, whose solution is well known [38, 40]. Equations (37) and (38) are the radial and angular parts of the KGE, respectively, whose solutions shall be discussed shortly. Equation (37) has no analytical solution for $l \neq 0$ due to the centrifugal term [38]. Therefore, we must take a proper approximation [51] to the centrifugal term as

$$\frac{1}{r^2} \approx 4\alpha^2\left(\frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}\right). \tag{40}$$

This approximation is valid for a short-ranged potential, and the values of the screening parameter (α) must be small. Thus, the range of validity of α is $0 < \alpha < 1$. Substituting Eq. (40) into Eq. (37), we have

$$\frac{d^2R}{dr^2} + \left[4\alpha(E+M)\frac{e^{-2\alpha r}}{1-e^{-2\alpha r}} + (E^2 - M^2) - \frac{4\lambda\alpha^2 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}\right]R(r) = 0. \tag{41}$$

Set

$$s = e^{-2\alpha r}. \tag{42}$$

Substituting Eqs. (42) into Eq. (41), we have

$$\frac{d^2R}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dR}{ds} + \frac{1}{4\alpha^2 s^2(1-s)^2}\times\left[\begin{array}{l} s^2\left(E^2 - M^2 - 4\alpha V_0 E - 4\alpha V_0 M\right) + \\ + s\left(\begin{array}{l}-2E^2 + 2M^2 + \\ + 4\alpha V_0 E + 4\alpha V_0 M - 4\lambda\alpha^2\end{array}\right) + \\ + E^2 - M^2 \end{array}\right]R(s) = 0, \tag{43}$$

where M is the mass of the particle. Let

$$-\bar{\beta}^2 = \frac{E^2 - M^2}{4\alpha^2}, \quad \delta^2 = 4\alpha V_0 (E + M), \quad W = -4\lambda\alpha^2. \tag{44}$$

Equation (43) becomes

$$\frac{d^2R}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dR}{ds} + \frac{1}{s^2(1-s)^2}[-(\bar{\beta}^2 + \delta^2)s^2 + s\left(2\bar{\beta}^2 + \delta^2 + W\right) - \bar{\beta}^2]R(s) = 0. \tag{45}$$

Comparing Eq. (45) with Eq. (14), and using Eq. (17) and (19), we have

$$\xi_1 = \bar{\beta}^2 + \delta^2; \quad \xi_2 = 2\bar{\beta}^2 + \delta^2 + W; \quad \xi_3 = \bar{\beta}^2, \tag{46}$$

and the other parameters are as follows:

$$\begin{aligned} &c_1 = 1, \quad c_2 = 1, \quad c_3 = 1, \\ &c_4 = 0, \quad c_5 = -\frac{1}{2}, \quad c_6 = \frac{1}{4} + \xi_1, \\ &c_7 = -\xi_2, \quad c_8 = \xi_3, \\ &c_9 = \xi_1 - \xi_2 + \xi_3 + \frac{1}{4}. \end{aligned} \tag{47}$$

Substituting Eq. (47) into Eq. (24), we have

$$n^2 + \frac{(2n+1)}{2} + (2n+1)\left(\sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4}} + \sqrt{\xi_3}\right) - \xi_2 + 2\xi_3 + 2\sqrt{\xi_3\left(\xi_1 - \xi_2 + \xi_3 + \frac{1}{4}\right)} = 0, \tag{48}$$

$$\xi_1 - \xi_2 + \xi_3 + \frac{1}{4} = \frac{1}{4} - W = \frac{1}{4} + 4\lambda\alpha^2 = \mathrm{A}^2, \tag{49}$$

$$-\xi_2 + 2\xi_3 = -\delta^2 + W = -\delta^2 - 4\lambda\alpha^2. \tag{50}$$

Substituting Eqs. (49)–(50) into Eq. (48), we have

$$n^2 + \frac{(2n+1)}{2} + (2n+1)(A + \bar{\beta}) + 2A\bar{\beta} = \delta^2 - W. \tag{51}$$

Rearranging Eq. (51) and squaring both sides, we obtain the energy eigenvalues for the radial part of the KGE as

$$E^2 - M^2 = -4\alpha^2 \times \frac{\left[\delta^2 + 4\lambda\alpha^2 - n^2 - \frac{(2n+1)}{2} - (2n+1)A\right]^2}{\left[(2n+1)^2 + 4A(2n+1) + 4A^2\right]}. \tag{52}$$

