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ASYMPTOTIC WAVE SOLUTIONS FOR THE MODEL OF A MEDIUM WITH VAN DER POL OSCILLATORS

A one-dimensional mathematical model for a complex medium with van der Pol oscillators has been studied. Using the Bogolyubov–Mitropolsky method, the wave solutions for a weakly nonlinear model are derived, with their amplitudes being described by a three-dimensional dynamical system analyzed in more details by numerical and qualitative methods. In particular, periodic, multiperiodic, and chaotic trajectories are found in the phase space of the dynamical system. Bifurcations of those regimes were considered using the Poincaré section technique. Exact solutions are derived in the case where the three-dimensional system for amplitudes is reduced to the two-dimensional one.

Keywords: nonlinear waves, van der Pol oscillator, chaotic attractor.

1. Introduction

The majority of natural media are hierarchical formations of structural elements, the dynamics of which cannot be neglected under certain loading conditions [1, 2]. For the description of such structured media, a mathematical model dealing with two continua was used: a background medium and particle oscillators coupled with it [3, 4]. Those models were extended onto nonlinear media in works [5–7] and nonlocal ones in works [8, 9]. In the cited works, the nonlinearity and the non-locality of a background medium were taken into account in the corresponding equations of state, whereas the dynamics of oscillating inclusions remained linear. In work [10], it was proposed to describe the dynamics of oscillating inclusions by nonlinear equations, in particular, by the van der Pol equations. In this work, the case of a linear background medium together with the nonlinear dynamics of oscillating inclusions, which are described by three forms of the van der Pol equations, is studied. Hence, the mathematical model of a complex medium looks like

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} - \sum_{j=1}^{N} m_j \frac{\partial^2 w_j}{\partial t^2}, \quad \sigma = \frac{E}{\rho}\frac{\partial u}{\partial x},$$
$$\frac{\partial^2 w_k}{\partial t^2} + F_k\left(w_k - u\right) = 0,\ k = 1, ..., N, \tag{1}$$

where u is a shift of the background medium characterized by the density ρ, σ is the strain, E is Young's modulus, w_k is a shift of the oscillating inclusion characterized by the density $m_k\rho$ and the characteristic frequency ω_k, and t and x are the time and space variables, respectively. The operator F_k, which describes the dynamics of the partial oscillator, has the following form:

$$F_k\left(y\right) = -\left(\lambda_k - \mu_k y^2\right)\frac{\partial y}{\partial t} + \omega_k^2 y,$$

where $k = 1, ..., N$, i.e. the oscillating inclusion is considered as a set of van der Pol oscillators.

Using the characteristic quantities τ, c_0, and u_0, we can make model (1) to be dimensionless, by executing the transformations $t \to \tau t$, $x \to \tau c_0 x$, $u \to u_0 u$, $w \to$

$\to u_0 w$, $\lambda_k \to \lambda_k \tau^{-1}$, $\omega_k \to \omega_k \tau^{-1}$, $\mu_k = \lambda_k \tau^{-1} u_0^{-2}$. Then, model (1) looks like

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \sum_{j=1}^{N} m_j \frac{\partial^2 w_j}{\partial t^2},\\ &\frac{\partial^2 w_k}{\partial t^2} - \lambda_k \left[1 - (w_k - u)^2\right] \frac{\partial (w_k - u)}{\partial t} + \\ &+ \omega_k^2 (w_k - u) = 0,\end{aligned} \tag{2}$$

where $c^2 = E\rho^{-1} c_0^{-2}$. Let us consider the wave solutions of model (2),

$$u = U(s),\ w_k = W_k(s),\ s = x + Dt, \tag{3}$$

and analyze the dependence of their structure on the parameters λ_k and D.

