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COVARIANT DIFFERENTIAL IDENTITIES AND CONSERVATION LAWS IN METRIC-TORSION THEORIES OF GRAVITATION

The general manifestly generally covariant formalism for constructing the conservation laws and the conserved quantities in arbitrary metric-torsion theories of gravitation, which recently has been elaborated by the authors, is presented.

K e y w o r d s: diffeomorphic invariance, manifest covariance, differential identities, conservation laws, stress-energy-momentum tensors, spin tensors, metric-torsion theories, gravity, Riemann–Cartan geometry.

1. Introduction

Last decades, one can see the unprecedented active development of alternative theories of gravity, which modify general relativity (GR) in various ways. Among them, there are scalar-tensor theories, the Einstein–Cartan theory, the Lovelock theory in the general form, as well as its special cases such as very popular Einstein–Gauss–Bonnet theory, metric-affine theories, supergravity, $f(R)$-theories, Chern–Simons modifications of GR, Lovelock–Cartan theories, topologically massive gravity, topologically massive supergravity, new massive gravity, critical gravity, chiral gravity, various topological gauge theories of gravity and supergravity, *etc.*

Constructing the conservation laws (CLs) and the conserved quantities (CQs) in an arbitrary field theory, including gravitational theories, is a main problem. Many above-listed theories presented in the second-order formalism are the metric-torsion theories. Therefore, there is a demand for *universal* expressions for CLs and CQs. Recently, such a formalism was developed by the authors [1–3]. The formalism itself is an initially manifestly covariant modification of the Noether-like approach suggested by Bergmann [4, 5] and Mitskevich [6–8]. In the present work, we give a brief derivation of this formalism and the new results obtained in its framework. The novelty of our results is in the following: – *Universality*. We consider *arbitrary* diffeomorphically invariant

classical field theories, Lagrangians of which contain the derivatives of field variables (tensor densities of *arbitrary*, but fixed ranks and weights) up to the second order;

– *Manifest general covariance.* We develop manifestly generally covariant formalism, first, using *initially* generally covariant expressions (without using auxiliary structures, such as a background metric); second, all of our calculations are manifestly generally covariant at *each step*;

– *The torsion field is taken into account.* The space-time under consideration is presented by an *arbitrary Riemann–Cartan space*. Both the torsion tensor and the metric tensor are the dynamical fields, the torsion coupling in the Lagrangian can be both minimal (through connection) and non-minimal (explicit).

Below, we use the following notation: Greek indices α, β, ..., μ, ν, ... take values of 0, 1, ..., D and enumerate space-time coordinates $x \overset{\text{def}}{=} \{x^\alpha\}$, partial $\partial \overset{\text{def}}{=} \{\partial_\alpha\} \overset{\text{def}}{=} \{\partial/\partial x^\alpha\}$ and covariant $\nabla \overset{\text{def}}{=} \{\nabla_\alpha\}$, $\overset{*}{\nabla} \overset{\text{def}}{=} \{\overset{*}{\nabla}_\alpha\}$ derivatives, and the space-time tensor components of fields as well. The coordinate x^0 is a time one, whereas the coordinates $\mathbf{x} \overset{\text{def}}{=} \{x^1, ..., x^D\}$ are space ones. Capital Latin indices A, B, ..., are collective and enumerate components of the full set of physical fields $\mathbf{\Phi} \overset{\text{def}}{=} \{\Phi^A(x)\}$ (containing both gravitational and matter fields) and are related to 1, 2, ..., N. At last, small Latin indices from the beginning of the alphabet a, b, ..., h enumerate components of the

matter (non-gravitational) fields $\boldsymbol{\varphi} \overset{\text{def}}{=} \{\varphi^a(x)\}$ and take values of 1, 2, ..., n.

As usual, for a twice repeated index, the Einstein summation rule is assumed. The indices in parentheses need to be symmetrized; whereas, the indices in brackets needs to be antisymmetrized. Two vertical lines inside the brackets ( ) and [ ] mean that the indices between them do not participate in symmetrization/antisymmetrization. For example,

$$A_{(\alpha\beta)} = \frac{1}{2}\left(A_{\alpha\beta} + A_{\beta\alpha}\right),$$

$$A_{[\alpha|\beta|\gamma]} = \frac{1}{2}\left(A_{\alpha\beta\gamma} - A_{\gamma\beta\alpha}\right).$$

The speed of light in vacuum is set to 1.

