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DYNAMICS OF A NON-ROTATING TEST NULL STRING IN THE GRAVITATIONAL FIELD OF A CLOSED "THICK" NULL STRING RADIALLY EXPANDING OR COLLAPSING IN THE PLANE $z = 0$

The dynamics of a test null string moving in the gravitational field of a closed "thick" null string radially expanding or collapsing in the plane $z = 0$ is considered, provided that the former string does not rotate initially.

K e y w o r d s: "thick" null string, gravitational field.

1. Introduction

The idea of that the researches of multidimensional objects, including strings, may form a basis for our understanding of the Nature has already been expressed rather neatly in modern physics. One of the directions of those researches in the string theory deals with the role of such objects in cosmology. The gauge theories of Grand Unification, which are based on the idea of a spontaneous symmetry breaking, predict a possibility of the formation of one-dimensional topological defects in the course of phase transitions in the early Universe. Those objects were called space strings [1–7].

In work [8], it was shown that the presence of such objects in the Universe does not contradict the existence of the observed microwave relic radiation. Null strings realize the zero-tension limit in the string theory [5, 7]. Therefore, since the tension is measured in units of the Planck mass, M, scale, the zero-tension limit corresponds, from the physical viewpoint, to the asymptotically large energy scale, $E \gg M$. From this viewpoint, the null strings, which realize a high-temperature phase of strings, could arise at the Big Bang moment and, hence, affect the observed structure of the Universe. In particular, in work [9], it was demonstrated that, by considering the gas of null strings as a dominant source of the gravitation in D-dimensional Friedmann–Robertson–Walker spaces with $k = 0$, one can describe the inflation mechanism typical of those spaces.

In a number of works, the gas of relic null strings is considered as one of the probable candidates for the role of a carrier of the so-called "dark" matter, the existence of which in the Universe can be regarded as an proved fact. Although, the object of research in the quoted examples is not a separate null string, but a gas of null strings, the properties of this gas still remain unclear. In our opinion, the first step to understanding the properties of the gas of null strings may consist in the solution of the problems concerning the gravitational field generated by a null string moving along different trajectories, as well as the dynamics of a test null string in such gravitational fields.

For instance, let the equations of motion for a test null string have solutions that can be interpreted as moving test null strings with time-independent shapes determined by initial conditions. At the same time, the trajectory of this null string is similar to that of the null string generating the gravitational field. Then, we may say that there exists a state (a phase) of the ideal gas consisting of identical null strings. The existence of such a gas may form a basis for the formulation of various multistring problems.

In this work, the dynamics of a test null string in a gravitational field generated by a closed "thick" null string that either radially expands or radially collapses in the plane $z = 0$ is considered. In this research, we are interested first of all in the presence of solutions for the equations of motion that would give rise to the possibility of the existence of a state (a phase) of the ideal gas consisting of identical null strings in this gravitational field. We are also interested in the features of the interaction between the test null string and the null string generating the gravitational field.

2. Equation of Motion for a Null String

The quadratic form describing the gravitational field of a closed "thick" null string radially expanding or radially collapsing in the plane $z = 0$ can be presented as follows [10, 11]:

$$dS^2 = e^{2\nu}\left((dt)^2 - (d\rho)^2\right) - B(d\theta)^2 - e^{2\mu}(dz)^2. \quad (1)$$

Here,

$$e^{2\nu(\eta,z)} = \frac{|\lambda_{,\eta}|}{(\lambda(\eta))^2}\left(\frac{\alpha(\eta)+\lambda(\eta)f(z)}{\lambda(\eta)^{1/(1-\chi)}}\right)^{2-\sqrt{4-2\chi}}, \quad (2)$$

$$B(\eta,z) = \left(\frac{\alpha(\eta)+\lambda(\eta)f(z)}{\lambda(\eta)^{1/(1-\chi)}}\right)^{\sqrt{4-2\chi}}, \quad (3)$$

$$e^{2\mu(\eta,z)} = (f(z))^2\left(\frac{\alpha(\eta)+\lambda(\eta)f(z)}{\lambda(\eta)^{1/(1-\chi)}}\right)^{2-\sqrt{4-2\chi}}, \quad (4)$$

$$\eta = t \pm \rho, \quad (5)$$

the sign "+" corresponds to the radial collapse, and the sign "−" to the radial expansion of the null string in the plane $z = 0$; the functions $\lambda(\eta)$ and $\alpha(\eta)$ are coupled by the relation

$$\lambda(\eta) = (1-\alpha(\eta))/f_0, \quad f_0 = \text{const}, \quad (6)$$

$\lambda_{,\eta} = \frac{d\lambda(\eta)}{d\eta}$, $f_{,z} = \frac{df(z)}{dz}$, $\chi = 8\pi G$, and the functions $\alpha(\eta)$ and $f(z)$ are finite and, at every $\eta \in$ $\in (-\infty,+\infty)$ and $z \in (-\infty,+\infty)$, acquire values within the intervals

$$0 < \alpha(\eta) < 1, \quad 0 < f(z) < f_0. \quad (7)$$

The limiting cases are [10, 11]

$$\lambda(\eta)|_{\eta\in(-\infty,-\Delta\eta)\cup(+\Delta\eta,+\infty)} \to 0, \qquad \lambda(\eta)|_{\eta\to 0} \to \frac{1}{f_0}, \quad (8)$$

$$f(z)|_{z\in(-\infty,-\Delta z)\cup(+\Delta z,+\infty)} \to f_0, \qquad f(z)|_{z\to 0} \to 0, \quad (9)$$

where $\Delta\eta$ and Δz are small positive constants that determine the "thickness" of the "thick" null string generating the gravitational field ($\Delta\eta \ll 1$, $\Delta z \ll 1$). In the limiting case of the contraction into a one-dimensional object (a null string), the following conditions (at $\Delta\eta \to 0$ and $\Delta z \to 0$) have also to be satisfied:

$$\left|\frac{\alpha_{,\eta}}{\alpha(\eta)}\right|_{\eta\to 0} \to \infty, \quad \left.\frac{f_{,z}}{f(z)}\right|_{z\to 0} \to 0, \qquad \left.\frac{\alpha_{,\eta}}{\alpha(\eta)}\frac{f_{,z}}{f(z)}\right|_{\eta\to 0, z\to 0} \to 0. \quad (10)$$

As an example, the following functions [10, 11] satisfy conditions (10):

$$\alpha(\eta) = \exp\left(\frac{-1}{\epsilon+(\xi\eta)^2}\right), \qquad f(z) = f_0\exp\left(-\gamma\left(1-\exp\left(\frac{-1}{(\zeta z)^2}\right)\right)\right). \quad (11)$$