2. Construction of the Asymptotic Solution for Model (2)

Substituting expressions (3) into system (2), we obtain a system of ordinary differential equations

$$\begin{aligned}&W_k'' + \Omega_k^2 (W_k - U) = R_k (W_k - U),\\ &U = \sum_{j=1}^{N} \varphi_j W_j,\end{aligned} \tag{4}$$

where $R_k(y) = \alpha_k (1 - y^2) y'$, $\alpha_k = \frac{\lambda_k}{D}$, $\Omega_k = \frac{\omega_k}{D}$, $\varphi_k = \frac{D^2}{c^2 - D^2} m_k$, $(...)' = d(...)/ds$, $k = 1, ..., N$. Let us consider the wave solutions for the model with regard for the dynamics of oscillators of three types, i.e. $N = 3$. The dynamic system (4) belongs to the model class of coupled oscillators. The main interest in the researches of those models consists in new effects, which are governed by the type of interaction between partial oscillators. As a rule, those models involve either the oscillator coupling with the nearest neighbor [11, 12] or oscillators globally coupled through the average field [13–15]. System (4) is an example of the latter variant. The major means to study such models are asymptotic and numerical methods.

Let us consider the case of weak nonlinearity, when $\alpha_k = \varepsilon \alpha_k$, where $\varepsilon \ll 1$. In this case, the Bogolyubov–Mitropolsky method [16, 17] can be applied to system (4). At $\varepsilon = 0$, the solutions of system (4) are sought in the form

$$\begin{aligned}&W_1 = a r_{11} \sin\theta_1 + b r_{12} \sin\theta_2 + c r_{13} \sin\theta_3,\\ &W_2 = a r_{21} \sin\theta_1 + b r_{22} \sin\theta_2 + c r_{23} \sin\theta_3,\\ &W_3 = a r_{31} \sin\theta_1 + b r_{32} \sin\theta_2 + c r_{33} \sin\theta_3,\end{aligned} \tag{5}$$

where a, b, c, r_{ij}, k_i, and β_i are constants, $\theta_i = k_i s + \beta_i$, and $i, j = 1, 2, 3$. Substituting expressions (5) into Eq. (4), we obtain, for $\varepsilon = 0$, a system of linear equations for r_{ij}. The condition of consistency is the equation

$$\begin{vmatrix} K_1 & \Omega_1^2 \varphi_2 & \Omega_1^2 \varphi_3 \\ \Omega_2^2 \varphi_1 & K_2 & \Omega_2^2 \varphi_3 \\ \Omega_3^2 \varphi_1 & \Omega_3^2 \varphi_2 & K_3 \end{vmatrix} = 0, \tag{6}$$

where $K_i = k^2 + \Omega_i^2 (\varphi_i - 1)$, $i = 1, 2, 3$. Taking condition (6) into account and adopting $r_{1j} = 1$, $j = 1, 2, 3$, we obtain

$$\begin{aligned}&r_{2j} = \frac{\Omega_2^2 \left(k_j^2 - \Omega_1^2\right)}{\Omega_1^2 \left(k_j^2 - \Omega_2^2\right)},\\ &r_{3j} = \frac{k_j^4 + k_j^2 \left(\Omega_1^2 (1 - \varphi_1) + \Omega_2^2 (1 - \varphi_2)\right)}{\Omega_1^2 \left(k_j^2 - \Omega_2^2\right) \varphi_3} - \\ &- \frac{\Omega_1^2 \Omega_2^2 (1 - \varphi_1 - \varphi_2)}{\Omega_1^2 \left(k_j^2 - \Omega_2^2\right) \varphi_3}.\end{aligned} \tag{7}$$

At small $\varepsilon \neq 0$, we suppose the solution of system (4) to be determined by expressions (5), in which a, b, c, and β_i are functions of the "slow" variable εs. We also adopt that

$$\begin{aligned}&\frac{dW_1}{ds} = a k_1 \cos\theta_1 + b k_2 \cos\theta_2 + c k_3 \cos\theta_3,\\ &\frac{dW_2}{ds} = a r_{21} k_1 \cos\theta_1 + \\ &+ b r_{22} k_2 \cos\theta_2 + c r_{23} k_3 \cos\theta_3,\\ &\frac{dW_3}{ds} = a r_{31} k_1 \cos\theta_1 + \\ &+ b r_{32} k_2 \cos\theta_2 + c r_{33} k_3 \cos\theta_3,\end{aligned} \tag{8}$$