Important definitions and relations in the Riemann–Cartan geometry are given in Ref. [1]. Now, we introduce the necessary notation only. The torsion tensor $\mathbf{T} \overset{\text{def}}{=} \{T^\lambda{}_{\mu\nu}\}$ and the curvature tensor $\mathbf{R} \overset{\text{def}}{=} \{R^\kappa{}_{\lambda\mu\nu}\}$ are presented as

$$T^\lambda{}_{\mu\nu} = -2\Gamma^\lambda{}_{[\mu\nu]};$$

$$R^\kappa{}_{\lambda\mu\nu} = 2\left(\partial_{[\mu}\Gamma^\kappa{}_{\lambda\nu]} + \Gamma^\kappa{}_{\alpha[\mu}\Gamma^\alpha{}_{\lambda\nu]}\right).$$

Here, the connection $\boldsymbol{\Gamma} \overset{\text{def}}{=} \{\Gamma^\lambda{}_{\mu\nu}\}$ is defined by a metric compatible condition

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\alpha{}_{\mu\lambda}g_{\alpha\nu} - \Gamma^\alpha{}_{\nu\lambda}g_{\mu\alpha} = 0,$$

where the standard covariant derivative $\boldsymbol{\nabla} \overset{\text{def}}{=} \{\nabla_\lambda\}$ is used, and $\mathbf{g} \overset{\text{def}}{=} \{g_{\mu\nu}\}$ is the metric tensor. The modified covariant derivative $\overset{*}{\boldsymbol{\nabla}} \overset{\text{def}}{=} \{\overset{*}{\nabla}_\lambda\}$ is

$$\overset{*}{\nabla}_\lambda \overset{\text{def}}{=} \nabla_\lambda + T_\lambda; \qquad T_\lambda \overset{\text{def}}{=} T^\alpha{}_{\lambda\alpha}.$$

2. General Consideration

2.1. General Noether identity. Generalized Noether's current and charge

We consider a classical field theory determined by the action functional

$$I[\boldsymbol{\Phi}; \Sigma_{1,2}] = \int\limits_{\Sigma_1}^{\Sigma_2} dx\, \sqrt{-g}\mathscr{L} \tag{1}$$

in the Riemann–Cartan space $\mathcal{C}(1,D)$. Here, $dx \overset{\text{def}}{=} dx^0 dx^1... \, dx^D$; the integration is provided over an

arbitrary $(D+1)$-dimensional volume in $\mathcal{C}(1,D)$ restricted by two space-like D-dimensional hypersurfaces Σ_1 and Σ_2; the Lagrangian $\mathscr{L}$ is a local function of the set of field variables $\boldsymbol{\Phi}(x) = \{\Phi^A(x); A = \overline{1,N}\}$ and their first and second derivatives. We assume that the Lagrangian $\mathscr{L}$ is a generally covariant scalar and functional variation $\delta_\Phi I$ of the action functional I,

$$\delta_\Phi I[\boldsymbol{\Phi}; \Sigma_{1,2}] \overset{\text{def}}{=} I[\boldsymbol{\Phi} + \delta\boldsymbol{\Phi}; \Sigma_{1,2}] - I[\boldsymbol{\Phi}; \Sigma_{1,2}] =$$

$$= \int\limits_{\Sigma_1}^{\Sigma_2} dx\, \delta\left(\sqrt{-g}\mathscr{L}\right), \tag{2}$$

has the following structure:

$$\delta_\Phi I = \int\limits_{\Sigma_1}^{\Sigma_2} dx\, \sqrt{-g}\frac{\Delta I}{\Delta\Phi^A}\delta\Phi^A +$$

$$+ \int\limits_{\Sigma_1}^{\Sigma_2} dx\, \sqrt{-g}\overset{*}{\nabla}_\mu \left\{K^\mu|_A\delta\Phi^A + L^{\beta\mu}|_A\nabla_\beta\delta\Phi^A\right\}. \tag{3}$$

Hereinafter, $\Delta I/\Delta\Phi^A$ is defined by the variational derivative $\delta I/\delta\Phi^A$, which is the operator of the equations of motion,

$$\frac{\Delta I}{\Delta\Phi^A} \overset{\text{def}}{=} \frac{1}{\sqrt{-g}}\frac{\delta I}{\delta\Phi^A}, \tag{4}$$

$\mathbf{K} \overset{\text{def}}{=} \{K^\mu|_A\}$ and $\mathbf{L} \overset{\text{def}}{=} \{L^{\beta\mu}|_A\}$ are local functions of the field variables $\boldsymbol{\Phi}$ and their first and second derivatives and are defined in an unique way (without ambiguities) by the Lagrangian $\mathscr{L}$.