Here, the constants ξ and ζ determine the size (the "thickness") of the "thick" null string that generates the gravitational field (depending on η and z, respectively), and the positive constants ϵ and μ provide the satisfaction of conditions (10) at $\eta \to 0$ and $z \to 0$, namely,

$$\xi, \zeta, \mu \to \infty, \quad \epsilon \to 0. \quad (12)$$

The dynamics of a null string in the pseudo-Riemannian space is governed by the following system of equations:

$$x^m_{,\tau\tau} + \Gamma^m_{pq}x^p_{,\tau}x^q_{,\tau} = 0, \quad (13)$$

$$g_{mn}x^m_{,\tau}x^n_{,\tau} = 0, \quad g_{mn}x^m_{,\tau}x^n_{,\sigma} = 0, \quad (14)$$

where g_{mn} and Γ^m_{pq} are the metric tensor and the Christoffel symbols, respectively, of the external space; $x^m_{,\tau} = \partial x^m/\partial\tau$ and $x^n_{,\sigma} = \partial x^n/\partial\sigma$, the indices m, n, p, and q take integer values from 0 to 3; the functions $x^m(\tau,\sigma)$ determine the trajectory of motion of the null string; τ and σ are parameters on the world surface of the null string; σ is the space-like parameter that marks the points on the null string, and τ

is a time-like parameter. In the cylindrical coordinate system,

$$x^0 = t, \quad x^1 = \rho, \quad x^2 = \theta, \quad x^3 = z,$$

and the functions $x^m(\tau, \sigma)$, which determine the trajectories of motion for the null string generating the gravitational field and are considered in this work, have the following form:

$$t = \tau, \quad \rho = \mp\tau, \quad \theta = \sigma, \quad z = 0, \tag{15}$$

where the sign "−" corresponds to the collapse of the null string in the plane $z = 0$ and the corresponding parameter $\tau \in (-\infty, 0]$, whereas the sign "+" corresponds to the radial expansion of the null string in the plane $z = 0$ and, in this case, $\tau \in [0, +\infty)$.

From equalities (15), it follows that, in the case $\rho = \tau$ ($\tau \in [0, +\infty)$), the null string that generates the gravitational field is in the plane $z = 0$ and has an infinitesimally small radius at the initial time moment. As the time t grows, the null string, remaining in the same plane $z = 0$, only increases its radius, i.e. it radially expands in the plane $z = 0$. On the other hand, in the case $\rho = -\tau$ ($\tau \in (-\infty, 0]$), the null string that generates the gravitational field is in the plane $z = 0$ and has an infinitely large radius at the initial time moment. As the time t grows, this null string, remaining in the same plane $z = 0$, only decreases its radius, i.e. it radially collapses in the plane $z = 0$.

The equations of motion for the test null string moving in the gravitational field of the null string that radially expands or radially collapses in the plane $z = 0$, look like

$$\eta_{,\tau\tau} + 2\nu_{,\tau}\eta_{,\tau} = 0, \tag{16}$$

$$q_{,\tau\tau} + 2\nu_{,z}q_{,\tau}z_{,\tau} + e^{-2\nu}B_{,\eta}(\theta_{,\tau})^2 +$$

$$+ 2e^{2(\mu-\nu)}\mu_{,\eta}(z_{,\tau})^2 = 0, \tag{17}$$

$$z_{,\tau\tau} + e^{2(\nu-\mu)}\nu_{,z}\eta_{,\tau}q_{,\tau} + 2\mu_{,\eta}\eta_{,\tau}z_{,\tau} -$$

$$- \frac{e^{-2\mu}}{2}B_{,z}(\theta_{,\tau})^2 + \mu_{,z}(z_{,\tau})^2 = 0, \tag{18}$$

$$\theta_{,\tau\tau} + \frac{B_{,\tau}}{B}\theta_{,\tau} = 0, \tag{19}$$

$$e^{2\nu}\eta_{,\tau}q_{,\tau} - B(\theta_{,\tau})^2 - e^{2\mu}(z_{,\tau})^2 = 0, \tag{20}$$

$$\frac{1}{2}e^{2\nu}\left(\eta_{,\tau}q_{,\sigma} + \eta_{,\sigma}q_{,\tau}\right) - B\theta_{,\tau}\theta_{,\sigma} - e^{2\mu}z_{,\tau}z_{,\sigma} = 0, \tag{21}$$

where

$$q = t \mp \rho, \tag{22}$$

the sign "−" corresponds to the radial collapse, and the sign "+" to the radial expansion of the null string that generates the gravitational field in the plane $z = 0$.

When integrating Eq. (16), the following two cases have to be considered:

$$\eta_{,\tau} = 0, \to \eta = \eta(\sigma), \tag{23}$$

$$\eta_{,\tau} \neq 0, \to \eta = \eta(\tau, \sigma). \tag{24}$$

3. Solution of the Equations of Motion for the Test Null String in the Case $\eta_{,\tau} = 0$

In case (23), Eq. (20) looks like

$$B(\theta_{,\tau})^2 + e^{2\mu}(z_{,\tau})^2 = 0. \tag{25}$$

Since the functions $B = B(\eta, z)$ and $e^{2\mu(\eta,z)}$ are positive at every $\eta \in (-\infty, +\infty)$ and $z \in (-\infty, +\infty)$, it follows from Eq. (20) that

$$z_{,\tau} = 0, \to z = z(\sigma), \tag{26}$$

$$\theta_{,\tau} = 0, \to \theta = \theta(\sigma). \tag{27}$$

Under conditions (23), (26), and (27), Eqs. (16), (18), and (19) are satisfied identically, and Eqs. (17) and (21) take the forms

$$q_{,\tau\tau} = 0, \tag{28}$$

$$q_{,\tau}\eta_{,\sigma} = 0, \tag{29}$$

respectively. Integrating Eq. (28), we obtain

$$q_{,\tau} = P_q(\sigma), \to q(\tau, \sigma) = q_0(\sigma) + P_q(\sigma)\tau, \tag{30}$$

where $q_0(\sigma)$ and $P_q(\sigma)$ are integration "constants". One should pay attention that

$$P_q(\sigma) \neq 0, \tag{31}$$

because, otherwise, we have $q = q_0(\sigma)$. The latter together with Eqs. (23), (26), and (27) means the realization of a static solution for the null string, which is impossible.