Using Eq. (25), (26), and (29), we obtain the corresponding wave function of the radial part as

$$R(s) = N_{nl} s^{\sqrt{\xi_3}} (1-s)^{\frac{1}{2} + \sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4}}} \times P_{nl}^{\left(2\sqrt{\xi_3}, 2 + 2\sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4}}\right)}(1 - 2s), \tag{53}$$

where N_{nl} is the normalization constant. As a further guide to interested readers, the calculation of the normalization constant is stated in Refs. [52, 53]. The eigenvalues and the eigenfunctions of the polar part of the Klein–Gordon equation can be obtained, in this case, by making use of Eq. (38) as

$$-\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} - \frac{\hbar^2}{\mu}(E + M) \times \left[\frac{\gamma + \beta\cos^2\theta + \eta\cos^4\theta}{\cos^2\theta\sin^2\theta}\right]\Theta + \left(\lambda - \frac{m^2}{\sin^2\theta}\right)\Theta = 0. \tag{54}$$

Let

$$p = \cos^2\theta. \tag{55}$$

Substituting Eq. (55) into Eq. (54), we have

$$\frac{d^2\Theta(p)}{dp^2} + \frac{\left(\frac{1}{2} - 3/2p\right)}{p(1-p)}\frac{d\Theta(p)}{dp} + \frac{1}{4p^2(1-p)^2} \times \left[\begin{array}{l} \left(2\frac{(E+M)\hbar\eta}{\mu} - \lambda\right)p^2 + \\ + \left(\frac{2(E+M)\hbar\beta}{\mu} + \lambda - m^2\right)p + \\ + \frac{2(E+M)\hbar\gamma}{\mu} \end{array}\right]\Theta(p) = 0, \tag{56}$$

where m is the magnetic quantum number, and M is the particle mass. Let

$$-N^2 = \frac{2(E+M)\hbar}{4\mu}. \tag{57}$$

Equation (57) becomes

$$\frac{d^2\Theta(p)}{dp} + \frac{\left(\frac{1}{2} - \frac{3}{2}p\right)}{p(1-p)}\frac{d\Theta(p)}{dp} + \frac{1}{p^2(1-p)^2} \times \left[-\left(N^2\eta + \frac{\lambda}{4}\right)p^2 + \left(-N^2\beta + \frac{\lambda}{4} - \frac{m^2}{4}\right) - N^2\gamma\right] \times \Theta(p) = 0. \tag{58}$$

Comparing Eq. (58) with Eq. (14) and using Eqs. (17) and (19), we have

$$c_1=\frac{1}{2},\quad c_2=\frac{3}{2},\quad c_3=1,\quad c_4=\frac{1}{4},\quad c_5=\frac{1}{4},$$
$$c_6=\frac{1}{16}+\xi_1,\quad c_7=-\frac{1}{8}-\xi_2,\quad c_8=\frac{1}{16}+\xi_3,\quad (59)$$
$$c_9=\xi_1-\xi_2+\frac{9}{4}\xi_3+\frac{5}{64}$$

and

$$\xi_1=N^2\eta+\frac{\lambda}{4},\quad \xi_2=-N^2\beta+\frac{\lambda}{4}-\frac{m^2}{4},\quad \xi_3=N^2\gamma. \quad (60)$$

Substituting Eq. (59) into Eq. (24) gives

$$\frac{n}{2}+n^2-\frac{1}{4}(2n+1)+ +(2n+1)\left[\sqrt{\xi_1-\xi_2+\frac{9}{4}\xi_3+\frac{5}{64}}+\sqrt{\frac{1}{16}+\xi_3}\right]- -\frac{1}{8}-\xi_2+\frac{1}{8}+2\xi_3+ +2\sqrt{\left(\frac{1}{16}+\xi_3\right)\left(\xi_1-\xi_2+\frac{9}{4}\xi_3+\frac{5}{64}\right)}=0. \quad (61)$$

But, we can use

$$\xi_1-\xi_2+\frac{9}{4}\xi_3+\frac{5}{64}=N^2\left(\eta+\beta+\frac{9}{4}\gamma\right)+\frac{m^2}{4}+\frac{5}{64}, \quad (62)$$

and

$$-\xi_2+2\xi_3=N^2(\beta+2\gamma)-\frac{\lambda}{4}+\frac{m^2}{4}. \quad (63)$$

Using Eqs. (62) and (63), Rq. (61) can now be simplified as follows:

$$n^2-\frac{1}{4}+(2n+1)\left[\sqrt{N^2\left(\eta+\beta+\frac{9}{4}\gamma\right)+\frac{m^2}{4}+\frac{5}{64}}+ +\sqrt{\frac{1}{16}+N^2\gamma}\right]+N^2(\beta+2\gamma)-\frac{\lambda}{4}+\frac{m^2}{4}+ +2\sqrt{\left(\frac{1}{16}+\xi_3\right)N^2\left(\eta+\beta+\frac{9}{4}\gamma\right)+\frac{m^2}{4}+\frac{5}{64}}=0. \quad (64)$$