Then, due to the invariance of the action functional I (1) with respect to general/arbitrary infinitesimal diffeomorphisms

$$\begin{cases} \delta x^\mu = \delta\xi^\mu(x); & (5) \\ \delta_\xi\Phi^A(x) = \Phi_\alpha|^A\delta\xi^\alpha + \Phi_\alpha{}^\beta|^A\nabla_\beta\delta\xi^\alpha, & (6) \end{cases}$$

the *general Noether identity* (the main identity) follows:

$$\overset{*}{\nabla}_\mu J^\mu[\delta\boldsymbol{\xi}] + \frac{\Delta I}{\Delta\Phi^A}\delta_\xi\Phi^A \equiv 0. \tag{7}$$

In formula (6), the quantities $\{\Phi_\alpha|^A\}$, $\{\Phi_\alpha{}^\beta|^A\}$ are local functions of $\boldsymbol{\Phi}$ and their derivatives, which are

defined uniquely by the transformation properties of $\boldsymbol{\Phi}$, and

$$J^\mu[\delta\boldsymbol{\xi}] \stackrel{\text{def}}{=} \left\{K^\mu|_A\Phi_\alpha|^A + \mathscr{L}\delta_\alpha^\mu + L^{\nu\mu}|_A\nabla_\nu\Phi_\alpha|^A\right\}\delta\xi^\alpha +$$
$$+ \left\{K^\mu|_A\Phi_\alpha{}^\beta|^A + L^{\beta\mu}|_A\Phi_\alpha|^A + L^{\nu\mu}|_A\nabla_\nu\Phi_\alpha{}^\beta|^A\right\}\times$$
$$\times\nabla_\beta\delta\xi^\alpha + \left\{L^{\gamma\mu}|_A\Phi_\alpha{}^\beta|^A\right\}\nabla_\gamma\nabla_\beta\delta\xi^\alpha. \tag{8}$$

Unlike the standard Noether currents obtained by the recipe of the 1-st Noether theorem in the theories with global symmetries, the current $\mathbf{J}[\delta\boldsymbol{\xi}] \stackrel{\text{def}}{=} \{J^\mu[\delta\boldsymbol{\xi}]\}$ (8) depends on an arbitrary *displacement vector* field $\delta\boldsymbol{\xi} \stackrel{\text{def}}{=} \{\delta\xi^\mu\}$ and its covariant derivatives. For this reason, we will call $\mathbf{J}[\delta\boldsymbol{\xi}]$ as the *generalized Noether current*, and a correspondent conserved charge

$$Q[\delta\boldsymbol{\xi};\Sigma] \stackrel{\text{def}}{=} \int\limits_\Sigma d\sigma_\mu\, J^\mu[\delta\boldsymbol{\xi}] \tag{9}$$

as the *generalized Noether charge*.

If the equations of motion $\Delta I/\Delta\Phi^A = 0$ hold, then identity (7) transforms into the continuity equation

$$\overset{*}{\nabla}_\mu J^\mu[\delta\boldsymbol{\xi}] = 0 \quad \text{(on the } \boldsymbol{\Phi}\text{-equations).} \tag{10}$$

Additionally, if the field variables and their derivatives vanish fast enough at a spatial infinity, then the last equation leads to the conservation of the generalized charge

$$\delta Q[\delta\boldsymbol{\xi};\Sigma]/\delta\Sigma(x) = 0 \quad \text{(on the } \boldsymbol{\Phi}\text{-equations),} \tag{11}$$

meaning that its value is the same on each of the hypersurfaces Σ.

2.2. The Klein and Noether identities

After some transformations, the generalized Noether current $\mathbf{J}[\delta\boldsymbol{\xi}]$ (8) can be led to the next *standard* form:

$$J^\mu[\delta\boldsymbol{\xi}] = U_\alpha{}^\mu\delta\xi^\alpha + M_\alpha{}^{\beta\mu}\nabla_\beta\delta\xi^\alpha + N_\alpha{}^{\beta\gamma\mu}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha, \tag{12}$$