Under conditions (23) and (30), Eq. (29) takes the form

$$\eta_{,\sigma} P_q(\sigma) = 0. \quad (32)$$

From whence, taking Eqs. (5) and (31) into account, we have

$$\eta_{,\sigma} = 0, \rightarrow \eta = t \pm \rho = \text{const}. \quad (33)$$

The solution described by Eqs. (26), (27), (30), and (33) means that, under condition (23), the closed test null string moves in the same direction as the null string generating the gravitational field, i.e. it expands in the case $\eta = t - \rho$ and collapses in the case $\eta = t + \rho$. At every fixed time moment t, all points of the closed test null string are equidistant from the axis z. Moreover, as follows from equality (26), the test null string is not localized in the single plane z in the general case. In other words, the obtained solution describes a closed test null string that, at every fixed time moment, is completely localized between two planes, $z = z_1 = \text{const}$ and $z = z_2 = \text{const}$, where $z_1 = \min z(\sigma)$ and $z_2 = \max z(\sigma)$, where $\sigma \in [0, 2\pi]$, on the surface of a cylinder with the radius $\rho = \mp t + \text{const}$, where the sign "$-$" corresponds to the collapse of the test null string (in this case, $t \in (-\infty, 0]$), and the sign "$+$" to the radial expansion of the test null string (in this case, $t \in [0, +\infty)$). At the same time, if we fix $z(\sigma) = z_0 = \text{const}$ in Eq. (27), this case describes the radial expansion or collapse of the test null string completely remaining in the plane $z = z_0$ at every time moment and preserving the circular shape.

Hence, it follows from the obtained solution that there may exist a state for the gas of null strings, in which closed circular null strings located in parallel planes $z = \text{const}$ (the polarization effect) radially expand or collapse simultaneously preserving their shape, i.e. without interaction (the phase of ideal gas of null strings).

4. Solution of the Equations of Motion for a Test Null String in the Case $\eta_{,\tau} \neq 0$

Integrating Eq. (19), we obtain

$$\theta(\tau, \sigma) = \theta_0(\sigma) + P_\theta(\sigma) \int (B)^{-1} d\tau, \quad (34)$$

where the functions $\theta_0(\sigma)$ and $P_\theta(\sigma)$ (the integration "constants") determine, with respect to the variable θ, the positions and the velocities, respectively, of null string points at the initial time moment. From equality (34), it follows that, in the case where $P_\theta(\sigma) = 0$ at the initial time moment, i.e. the closed test null string does not rotate, its further dynamics will also evolve without rotation, so that

$$\theta = \theta(\sigma). \quad (35)$$

In this work, we have found a solution of the equations of motion for the closed test null string in case (24) and under the condition that its rotation is absent at the initial time moment, i.e. provided that

$$\eta = \eta(\tau, \sigma), \quad \theta = \theta(\sigma). \quad (36)$$

In this case, the variable η depends on the parameter τ (it changes in time). Therefore, Eq. (36) describes the motion of the test null string "toward" the null string that generates the gravitational field. However, the polar angle corresponding to every point of the test null string does not vary in time.

If the test null string moves "toward" the null string that generates the gravitational field, the η-value only increases. Therefore,

$$\eta_{,\tau} > 0. \quad (37)$$

The case $\eta_{,\tau} < 0$ describes the motion of a test null string in the same direction as the null string generating the gravitational field, but at a higher velocity, i.e. at a velocity higher than the speed of light, which is impossible.

Under conditions (36), Eq. (20) looks like

$$e^{2\nu} \eta_{,\tau} q_{,\tau} = e^{2\mu} (z_{,\tau})^2. \quad (38)$$

From whence, taking Eq. (37) and the positive definiteness of metric functions into account, it follows that

$$q_{,\tau} \geq 0. \quad (39)$$

In the case

$$q_{,\tau} = 0, \rightarrow q = q_0(\sigma), \quad (40)$$

where $q_0(\sigma)$ is the integration "constant", Eq. (38) gives rise to

$$z_{,\tau} = 0, \rightarrow z = z_0(\sigma), \quad (41)$$

where $z_0(\sigma)$ is the integration "constant". Under conditions (36), (40), and (41), the equations of motion (17)–(20) for the test null string are satisfied identically, and Eq. (21) takes the form

$$\eta_{,\tau} q_{,\sigma} = 0. \tag{42}$$

WFrom whence, taking Eqs. (22), (37), and (40) into account, we have

$$q = t \mp \rho = q_0 = \text{const}. \tag{43}$$

To summarize, Eqs. (36) and (40) describe the motion of a closed test null string with arbitrary shape "toward" the null string generating the gravitational field. At every fixed time moment t, all points of the closed test null string are equidistant from the axis z, and the shape of the test null string given by the functions $z_0(\sigma)$ and $\theta_0(\sigma)$ remains invariant. If the test null string is completely located in the plane $z = z_0 = \text{const}$ at the initial time moment, its further dynamics evolves in this plane. The only possible shape for it is the circle. The radius of this circle can only increase in time in the case $\eta = t + \rho$ (the closed test null string radially expands in the plane $z = z_0$) or can only decrease in time in the case $\eta = t - \rho$ (the closed test null string radially collapses in the plane $z = z_0$).

Hence, requirement (40) brings about a solution testifying to the possibility for the gas of null strings to exists in a state composed of two non-interacting subsystems. In each subsystem, the closed circular null strings are located in parallel planes $z = \text{const}$ (the polarization effect). The null strings radially expand in one subsystem and, simultaneously, radially collapse in another one without changing their shape, i.e. without interaction.

Under conditions (36) and (37), the first integral of Eq. (16) looks like

$$\eta_{,\tau} = P_1(\sigma) e^{2\nu}, \tag{44}$$

where

$$P_1(\sigma) > 0 \tag{45}$$

is the integration "constant". One can show that, for the case

$$q_{,\tau} > 0, \tag{46}$$

(another option in inequality (39)) and taking Eq. (38) into account, the first integrals of Eqs. (17) and (18) take the form

$$|f_{,z} z_{,\tau}| = \frac{P_2(\sigma)}{P_1(\sigma)} \frac{|\lambda_{,\eta}|}{(\lambda(\eta))^2} \eta_{,\tau}, \tag{47}$$

$$q_{,\tau} = \left(\frac{P_2(\sigma)}{P_1(\sigma)}\right)^2 \frac{|\lambda_{,\eta}|}{(\lambda(\eta))^2} \eta_{,\tau}, \tag{48}$$

where the function $P_2(\sigma)$ (the integration "constant") determines the z-velocities of test null string points at the initial time moment; and, as follows from Eqs. (37), (45), and (47),

$$P_2(\sigma) > 0. \tag{49}$$

From Eqs. (47) and (48), it follows that, in the case of (36) and (46), the variables η, q, and z, which determine the position of the test null string at every fixed time moment, are no more independent, but interrelated.