Solving Eq. (64), λ becomes

$$\lambda=4n^2-1+ +4(2n+1)\left[\sqrt{N^2\left(\eta+\beta+\frac{9}{4}\gamma\right)+\frac{m^2}{4}+\frac{5}{64}}+ +\sqrt{\frac{1}{16}+N^2\gamma}\right]+4N^2(\beta+2\gamma)+m^2+ +8\sqrt{\left(\frac{1}{16}+N^2\gamma\right)\left(N^2\left(\eta+\beta+\frac{9}{4}\gamma\right)+\frac{m^2}{4}+\frac{5}{64}\right)}, \quad (65)$$

where $\lambda=l(l+1)$, and c_{10}, c_{11}, c_{12}, and c_{13} obtained with the use of Eqs. (27), (28), and (30) are as follows:

$$c_{10}=1+2\sqrt{\frac{1}{16}+\xi_3},$$
$$C_{11}=2+2\left(\sqrt{\xi_1-\xi_2+\frac{9}{4}\xi_3+\frac{5}{64}}+\sqrt{\frac{1}{16}+\xi_3}\right), \quad (66)$$
$$C_{12}=\frac{1}{4}+\sqrt{\frac{1}{16}+\xi_3},$$
$$C_{13}=-\frac{1}{4}-\left(\sqrt{\xi_1-\xi_2+\frac{9}{4}\xi_3+\frac{5}{64}}+\sqrt{\frac{1}{16}+\xi_3}\right).$$

The corresponding wave function of the angle-dependent part is obtained by substituting Eq. (59) and (66) into Eq. (31). Thus,

$$\Theta(p)=N_{lm}p^{\frac{1}{4}+\sqrt{\frac{1}{16}+\xi_3}}(1-p)^{\sqrt{\xi_1-\xi_2+\frac{9}{4}\xi_3+\frac{5}{64}}}\times \times P_{lm}^{\left(2\sqrt{\frac{1}{16}+\xi_3},\,2+2\sqrt{\xi_1-\xi_2+\frac{9}{4}\xi_3+\frac{5}{64}}\right)}(1-2p), \quad (67)$$

where N_{lm} is the normalization constant (see, e.g., [52, 53]). The total energy of the Yukawa ring-shaped potential is obtained by considering the effect of the angle-dependent part on the radial part. Substituting Eq. (65) into Eq. (52) yields the energy spectra for this system as

$$E_{nlm}^2-M^2=-4\alpha^2\times \times\frac{\left[\delta^2+4F\alpha^2-n^2-\frac{(2n+1)}{2}-(2n+1)A\right]^2}{\left[(2n+1)^2+4A(2n+1)+4A^2\right]}, \quad (68)$$

where

$$F=4n^2-1+4(2n+1)\times \times\left[\sqrt{N^2\left(\eta+\beta+\frac{9}{4}\gamma\right)+\frac{m^2}{4}+\frac{5}{64}}+\right.$$

$$+\sqrt{\frac{1}{16}+N^2\gamma}\Bigg] + 4N^2(\beta+2\gamma) + m^2 +$$
$$+8\sqrt{\left(\frac{1}{16}+N^2\gamma\right)\left(N^2\left(\eta+\beta+\frac{9}{4}\gamma\right)+\frac{m^2}{4}+\frac{5}{64}\right)}, \tag{69}$$

and

$$A = \left(4F\alpha^2 + \frac{1}{4}\right)^{1/2}. \tag{70}$$

4. Discussions

Using Eq. (68), the numerical solutions of the bound-state energy for the Yukawa ring-shaped potential with $M = 1$, $V_0 = 0.02$, $\beta = \gamma = \mu = \eta = 1$ are given in the table below for $\alpha = 0.02$, $\alpha = 0.04$, and $\alpha = 0.06$, respectively, with different quantum states (n, l, m). From the table, it is clear that the energy eigenvalues are all positive. This is in order as the bound-state energy could either be negative (particles) or positive (antiparticles). In the figure below, the variation of the energy eigenvalues in the quantum state n are discussed for a fixed $l = m = 0$ and $\alpha = 0.02$, $\alpha = 0.04$, and $\alpha = 0.06$. From the graph, it is observed that the bound-state energy decreases, as the principal quantum n increases.

Furthermore, some special cases can be deduced by adjusting some parameters of our potential, and their respective energy eigenvalues and the corresponding wave function could be studied with these adjusted parameters. Some of those special potentials are presented as follows.