provided the additional condition

$$\begin{aligned}&\frac{da}{ds} \sin\theta_1 + \frac{d\beta_1}{ds} a \cos\theta_1 + \frac{db}{ds} \sin\theta_2 + \frac{d\beta_2}{ds} b \cos\theta_2 + \\ &+ \frac{dc}{ds} \sin\theta_3 + \frac{d\beta_3}{ds} c \cos\theta_3 = 0,\\ &\frac{da}{ds} r_{21} \sin\theta_1 + \frac{d\beta_1}{ds} a r_{21} \cos\theta_1 + \frac{db}{ds} r_{22} \sin\theta_2 + \\ &+ \frac{d\beta_2}{ds} b r_{22} \cos\theta_2 + \frac{dc}{ds} r_{23} \sin\theta_3 + \frac{d\beta_3}{ds} c r_{32} \cos\theta_3 = 0,\\ &\frac{da}{ds} r_{31} \sin\theta_1 + \frac{d\beta_1}{ds} a r_{31} \cos\theta_1 + \frac{db}{ds} r_{32} \sin\theta_2 + \\ &+ \frac{d\beta_2}{ds} b r_{32} \cos\theta_2 + \frac{dc}{ds} r_{33} \sin\theta_3 + \frac{d\beta_3}{ds} c r_{33} \cos\theta_3 = 0.\end{aligned} \tag{9}$$

Substituting Eqs. (5)and (8) into Eq. (4) and taking Eq. (6) into account, we obtain the following system written in the normal form:

$$\frac{da}{ds} = \frac{\varepsilon\cos\theta_1}{k_1\delta}\big(R_1\left(r_{23}r_{32} - r_{22}r_{33}\right) + \\ + R_2\left(r_{33} - r_{32}\right) + R_3\left(r_{22} - r_{23}\right)\big),$$
$$\frac{db}{ds} = \frac{\varepsilon\cos\theta_2}{k_2\delta}\big(R_1\left(r_{21}r_{33} - r_{23}r_{31}\right) + \\ + R_2\left(r_{31} - r_{33}\right) + R_3\left(r_{23} - r_{21}\right)\big), \quad (10)$$
$$\frac{dc}{ds} = \frac{\varepsilon\cos\theta_3}{k_3\delta}\big(R_1\left(r_{22}r_{31} - r_{21}r_{32}\right) + \\ + R_2\left(r_{32} - r_{31}\right) + R_3\left(r_{21} - r_{22}\right)\big),$$

where

$$R_k = \alpha_k\left[1 - \left(W_k - \sum_{j=1}^{3}\varphi_j W_j\right)^2\right]\times \\ \times\frac{d}{ds}\left(W_k - \sum_{j=1}^{3}\varphi_j W_j\right),$$
$$\delta = r_{23}\left(r_{32} - r_{31}\right) + r_{21}\left(r_{33} - r_{32}\right) + r_{22}\left(r_{31} - r_{33}\right).$$

After the averaging according to the formula

$$\frac{d}{ds}(\star) = \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}(\star)d\theta_1 d\theta_2 d\theta_3,$$

system (10) reads

$$4\delta\frac{dx}{dT} = x\left(A_0 + A_1x + A_2y + A_3z\right),$$
$$4\delta\frac{dy}{dT} = y\left(B_0 + B_1x + B_2y + B_3z\right), \quad (11)$$
$$4\delta\frac{dz}{dT} = z\left(C_0 + C_1x + C_2y + C_3z\right),$$

where $x = a^2$, $y = b^2$, $z = c^2$, and $T = \varepsilon s$. The other parameters are