From Eqs. (8) and (9), it follows that, for Eqs. (47) and (48), the whole region of variation for the variables η and z can be divided into four domains depending on the sign of derivatives of the functions $\lambda(\eta)$ and $f(z)$:

(I) $\eta \in (-\infty, 0)$ and $z \in (0, +\infty)$, in which $f_{,z} > 0$ and $\lambda_{,\eta} > 0$;

(II) $\eta \in (-\infty, 0)$ and $z \in (-\infty, 0)$, in which $f_{,z} < 0$ and $\lambda_{,\eta} > 0$;

(III) $\eta \in (0, +\infty)$ and $z \in (0, +\infty)$, in which $f_{,z} > 0$ and $\lambda_{,\eta} < 0$; and

(IV) $\eta \in (0, +\infty)$ and $z \in (-\infty, 0)$, in which $f_{,z} < 0$ and $\lambda_{,\eta} < 0$.

Integrating Eq. (48) firstly at $\eta < 0$ (regions I and II; $\lambda_{,\eta} > 0$) and then at $\eta > 0$ (regions III and IV; $\lambda_{,\eta} < 0$), and matching the obtained solutions across the boundary $\eta = 0$ (using $\lambda(\eta)|_{\eta=0} = (f_0)^{-1}$), we have:

in regions I and II ($\eta < 0$),

$$q = q_0(\sigma) + 2 f_0 \left(\frac{P_2(\sigma)}{P_1(\sigma)}\right)^2 - \left(\frac{P_2(\sigma)}{P_1(\sigma)}\right)^2 (\lambda(\eta))^{-1}, \tag{50}$$

in regions III and IV ($\eta > 0$),

$$q = q_0(\sigma) + \left(\frac{P_2(\sigma)}{P_1(\sigma)}\right)^2 (\lambda(\eta))^{-1}, \tag{51}$$

where $q_0(\sigma)$ is the integration "constant".

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In each region, two possible directions of motion of the test null string along the axis z can be realized, $z_{,\tau} > 0$ and $z_{,\tau} < 0$. Therefore, the solution of Eq. (47) in each region can be presented in the form

$$f_L^i = U_L^i(\sigma) + \gamma_L^i \frac{P_2(\sigma)}{P_1(\sigma)} (\lambda(\eta))^{-1}, \tag{52}$$

where the subscript L takes values I to IV and corresponds to the number of the region, in which the found solution is realized; the superscript i takes values 0 (the case $z_{,\tau} > 0$, the test null string moves in the positive direction of the axis z); the constants γ_L^i equal

$$\begin{aligned} &\gamma_{\mathrm{I}}^0 = \gamma_{\mathrm{II}}^1 = \gamma_{\mathrm{III}}^1 = \gamma_{\mathrm{IV}}^0 = -1, \\ &\gamma_{\mathrm{I}}^1 = \gamma_{\mathrm{II}}^0 = \gamma_{\mathrm{III}}^0 = \gamma_{\mathrm{IV}}^1 = 1, \end{aligned} \tag{53}$$

the functions $U_L^i(\sigma)$, in view of the continuity of the obtained solution across the boundary $\eta = 0$, look like

$$\begin{aligned} &U_{\mathrm{I}}^0 = F_1(\sigma), \quad U_{\mathrm{III}}^0 = F_1(\sigma) - 2f_0 \frac{P_2(\sigma)}{P_1(\sigma)}, \\ &U_{\mathrm{I}}^1 = \widetilde{F}_1(\sigma), \quad U_{\mathrm{III}}^1 = \widetilde{F}_1(\sigma) + 2f_0 \frac{P_2(\sigma)}{P_1(\sigma)}, \\ &U_{\mathrm{II}}^0 = F_2(\sigma), \quad U_{\mathrm{IV}}^0 = F_2(\sigma) + 2f_0 \frac{P_2(\sigma)}{P_1(\sigma)}, \\ &U_{\mathrm{II}}^1 = \widetilde{F}_2(\sigma), \quad U_{\mathrm{IV}}^1 = \widetilde{F}_2(\sigma) - 2f_0 \frac{P_2(\sigma)}{P_1(\sigma)}, \end{aligned} \tag{54}$$

and the functions $F_1(\sigma)$, $\widetilde{F}_1(\sigma)$, $F_2(\sigma)$, and $\widetilde{F}_2(\sigma)$ are integration "constants".

From equality (52), it follows that the size, i.e. the radius, of the moving test null string is strictly related to its position with respect to the null string generating the gravitational field; i.e. it depends on the variable η. Analogously, since the function $f(z)$ on the left-hand side of equality (52) is finite and the function $\lambda(\eta)$ in the denominator of the right-hand side of this equality, in accordance with Eq. (8), tends to zero for $\eta \in (-\infty, -\Delta\eta) \cup (+\Delta\eta, +\infty)$, any choice of integration "constants" is always associated with a certain confined region symmetric in η, where equality (52) is satisfied. However, in this case, since there are no restrictions on the test null string coordinates z and t (in the general case, $\eta \in (-\infty, +\infty)$), only those test null strings that are located in this narrow region (the "interaction zone") are "visible", i.e. they interact with the null string generating the gravitational field. The same test null strings located at this moment beyond this zone, in accordance with Eq. (52), remain "invisible" for the null string that generates the gravitational field. Here, we cannot say that they do not interact, because, in the framework of the general theory of relativity, the absence of interaction manifests itself in the null string preservation without changing the trajectory of its motion. Whereas, in our case, it is impossible to say anything about the trajectory of motion of the test null string beyond this region. However, one cannot exclude that, at a certain time moment, such an "invisible" null string will enter this region, and its subsequent dynamics will be determined, at least until the time moment, when the test null string leaves it. In other words, the test null string, when entering this narrow "interaction zone", already has a prehistory, and its dynamics in this zone depends on the size, location, and direction of its motion along the axis z (it moves in the positive or negative direction of the axis z, i.e. $z_{,\tau} > 0$ or $z_{,\tau} < 0$), being determined by equality (52).

Under conditions (44) and (50)–(52), Eq. (21) takes the following form in each region determined by the subscript L:

$$\left(q_0(\sigma) + 2f_0 \left(\frac{P_2(\sigma)}{P_1(\sigma)} \right)^2 \right)_{,\sigma} + \\ + 2\frac{P_2(\sigma)}{P_1(\sigma)} F_L^i(\sigma)_{,\sigma} = 0. \tag{55}$$

Here, the indices i and L have the same interpretation and accept the same values as in Eq. (52), and the functions $F_L^i(\sigma)$ are

$$\begin{aligned} &F_{\mathrm{I}}^0 = F_{\mathrm{III}}^0 = -F_1(\sigma), \quad F_{\mathrm{I}}^1 = F_{\mathrm{III}}^1 = \widetilde{F}_1(\sigma), \\ &F_{\mathrm{II}}^0 = F_{\mathrm{IV}}^0 = F_2(\sigma), \quad F_{\mathrm{II}}^1 = F_{\mathrm{IV}}^1 = -\widetilde{F}_2(\sigma). \end{aligned} \tag{56}$$

The functions $P_k(\sigma)$, $k = 1, 2$, determine the initial momenta of the test null string points. As follows from equalities (52), the requirement

$$P_k(\sigma), F_k(\sigma), \widetilde{F}_k(\sigma) = \mathrm{const}, \quad k = 1, 2, \tag{57}$$

describes the case where the test null string shape is not changed (remains to be a circle) in the course of motion, and the variations of the radius of the test null string and its position on the axis z are determined by the form of the functions $f(z)$ and $\lambda(\eta)$.