4.1. Yukawa potential

When we set $\gamma = \beta = \eta = 0$ in Eq. (32), the Yukawa ring-shaped potential reduces to Yukawa potential and the energy eigenvalues are obtained as follows [21]:

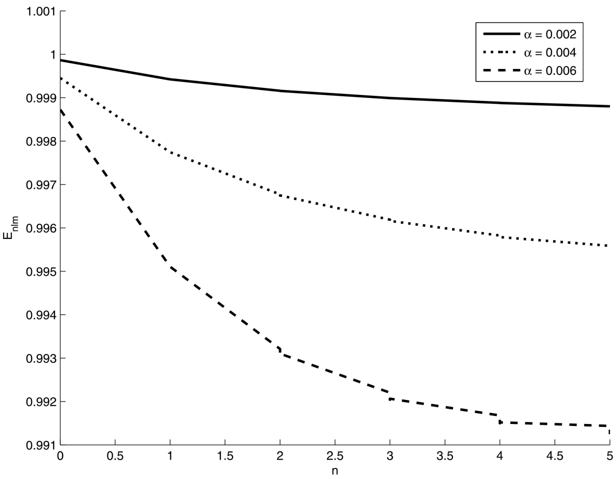
$$E^2_{nlm} - M^2 = -4\alpha^2 \times$$
$$\times \frac{\left[\delta^2 + 4Q\alpha^2 - n^2 - \frac{(2n+1)}{2} - (2n+1)A\right]^2}{\left[(2n+1)^2 + 4A(2n+1) + 4A^2\right]}, \tag{71}$$

where

$$Q = 4n^2 - 1 + 4(2n+1)\left[\sqrt{\frac{m^2}{4}+\frac{5}{64}}+\frac{1}{4}\right] + m^2 +$$
$$+8\sqrt{\left(\frac{m^2}{4}+\frac{5}{64}\right)}. \tag{72}$$

4.2. Angle-dependent potential

When we set $V_0 = 0$, $\alpha = 0$ in Eq. (32), the Yukawa ring-shaped potential reduces to angle dependent potential. Thus, in this limit, the energy eigenvalues for



Energy (E) versus the quantum state number n with $l = m = 0$

Bound-state energy of Yukawa ring-shaped potential for $M = 1$, $V_0 = 0.02$, $\beta = \gamma = \mu = \eta = 1$, $\alpha = 0.02$, $\alpha = 0.04$, and $\alpha = 0.06$

n	l	m	E_{nlm}		
			$\alpha = 0.02$	$\alpha = 0.04$	$\alpha = 0.06$
0	0	0	0.999865824	0.999454126	0.998724631
1	0	0	0.999422666	0.997744429	0.995105907
2	0	0	0.999159122	0.996770694	0.993210389
2	1	0	0.999157305	0.996742223	0.99307115
2	1	1	0.999157693	0.996748347	0.993101483
2	1	–1	0.999157693	0.996748347	0.993101483
3	0	0	0.998992801	0.996189201	0.992204017
3	1	0	0.998990608	0.996155078	0.992039355
3	1	1	0.998990934	0.996160185	0.992064301
3	1	–1	0.998990934	0.996160185	0.992064301
4	0	0	0.998879969	0.995822657	0.991678541
4	1	0	0.998877521	0.995784834	0.991498421
4	1	1	0.998877800	0.995789181	0.991519371
4	1	–1	0.998877800	0.995789181	0.991519371
5	0	0	0.998799361	0.995585144	0.991437421
5	1	0	0.998796731	0.995544785	0.991247786
5	1	1	0.998796974	0.995548554	0.991265701
5	1	–1	0.998796974	0.995548554	0.991265701

pure angle dependent potential could be obtained by using this adjusted parameters of Eq. (68).

4.3. Coulomb potential

Also setting $\alpha = \gamma = \beta = \eta = 0$, the potential in Eq. (32) reduces to Coulomb potential [6] and the corresponding energy could be obtained by using these parameters in Eq. (68).

5. Conclusion

We have studied the approximate solutions of the Klein–Gordon equation with the Yukawa ring-shaped potential using the Nikiforov–Uvarov method when the scalar potential and the vector potential are equal. The bound states energy eigenvalues and the corresponding wave functions are obtained by using a proper approximation [49].

We also show that the results can be used to evaluate the energy eigenvalues of the Yukawa potential and when $\gamma = \beta = \eta = 0$, they are in good agreement with the results of [36].

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ВЗАЄМОДІЯ БЕЗСПІНОВИХ ЧАСТИНОК
З КІЛЬЦЕВИМ ПОТЕНЦІАЛОМ ЮКАВИ

Р е з ю м е

Ми отримали наближені рішення рівняння Клейна–Гордона з кільцевим потенціалом Юкави методом Никифорова–Уварова в спеціальному випадку рівних скалярного і векторного потенціалів. Знайдено енергії і відповідні хвильові функції зв'язаних станів. Показано, що результати можуть бути використані для оцінки енергій для потенціалів Юкави і Кулона та потенціалу, що залежить від кута. Числові результати представлені в таблиці і на рисунку та можуть бути корисними при описі інших систем. При виборі параметрів потенціалів, зазначених в таблиці, показано, що взаємодія безспінових (Клейна–Гордона) частинок з кільцевим потенціалом Юкави приводить до позитивних власних значень енергії для різних квантових станів.