$$A_0 = 4\big(\alpha_1H_{11}(r_{23}r_{32} - r_{22}r_{33}) + \alpha_2H_{21}(r_{33} - r_{32}) + \\ + \alpha_3H_{31}(r_{22} - r_{23})\big),$$
$$A_1 = \alpha_1H_{11}^3(r_{23}r_{32} - r_{22}r_{33}) - \alpha_2H_{21}^3(r_{33} - r_{32}) - \\ - \alpha_3H_{31}^3(r_{22} - r_{23}),$$
$$A_2 = 2(\alpha_1H_{11}H_{12}^2(r_{22}r_{33} - r_{23}r_{32}) + \\ + \alpha_2H_{21}H_{22}^2(r_{32} - r_{33}) - \alpha_3H_{31}H_{32}^2(r_{22} - r_{23})),$$
$$A_3 = 2(\alpha_1H_{11}H_{13}^2(r_{22}r_{33} - r_{23}r_{32}) + \\ + \alpha_2H_{21}H_{23}^2(r_{32} - r_{33}) - \alpha_3H_{31}H_{33}^2(r_{22} - r_{23})),$$
$$B_0 = 4\big(-\alpha_1H_{12}(r_{23}r_{31} - r_{21}r_{33}) + \\ + \alpha_2H_{22}(r_{31} - r_{33}) - \alpha_3H_{32}(r_{21} - r_{23})\big),$$
$$B_1 = -2(\alpha_1H_{12}H_{11}^2(r_{21}r_{33} - r_{23}r_{31}) + \\ + \alpha_2H_{22}H_{21}^2(r_{31} - r_{33}) - \alpha_3H_{32}H_{31}^2(r_{21} - r_{23})),$$
$$B_2 = \alpha_1H_{12}^3(r_{23}r_{31} - r_{21}r_{33}) - \alpha_2H_{22}^3(r_{31} - r_{33}) + \\ + \alpha_3H_{32}^3(r_{21} - r_{23}),$$
$$B_3 = -2(-\alpha_1H_{12}H_{13}^2(r_{23}r_{31} - r_{21}r_{33}) + \\ + \alpha_2H_{22}H_{23}^2(r_{31} - r_{33}) - \alpha_3H_{32}H_{33}^2(r_{21} - r_{23})),$$
$$C_0 = 4\big(\alpha_1H_{13}(r_{22}r_{31} - r_{21}r_{32}) - \\ - \alpha_2H_{23}(r_{31} - r_{32}) + \alpha_3H_{33}(r_{21} - r_{22})\big),$$
$$C_1 = 2(\alpha_1H_{13}H_{11}^2(r_{21}r_{32} - r_{22}r_{31}) + \\ + \alpha_2H_{23}H_{21}^2(r_{31} - r_{32}) - \alpha_3H_{33}H_{31}^2(r_{21} - r_{22})),$$
$$C_2 = 2(-\alpha_1H_{13}H_{12}^2(r_{22}r_{31} - r_{21}r_{32}) + \\ + \alpha_2H_{23}H_{22}^2(r_{31} - r_{32}) - \alpha_3H_{33}H_{32}^2(r_{21} - r_{22})),$$
$$C_3 = -\alpha_1H_{13}^3(r_{22}r_{31} - r_{21}r_{32}) + \alpha_2H_{23}^3(r_{31} - r_{32}) - \\ - \alpha_3H_{33}^3(r_{21} - r_{22}),$$

where

$$H_{1j} = \varphi_1 - 1 + \varphi_2r_{2j} + \varphi_3r_{3j},$$
$$H_{2j} = \varphi_1 + (\varphi_2 - 1)r_{2j} + \varphi_3r_{3j},$$
$$H_{3j} = \varphi_1 + \varphi_2r_{2j} + (\varphi_3 - 1)r_{3j}.$$

Hence, we come to studying the structure of the first octant in the phase space of system (11).

3. Analysis of Two-Dimensional Subsystems of the Dynamic System (11)

It is evident that, on the coordinate planes, the three-dimensional system (11) is reduced to two-dimensional ones. For instance, on the plane $z = 0$, it looks like

$$\frac{dx}{dT} = x\left(A_0 + A_1x + A_2y\right),$$
$$\frac{dy}{dT} = y\left(B_0 + B_1x + B_2y\right), \quad z = 0. \quad (12)$$

The dynamic system (12) describes the amplitude dynamics for the two-frequency solution (5). Carrying out the scaling