Note that, under condition (57), Eq. (55) is reduced to a single requirement,

$$q_0(\sigma)_{,\sigma} = 0, \rightarrow q_0(\sigma) = q_0 = \text{const}. \tag{58}$$

From Eqs. (50) and (51), it follows that the constant q_0 defines the surface, on which the test null string and the null string generating the gravitational field "meet," while moving "toward" each other.

Using Eq. (2), let us express Eq. (44) in the form

$$\eta_{,\tau}\frac{|\lambda_{,\eta}|\left(\alpha(\eta)+\lambda(\eta)f(z)\right)^{2-\sqrt{4-2\chi}}}{(\lambda(\eta))^{2+(2-\sqrt{4-2\chi})/(1-\chi)}} = P_1(\sigma), \tag{59}$$

Since the function $0 < \alpha(\eta) + \lambda(\eta)f(z) < 1$ at any η and z, and the constant $\chi = 8\pi G \ll 1$, so that the difference $2-\sqrt{4-2\chi} \approx 0$, Eq. (59) can be presented in the form

$$\eta_{,\tau}\frac{|\lambda_{,\eta}|}{(\lambda(\eta))^2} = P_1(\sigma).$$

Integrating this equation, we obtain the following relations between the variable η and the parameters τ and σ on the world surface of the test null string:

in regions I and II ($\eta < 0$),

$$(\lambda(\eta))^{-1} = \eta_0(\sigma) - P_1((\sigma)\tau; \tag{60}$$

in regions III and IV ($\eta > 0$),

$$(\lambda(\eta))^{-1} = \widetilde{\eta}_0(\sigma) + P_1((\sigma)\tau. \tag{61}$$

Here, the integration "constants" $\eta_0(\sigma)$ and $\widetilde{\eta}_0(\sigma)$ determine the value of parameter τ, at which the test null string moving "toward" the null string generating the gravitational field meets the latter on the same surface. For instance, under condition (57), by fixing

$$\eta_0(\sigma) = \widetilde{\eta}_0(\sigma) = f_0 = \text{const} \tag{62}$$

in Eqs. (60) and (61), we obtain that, at $\eta = 0$, the parameter $\tau = 0$. Moreover,

in regions I and II ($\eta < 0$) at $\eta \in (-\infty, 0)$, the parameter $\tau \in (-\infty, 0)$;

in regions III and IV ($\eta > 0$) at $\eta \in (0, +\infty)$, the parameter $\tau \in (0, +\infty)$;

Under conditions (57), (58), and (62), the variables η and q determined by equalities (50), (51), (60), and (61) depend only on the parameter τ by means of the relations

$$\eta = \Lambda(f_0 \mp P_1\tau), \tag{63}$$

$$q = q_0 + f_0\left(\frac{P_2(\sigma)}{P_1(\sigma)}\right)^2 + \frac{(P_2(\sigma))^2}{P_1(\sigma)}\tau, \tag{64}$$

where the choice of the sign in Eq. (63) is related to the region of the test null string location (the sign "$-$" at $\eta < 0$, and "+" at $\eta > 0$), and the function $\Lambda(f_0 \mp P_1\tau)$ is determined by the explicit form of the function $\lambda(\eta)$; for example, for expression (11),

$$\Lambda(f_0 \mp P_1\tau) = \mp\frac{1}{\xi}\sqrt{ln^{-1}\left(1-\frac{f_0}{f_0 \mp P_1\tau}\right) - \epsilon}.$$

Note that equalities (52) put restrictions on the values of parameter τ, i.e. they determine the boundaries of the region, in which the moving test null string becomes "visible" for the null string generating the gravitational field and interacts with it.

5. Examples of Test Null String Motion at $\eta = t - \rho$

In the case $z_{,\tau} > 0$ in regions I and III, according to Eqs. (52–(54), (63), and (64), the solution of the equations of motion for the test null string is

$$t - \rho = \mp\frac{1}{\xi}\sqrt{ln^{-1}\left(1-\frac{f_0}{f_0 \mp P_1\tau}\right) - \epsilon}, \tag{65}$$

$$t + \rho = q_0 + f_0\left(\frac{P_2(\sigma)}{P_1(\sigma)}\right)^2 + \frac{(P_2(\sigma))^2}{P_1(\sigma)}\tau, \tag{66}$$

$$f(z) = F_1 - \frac{P_2}{P_1}f_0 \mp P_2\,|\tau|\,. \tag{67}$$

In equalities (65) and (67), the upper sign is selected for region I ($\tau \in (-\infty, 0)$) and the lower one for region III ($\tau \in (0, +\infty)$). The interaction zone boundaries in those regions are determined as follows. In region I, this is the minimum possible value of the right-hand side in equality (67). Here, this value equals zero and is reached at $|\tau| = F_1(P_2)^{-1} - f_0(P_1)^{-1} > 0$ (the leftmost boundary of the interaction zone). In region III, this is the maximum possible value of the right-hand side in equality (67). Here, this value equals f_0 and is reached at $\tau \rightarrow f_0((P_2)^{-1}+(P_1)^{-1}) -$

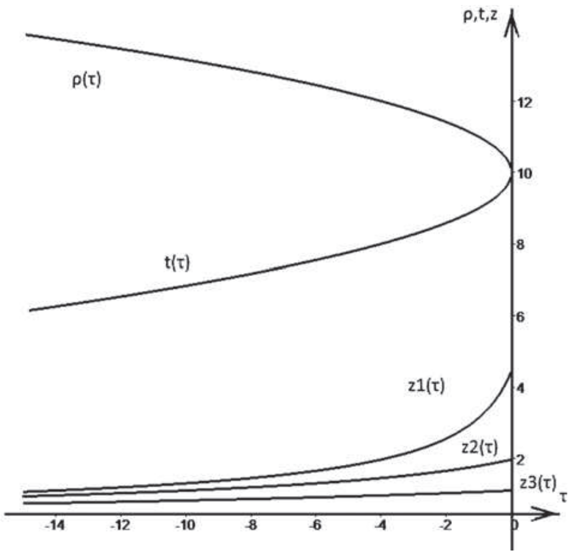

Fig. 1. Plots of the functions $t(\tau)$, $\rho(\tau)$, and $z(\tau)$ in region I at $f_0 = 100$, $P_1 = f_0^2$, $P_2 = 1$, $\xi = \zeta = \mu = 5$, $\epsilon = 10^{-7}$, $q_0 = 20$, and for various $F_1 = 99$ ($z_1(\tau)$), 95 ($z_2(\tau)$), and 85 ($z_3(\tau)$)

$- F_1(P_2)^{-1} > 0$ (the rightmost boundary of the interaction zone).