$\nu\varepsilon s = T$, $x = \xi\bar{x}$, $y = \eta\bar{y}$,

$$\nu = -\frac{A_0}{4\delta},\ \xi = -\frac{A_0}{A_1},\ \eta = -\frac{A_0}{A_2},$$

the dynamic system (12) is transformed into the form (the bars over the variables are omitted)

$$\begin{aligned} \frac{dx}{dT} &= x\left(x + y - 1\right), \\ \frac{dy}{dT} &= y\left(\mu_1 x + \mu_2 y - \mu_3\right), \end{aligned} \tag{13}$$

where

$$\mu_1 = \frac{B_1}{A_1},\ \mu_2 = \frac{B_2}{A_2},\ \mu_3 = \frac{B_0}{A_0}.$$

Without making a detailed analysis of this amplitude system (this was done in work [10]), let us point to the main properties of the dynamic system (13).

System (13) has four stationary points with the coordinates

$$O\left(0;0\right), X\left(1;0\right), Y\left(0;\frac{\mu_3}{\mu_2}\right), Q\left(\frac{\mu_2-\mu_3}{\mu_2-\mu_1};\frac{\mu_3-\mu_1}{\mu_2-\mu_1}\right).$$

In view of the sense of the variables x and y, it follows that points Y and Q lie in the first quadrant of the phase plane if

$$\frac{\mu_3}{\mu_2} \geq 0 \quad \text{and} \quad \frac{\mu_2-\mu_3}{\mu_2-\mu_1} \geq 0, \quad \frac{\mu_3-\mu_1}{\mu_2-\mu_1} \geq 0,$$

respectively.

The eigenvalues of the linearization matrix

$$J\left(x_0;y_0\right) = \begin{pmatrix} 2x_0 + y_0 - 1 & x_0 \\ \mu_1 y_0 & \mu_1 x_0 + 2\mu_2 y_0 - \mu_3 \end{pmatrix}$$

are as follows:
– for point O, $\lambda_O = (\ -1; -\mu_3)$;
– for point X, $\lambda_X = (1; \mu_1 - \mu_3)$;
– for point Y, $\lambda_Y = \left(\mu_3; \mu_3\mu_2^{-1} - 1\right)$;
– for point Q, $\lambda_Q = \dfrac{\mu_2 - \mu_1\mu_2 - \mu_3 + \mu_2\mu_3 \pm \sqrt{\Delta}}{2\left(\mu_2 - \mu_1\right)}$,
where $\Delta = \left(\mu_2 - \mu_1\mu_2 - \mu_3 + \mu_2\mu_3\right)^2 + 4(\mu_2 - \mu_1)(\mu_2 - \mu_3)(\mu_1 - \mu_3)$.
The stationary points X and Y evidently correspond to the existence of harmonic modes with frequencies k_1 and k_2 in the system. Point Q corresponds to a biharmonic mode.

As was shown in work [18], system (13) has no closed trajectories, because there exists the function

$$B\left(x,y\right) = x^{p-1}y^{q-1},\ p = \frac{\mu_2 - \mu_1\mu_2}{\mu_1 - \mu_2},\ q = \frac{\mu_2 - 1}{\mu_1 - \mu_2},$$

for which

$$G = \frac{\partial}{\partial x}(Bx(x+y-1)) + \frac{\partial}{\partial y}(By(\mu_1 x + \mu_2 y - \mu_3)) = \frac{\mu_2(\mu_1 - \mu_3 - 1) + \mu_3}{\mu_1 - \mu_2}B.$$

The curve $G = 0$ has no branches, in particular, in the first quadrant if $\mu_2\left(\mu_1 - \mu_3 - 1\right) + \mu_3 \neq 0$. Therefore, according to the Dulac criterion, the first quadrant does not contain closed trajectories. In work [10], all typical phase portraits of the dynamic system and their dependences on the parameter D were analyzed, and special cases were indicated, when the exact solutions of the dynamic dynamic system can be found. In particular, the exact solution

$$y\left(x\right) = \frac{x\left(x-1\right)}{-x \pm \sqrt{x^2 + \text{const}\ \left(x-1\right)x^{2-2\mu_3}}} \tag{14}$$

was found under the additional conditions

$$\mu_2 = 2\mu_3, \quad \mu_1 = \frac{1}{2}\left(1 + 2\mu_3\right), \tag{15}$$

which is absent from work [19].