The functions $t(\tau)$, $\rho(\tau)$, and $z(\tau)$ are plotted in Figs. 1 (for region I) and 2 (for region III) in the case $z_{,\tau} > 0$, the certain fixed values of constants P_1 and P_2, and three different values of constant F_1. From those figures, it follows that the test null string, when approaching the right boundary of the interaction zone (Fig. 2), becomes always pushed out by the gravitational field (the variable z) to the infinity within a very short time interval.

In regions II and IV in the case $z_{,\tau} > 0$, equality (52), making allowance for Eqs. (53), (54), and (60)–(62), takes the form

$$f(z) = F_2 + \frac{P_2}{P_1} f_0 \pm P_2 \left|\tau\right|, \tag{68}$$

where the upper sign is selected for region II ($\tau \in (-\infty, 0)$), and the lower one for region IV ($\tau \in (0, +\infty)$). The boundaries of the interaction zone in those regions are determined analogously. In region II, this is the maximum value of the right-hand side of equality (68). Now, this maximum equals f_0 and is reached at $|\tau| \to (f_0((P_2)^{-1} - (P_1)^{-1}) - F_2(P_2)^{-1}) > 0$ (the leftmost boundary of the interaction zone). In region IV, this is the minimum

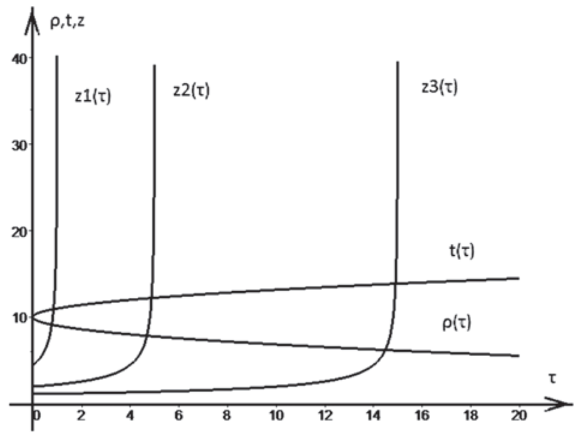

Fig. 2. Plots of the functions $t(\tau)$, $\rho(\tau)$, and $z(\tau)$ in region III at $f_0 = 100$, $P_1 = f_0^2$, $P_2 = 1$, $\xi = \zeta = \mu = 5$, $\epsilon = 10^{-7}$, $q_0 = 20$, and for various $F_1 = 99$ ($z_1(\tau)$), 95 ($z_2(\tau)$), and 85 ($z_3(\tau)$)

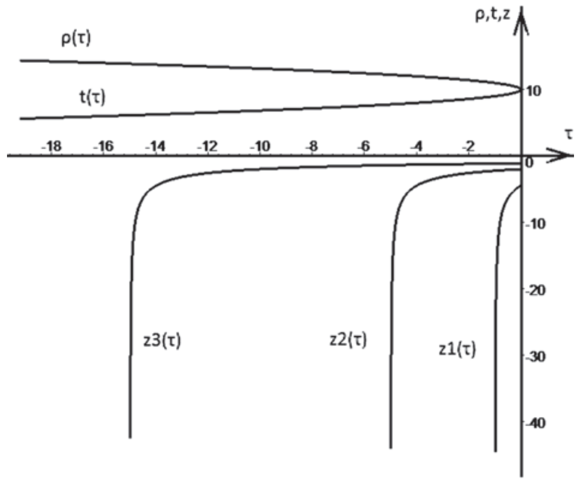

Fig. 3. Plots of the functions $t(\tau)$, $\rho(\tau)$, and $z(\tau)$ in region II at $f_0 = 100$, $P_1 = f_0^2$, $P_2 = 1$, $\xi = \zeta = \mu = 5$, $\epsilon = 10^{-7}$, $q_0 = 20$, and for various $F_2 = 99$ ($z_1(\tau)$), 95 ($z_2(\tau)$), and 85 ($z_3(\tau)$)

possible value of the right-hand side of equality (68), which equals zero and is reached at $\tau = F_2(P_2)^{-1} + f_0(P_1)^{-1} > 0$ (the rightmost boundary of the interaction zone).

In Figs. 3 (for region II) and 4 (for region IV), the functions $t(\tau)$, $\rho(\tau)$, and $z(\tau)$ are plotted in the case $z_{,\tau} > 0$, at the certain fixed values of constants P_1 and P_2, and for three different values of constant F_2. The figures demonstrate that the width and the position of the interaction zone for the test null string depend on the constant F_2.

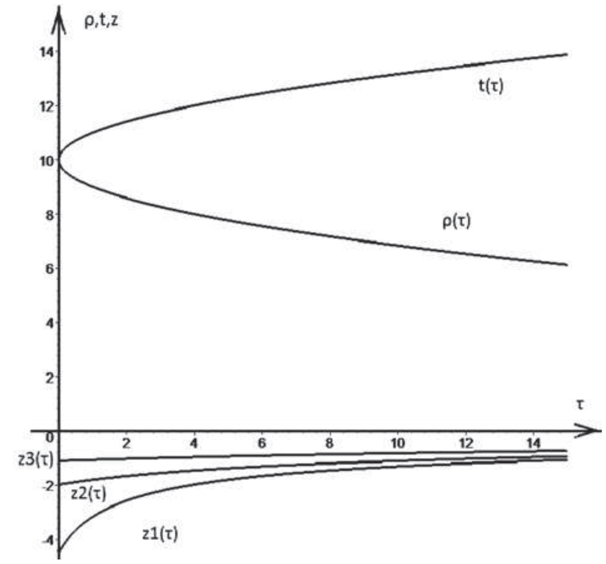

Fig. 4. Plots of the functions $t(\tau)$, $\rho(\tau)$, and $z(\tau)$ in region IV at $f_0 = 100$, $P_1 = f_0^2$, $P_2 = 1$, $\xi = \zeta = \mu = 5$, $\epsilon = 10^{-7}$, $q_0 = 20$, and for various $F_2 = 99$ ($z_1(\tau)$), 95 ($z_2(\tau)$), and 85 ($z_3(\tau)$)