4. Qualitative Analysis of Stationary Points in the Dynamic System (11)

Let us consider the phase space structure of the dynamic system (11) in a vicinity of the stationary point $S(x_0;y_0;z_0)$ with the coordinates that satisfy the system of linear algebraic equations

$$\begin{aligned} A_0 + A_1x_0 + A_2y_0 + A_3z_0 &= 0, \\ B_0 + B_1x_0 + B_2y_0 + B_3z_0 &= 0, \\ C_0 + C_1x_0 + C_2y_0 + C_3z_0 &= 0. \end{aligned} \tag{16}$$

According to Cramer's rule, $x_i = \Delta_i\Delta^{-1}$. Using the Andronov–Hopf theorem [20], let us determine conditions, under which oscillatory modes can emerge in a vicinity of the stationary point S. According to the theorem, one of the necessary conditions for a periodic solution to exist consists in that the linearization matrix

$$J = \begin{pmatrix} A_1x_0 & A_2x_0 & A_3x_0 \\ B_1y_0 & B_2y_0 & B_3y_0 \\ C_1z_0 & C_2z_0 & C_3z_0 \end{pmatrix}$$

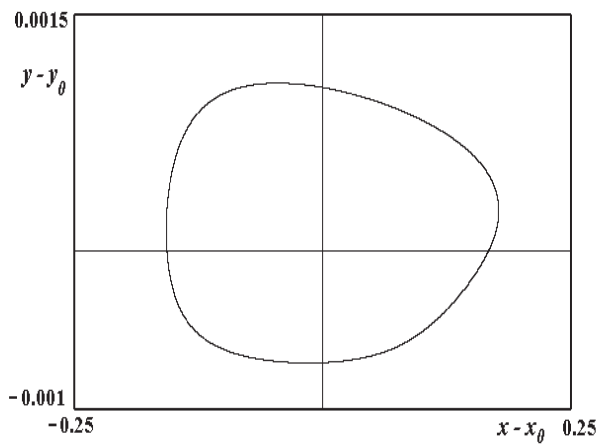

Fig. 1. Phase portrait of the limiting cycle in the phase space of the dynamic system (11) for $\lambda_3 = -0.035$

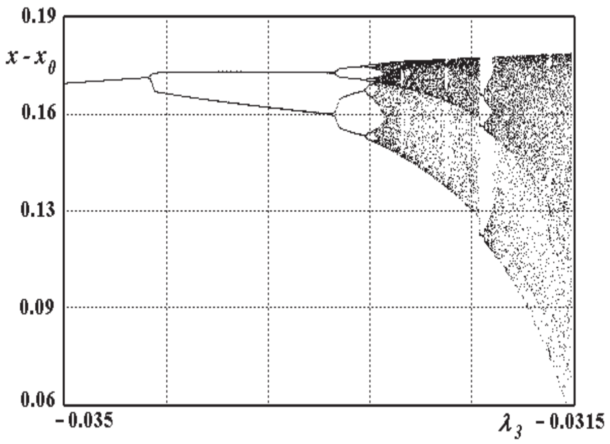

Fig. 2. Bifurcation diagram for the development of the limiting cycle of the dynamic system (11) with the growth of the parameter λ_3

should possess a pair of purely imaginary eigenvalues. Then the matrix J must satisfy the relation

$$\det J = \operatorname{tr} J \sum_{i=1}^{3} J_{ii}.$$

It is convenient to express this condition in the form

$$(A_1\Delta_1 + B_2\Delta_2 + C_3\Delta_3)\left(\frac{1}{\Delta_3}\begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} + \right.$$
$$\left. + \frac{1}{\Delta_2}\begin{vmatrix} A_1 & A_3 \\ C_1 & C_3 \end{vmatrix} + \frac{1}{\Delta_1}\begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix}\right) = 1. \quad (17)$$

Expression (17) describes the curve of neutral stability. It is a definite manifold in the parametric space. While crossing this manifold, the saddle-focus

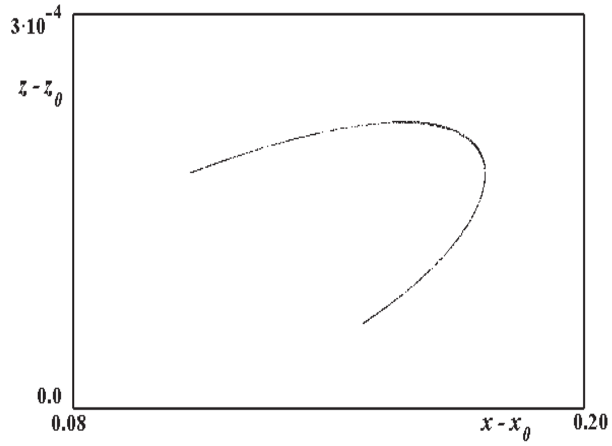

Fig. 3. Poincaré section of the chaotic attractor for $\lambda_3 = -0.032$

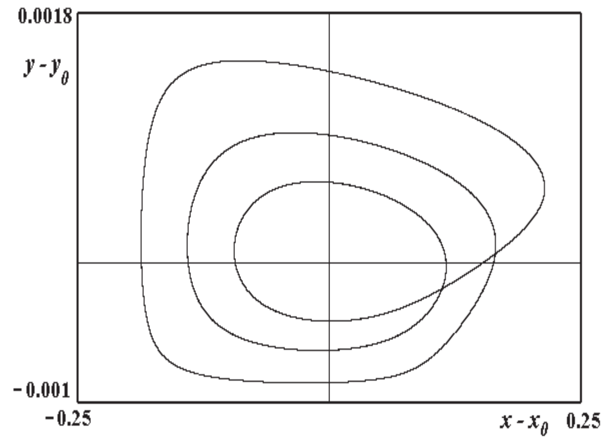

Fig. 4. Limiting cycle of the period $3T$ in the phase space of the dynamic system (11) for $\lambda_3 = -0.03213$

S with a stable one-dimensional manifold can be transformed into a limiting cycle. In order to study the limiting cycle while going away from the neutral stability curve, the methods of qualitative and numerical analyses turn out rather effective [21].

5. Numerical Analysis of System (11)

Let us so choose the values of model parameters that (i) solutions of Eq. (6) are real-valued and (ii) solutions of system (16) are positive, i.e. the stationary point S is located in the first octant. Taking those requirements into account, we fix the following parameters: $\omega_1 = 0.2$, $\omega_2 = 0.4$, $\omega_3 = 0.6$, $m_1 = 0.5$, $m_2 = 0.6$, $m_3 = 0.8$, $\lambda_1 = 1.8$, $\lambda_2 = 0.5$, $c = 1$, and $D = 1.3$.

It is convenient to choose one of the λ_i-parameters, e.g., λ_3, as a bifurcation one. Then the quantities k_i^2 and r_{ij} do not depend on λ_3, and their values are not changed in the course of numerical experiments. Hence,

$$\begin{aligned}&\{k_i^2\} = \{0.0290475; 0.121186; 0.766635\},\\&\{r_{2i}\} = \{-0.327842; 14.7132; 4.42268\},\\&\{r_{3i}\} = \{0.291313; -33.4464; -25.641\},\\&\delta = 229.779,\ i = 1, 2, 3.\end{aligned} \quad (18)$$

Substituting values (18) into condition (17), we obtain a high-order algebraic equation for λ_3. One of the roots of this equation, $\lambda_3 = \lambda_3^\star = -0.406295$, corresponds to the existence of two purely imaginary and one negative eigenvalues in the spectrum of the linearization matrix J. The analysis of eigenvalues of the matrix J also shows that the point S is a stable focus, if $\lambda_3 < \lambda_3^\star$, and an unstable one, if $\lambda_3 > \lambda_3^\star$.

Let us consider the behavior of trajectories of system (11) at $\lambda_3 > \lambda_3^\star$. Proceeding from $\lambda_3 = -0.035$, we can integrate system (11) with initial conditions selected near the stationary point S to be convinced that the trajectory of the system converges to the limiting cycle (Fig. 1).