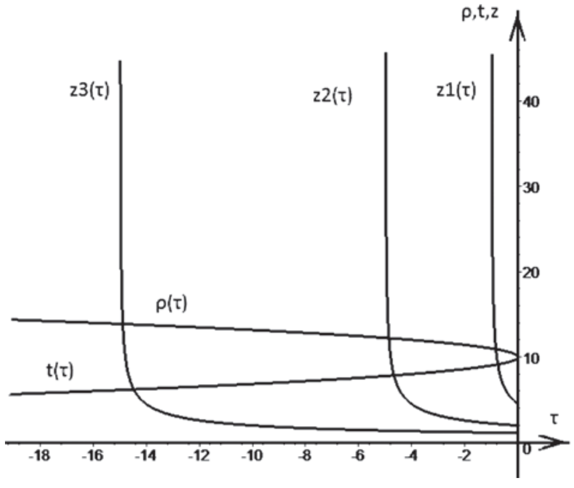

Fig. 5. Plots of the functions $t(\tau)$, $\rho(\tau)$, and $z(\tau)$ in region I at $f_0 = 100$, $P_1 = f_0^2$, $P_2 = 1$, $\xi = \zeta = \mu = 5$, $\epsilon = 10^{-7}$, $q_0 = 20$, and for various $\tilde{F}_1 = 99$ ($z_1(\tau)$), 95 ($z_2(\tau)$), and 85 ($z_3(\tau)$)

The test null string, irrespective of how far it is located along the axis z, when arriving at the left boundary of the interaction zone (Fig. 3), is always attracted by the gravitational field (the variable z) to the plane, where the null string generating the gravitational field is located, within a very short time interval.

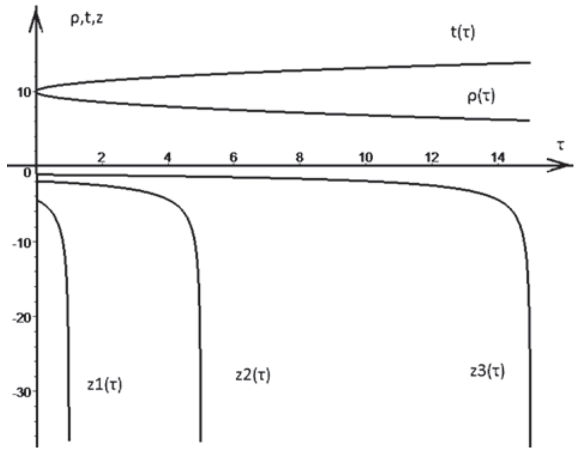

Fig. 6. Plots of the functions $t(\tau)$, $\rho(\tau)$, and $z(\tau)$ in region IV at $f_0 = 100$, $P_1 = f_0^2$, $P_2 = 1$, $\xi = \zeta = \mu = 5$, $\epsilon = 10^{-7}$, $q_0 = 20$, and for various $\tilde{F}_2 = 99$ ($z_1(\tau)$), 95 ($z_2(\tau)$), and 85 ($z_3(\tau)$)

Examples and the motion diagram of the test null string in the case $z_{,\tau} < 0$ are similar to those presented above. They can also be obtained by changing the direction of the axis z and making the corresponding modifications in the notations for the regions. For instance, in Figs. 5 and 6, the plots of the functions $t(\tau)$, $\rho(\tau)$, and $z(\tau)$ are depicted for regions I and III, respectively, in the case $z_{,\tau} < 0$. From those figures, it follows that

– the test null string, irrespective of how far it is located along the axis z, when arriving at the left boundary of the interaction zone in region I (Fig. 5), is always attracted within a very short time interval by the gravitational field (the variable z) to the plane, where the null string generating the gravitational field is located;

– the test null string located near the plane $z = 0$ and approaching the right boundary of the interaction zone (Fig. 6) is always pushed out to the infinity by the gravitational field (the variable z) within a very short time interval.

6. Examples of Test Null String Motion at $\eta = t + \rho$

From equalities (52)–(54), (63), and (64), it follows that the dynamics of the test null string in regions I to IV in the cases $\eta = t - \rho$ and $\eta = t + \rho$ is similar. For example, in the case $\eta = t + \rho$ and $z_{,\tau} > 0$, the solutions of the equations of motion for the test null string in regions I and III look like

$$t+\rho=\mp\frac{1}{\xi}\sqrt{ln^{-1}\left(1-\frac{f_0}{f_0\mp P_1\tau}\right)-\epsilon}, \tag{69}$$

$$t-\rho=q_0+f_0\left(\frac{P_2(\sigma)}{P_1(\sigma)}\right)^2+\frac{(P_2(\sigma))^2}{P_1(\sigma)}\tau, \tag{70}$$

$$f(z)=F_1-\frac{P_2}{P_1}f_0\mp P_2\left|\tau\right|. \tag{71}$$

In equalities (69) and (71), the upper sign is selected for region I ($\tau\in(-\infty,0)$), and the lower one for region III ($\tau\in(0,+\infty)$). The interaction zone boundaries in those regions determine the minimum possible value of the right-hand side in equality (71) in region I, which equals zero and is reached at $|\tau|=F_1(P_2)^{-1}-f_0(P_1)^{-1}>0$ (the leftmost boundary of the interaction zone), and the maximum possible value of the right-hand side in equality (67) in region III, which equals f_0 and is reached at $\tau\to f_0((P_2)^{-1}+(P_1)^{-1})-F_1(P_2)^{-1}>0$ (the rightmost boundary of interaction zone).

In Figs. 7 (for region I) and 8 (for region III), the functions $t(\tau)$, $\rho(\tau)$, and $z(\tau)$ are plotted for the case $\eta=t+\rho$ and $z_{,\tau}>0$, at the certain fixed values of constants P_1 and P_2, and for three different values of constant F_1. The figures demonstrate that the test null string, when approaching the right boundary of the interaction zone (Fig. 8), becomes always pushed out by the gravitational field (the variable z) to the infinity within a very short time interval.

From the given examples of the test null string motion, it follows that, in the case where the initial momenta of the test null string points along the axis z differ from zero ($P_2(\sigma)\neq0$), every test null string in the "interaction zone" is always either pushed out to the infinity (Figs. 2, 6, and 8) or attracted to the plane, where the null string generating the gravitational field is located (Figs. 3 and 5), irrespective of how far it is, by the gravitational field (the variable z) within a very short time interval. The specific scenario depends on the test null string position with respect to the plane, in which the null string generating the gravitational field is located, and the direction of the test null string motion along the axis z. In our opinion, the presence of trajectory sections with this anomalous behavior for every test null string in the "interaction zone" may indirectly testify that the ability to inflate (both the accelerated expansion and the accelerated collapse) can be an internal property of the gas of null strings. However, this statement requires an additional research.