The evolution of a limiting cycle with the growth of the parameter λ_3 is convenient to be studied with the help of the Poincaré section technique. Let the plane $y = 0$ be selected as a secant one. The coordinate x of the intersection point between the trajectory and the secant plane is reckoned along the ordinate axis, and the parameter λ_3 along the abscissa one. The analysis of the obtained bifurcation diagram (Fig. 2) shows that the limiting cycle undergoes a few doubling bifurcations followed by the emergence of a chaotic attractor with rather a typical structure (Fig. 3). One of the features in the chaotic diagram region is a window of periodicity for $\lambda_3 = -0.03213$, which corresponds to the existence of a periodic trajectory with the period $3T$ (Fig. 4). A jump of the oscillation amplitude is also observed at this point. Despite all that, the form of Poincaré sections remains similar to parabolic, as is shown in Fig. 3, with the changes manifesting themselves only in the addition of segments to the branches of this parabola.

6. Conclusions

Hence, the three-frequency wave solution for a weakly nonlinear model of the medium, Eq. (1), is described by an amplitude system which has periodic, multiperiodic, and chaotic solutions. It is evident that the increase in the number of partial oscillators in the model stimulates the growth of the phase space dimensionality for the amplitude system, which may result in the emergence of new modes.

Note that the three-frequency mode manifests itself when two partial oscillators, in the absence of a coupling, are in the limiting cycle mode ($\lambda_1 > 0$, $\lambda_2 > 0$), whereas the equilibrium position of the third oscillator is a stable focus ($\lambda_3 < 0$). The coupling between the oscillators through the background medium is capable of redistributing the energy between the oscillators, as well as creating new localized modes.

It should also be emphasized that the results obtained should be used with a certain caution, because, as follows from the analysis of expressions (7) and (10), the results were obtained under the condition that the solution frequencies, k_i^2, and the partial frequencies of the linear system, Ω_i^2, differ substantially (the absence of resonances). This means that the parameter ε must be smaller than the differences between the indicated frequencies.

Despite that the results obtained with the use of asymptotic methods have the known restrictions, we may assert that the account for processes in the medium model at the microstructure level makes it possible to describe the capability of such media to manifest their self-organization properties: the formation of multiperiodic localized waves, their bifurcations, and so forth.

The results obtained testify that the variety of, at least, wave solutions for model (1) is much wider than that of solutions for classical models, which do not consider a complicated rheology of media. Instead, the direct application of generalized models, like model (1), to the description of a physical object becomes more complicated. The separate formulations of problems – in particular, concerning the propagation of vibrations along a one-dimensional rod – were considered in work [22], where the necessity to use experimental data on the distribution of quantities m_k and ω_k was stressed.

Other statements of problems are devoted to the resonance phenomena in geomaterials [7], where the characteristic oscillator frequencies ω_k were identified with the dominating frequencies in geomedia [23]. Therefore, the development of similar models stimulates the planning of new experiments, outlines the scope of tasks for natural and numerical experiments, and allows the known data and the methods of their collection to be specified and ordered.

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АСИМПТОТИЧНІ ХВИЛЬОВІ
РОЗВ'ЯЗКИ МОДЕЛІ СЕРЕДОВИЩА
З ОСЦИЛЯТОРАМИ ВАН ДЕР ПОЛЯ

Р е з ю м е

У роботі розглядається одновимірна математична модель складного середовища, яка складається із хвильового рівняння для основного середовища та зв'язаних з ним рівнянь Ван дер Поля для коливних включень. Використовуючи метод Боголюбова–Митропольського, побудовані хвильові розв'язки слабконелінійної моделі, амплітуда яких описується тривимірною динамічною системою. Амплітудна система докладно вивчалась методами методами якісного та числового аналізу. Зокрема, було виявлено фазовому просторі системи періодичних, мультиперіодичних та хаотичних траєкторій, досліджено біфуркації цих режимів за допомогою техніки перерізів Пуанкаре, також було знайдено точні розв'язки у випадку редукції системи до двовимірної.