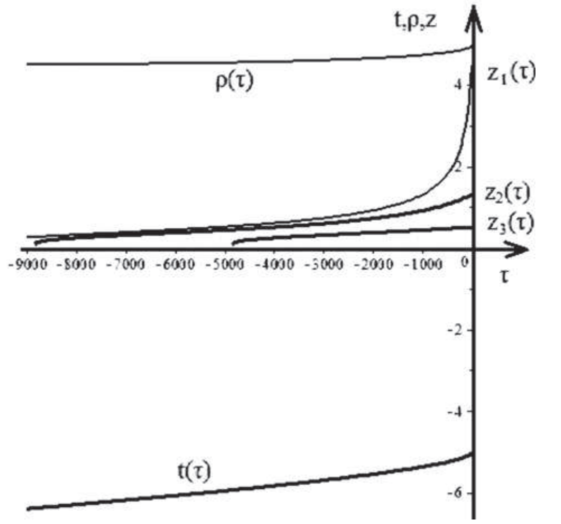

Fig. 7. Plots of the functions $t(\tau)$, $\rho(\tau)$, and $z(\tau)$ in region I in the case $\eta=t+\rho$ and $z_{,\tau}>0$ at $f_0=100$, $P_1=1$, $P_2=f_0^{-1}$, $\xi=\zeta=\mu=5$, $\epsilon=10^{-7}$, $q_0=-10$, and for various $F_1=100$ ($z_1(\tau)$), 90 ($z_2(\tau)$), and 50 ($z_3(\tau)$)

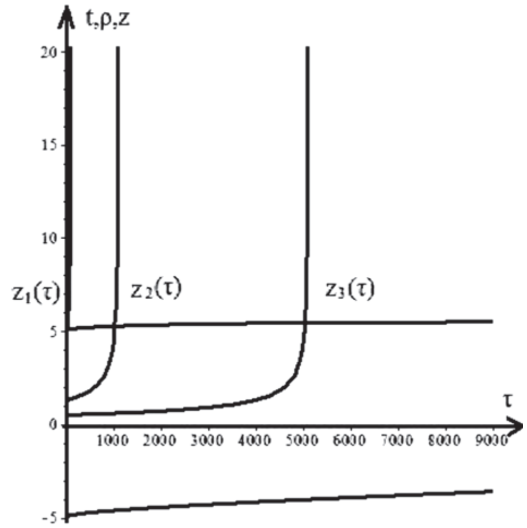

Fig. 8. Plots of the functions $t(\tau)$, $\rho(\tau)$, and $z(\tau)$ in region III in the case $z_{,\tau}>0$ at $f_0=100$, $P_1=1$, $P_2=f_0^{-1}$, $\xi=\zeta=\mu=5$, $\epsilon=10^{-7}$, $q_0=-10$, and for various $F_1=100$ ($z_1(\tau)$), 90 ($z_2(\tau)$), and 50 ($z_3(\tau)$)

7. Conclusions

The analysis of the solutions obtained for the equations of motion of a test null string moving in the gravitational field of a closed "thick" null string that either radially expands or radially collapses in the plane $z = 0$ has shown the following:

1. The test null string can move in the considered gravitational fields either in the same direction with the null string generating the gravitational field or "toward" it.

2. When the test null string moves in the same direction with the null string generating the gravitational field (i.e. if it either collapses in the case $\eta = t + \rho$ or radially expands in the case $\eta = t - \rho$), it is always located in one of the planes that are parallel to the plane, in which the null string generating the gravitational field is located (this fact can be interpreted as a polarization effect), preserving its initial shape in time, i.e. without interaction (the phase of the ideal gas of null strings).

3. When the test null string moves toward the null string generating the gravitational field, then

– in the case where the initial momentum of the test null string points along the axis z equals to zero ($P_2(\sigma) = 0$), all points of the closed test null string are equidistant from the axis z at every fixed time moment t, and the initial shape of the test null string remains constant in time. If, additionally, the test null string is located completely in the plane $z = z_0$ at the initial time moment, its whole further dynamics will develop in this plane, the circle will be the only allowable shape for this string, and the radius of this circle will only increase (in the case $\eta = t + \rho$; the test null string radially expands in the plane $z = z_0$) or decrease (in the case $\eta = t - \rho$; the test null string radially collapses in the plane $z = z_0$). Hence, there may exist a state (a phase) of the gas of null strings consisting of two noninteracting subsystems. In each of them, the closed circular null strings are located in parallel planes $z =$ const. The null strings radially expand in one subsystem and, simultaneously, radially collapse in the other, without changing their shape, i.e. without interaction.

– in the case where the initial momentum of the test null string points $P_2(\sigma) \neq 0$, the variables η, q, and z, which determine the test null string position at every fixed time moment, are no more independent, but coupled with one another. The radius of the test null string is strictly connected with its position with respect to the null string generating the gravitational field. For any choice of integration "constants", there always exists a "narrow" region (the "interaction zone"), where all equalities governing the test null string dynamics are satisfied. Only the test null strings located in this narrow region are "visible", i.e. they can interact with the null string generating the gravitational field.

4. The presence of trajectory sections, at which the test null string is either pushed out to the infinity or attracted from it, for every test null string in the "interaction zone" can indirectly testify that the ability to inflate (both the accelerated expansion and the accelerated collapse) can be an internal property of the gas of null strings. However, this statement needs an additional research.

5. By analyzing the results of this work, we may also suppose that, since separate regions in the gas of null strings (they include closed null strings that either radially expand or radially collapse simultaneously in parallel planes) are causally independent at the initial time moment, there may appear a domain structure in this gas. In other words, there may exist a large number of separated regions, in which the null strings either radially expand or radially collapse simultaneously in parallel planes (i.e. they are strictly polarized). The spatial orientation of those planes is random in every domain, without any correlation between neighbor domains. The conditions for such domains to emerge and exist, as well as the physical processes in the interdomain regions, can be a subject of further researches of the gas of null strings.

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ДИНАМІКА ПРОБНОЇ НУЛЬ-СТРУНИ, БЕЗ ОБЕРТАННЯ, В ГРАВІТАЦІЙНОМУ ПОЛІ "РОЗМАЗАНОЇ" ЗАМКНЕНОЇ НУЛЬ-СТРУНИ, ЯКА ПРЯМУЄ В ПЛОЩИНІ $z = 0$

Р е з ю м е

У роботі розглянуто динаміку пробної нуль-струни в гравітаційному полі замкненої "розмазаної" нуль-струни, яка радіально розширюється в площині $z = 0$ або радіально колапсує в площині $z = 0$, за умови, що в початковий момент часу обертання пробної нуль-струни було відсутнє.