where

$$\begin{cases} U_\alpha{}^\mu \stackrel{\text{def}}{=} \mathscr{L}\delta_\alpha^\mu + K^\mu|_A\Phi_\alpha|^A \\ \quad + L^{\kappa\mu}|_A\left(\nabla_\kappa\Phi_\alpha|^A + \dfrac{1}{2}R^\varepsilon{}_{\alpha\kappa\lambda}\Phi_\varepsilon{}^\lambda|^A\right); \tag{13} \\[2mm] M_\alpha{}^{\beta\mu} \stackrel{\text{def}}{=} K^\mu|_A\Phi_\alpha{}^\beta|^A + L^{\beta\mu}|_A\Phi_\alpha|^A \\ \quad + L^{\kappa\mu}|_A\left(\nabla_\kappa\Phi_\alpha{}^\beta|^A - \dfrac{1}{2}T^\beta{}_{\kappa\lambda}\Phi_\alpha{}^\lambda|^A\right); \tag{14} \\[2mm] N_\alpha{}^{\beta\gamma\mu} \stackrel{\text{def}}{=} L^{(\gamma|\mu}|_A\Phi_\alpha{}^{|\beta)}|^A. \tag{15} \end{cases}$$

Substituting (12) and (6) into (7) and taking into account that $\{\delta\xi^\alpha,\ \nabla_\beta\delta\xi^\alpha,\ \nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha,\ \nabla_{(\delta}\nabla_\gamma\nabla_{\beta)}\delta\xi^\alpha\}$ presents a set of independent and arbitrary quantities at every point, one obtains the system of identities

$$\begin{cases} \overset{*}{\nabla}_\mu U_\alpha{}^\mu - \dfrac{1}{2}M_\lambda{}^{\mu\nu}R^\lambda{}_{\alpha\mu\nu} - \\ \quad - \dfrac{1}{3}N_\kappa{}^{\lambda\mu\nu}\left(\nabla_\lambda R^\kappa{}_{\alpha\mu\nu} + \dfrac{1}{2}T^\sigma{}_{\mu\nu}R^\kappa{}_{\alpha\lambda\sigma}\right) \equiv -I_\alpha; \tag{16} \\[2mm] U_\alpha{}^\beta + \left(\overset{*}{\nabla}_\mu M_\alpha{}^{\beta\mu} + \dfrac{1}{2}M_\alpha{}^{\mu\nu}T^\beta{}_{\mu\nu}\right) + \\ \quad + \dfrac{1}{3}N_\alpha{}^{\lambda\mu\nu}\left(2R^\beta{}_{\lambda\mu\nu} + \nabla_\lambda T^\beta{}_{\mu\nu} + \right. \\ \quad \left. + \dfrac{1}{2}T^\sigma{}_{\mu\nu}T^\beta{}_{\lambda\sigma}\right) - N_\kappa{}^{\beta\mu\nu}R^\kappa{}_{\alpha\mu\nu} \equiv -I_\alpha{}^\beta; \tag{17} \\[2mm] M_\alpha{}^{(\beta\gamma)} + \overset{*}{\nabla}_\mu N_\alpha{}^{\beta\gamma\mu} + N_\alpha{}^{(\beta|\mu\nu}T^{|\gamma)}{}_{\mu\nu} \equiv 0; \tag{18} \\[2mm] N_\alpha{}^{(\beta\gamma\delta)} \equiv 0, \tag{19} \end{cases}$$

where

$$I_\alpha \stackrel{\text{def}}{=} \frac{\Delta I}{\Delta\Phi^A}\Phi_\alpha|^A; \quad I_\alpha{}^\beta \stackrel{\text{def}}{=} \frac{\Delta I}{\Delta\Phi^A}\Phi_\alpha{}^\beta|^A. \tag{20}$$

Originally, the system analogous to the above one has been obtained in a non-covariant form by Klein [9] for purely metric theories of gravity. Therefore, we will name system (16)–(19) as the *Klein identities*.

Statement 1. *The Klein identities* (16)–(19) *present a complete manifestly covariant universal system of differential identities, which is valid in an arbitrary diffeomorphically invariant field theory.*

Subtracting the divergence $\overset{*}{\nabla}_\beta$ of (17) from identity (16), we obtain the new identity

$$\overset{*}{\nabla}_\mu I_\alpha{}^\mu - I_\alpha \equiv 0 \tag{21}$$

that is the *Noether identity* rewritten in a manifestly covariant form.

Statement 2. *The (usual) Noether identity* (21) *is a consequence of the Klein identities* (16)–(19).

Analyzing system (16)–(19), one shows that the following is valid:

Statement 3. *Instead of the Klein system* (16)–(19), *one can use the equivalent Klein–Noether sys-*

tem of identities:

$$
\begin{cases}
\overset{*}{\nabla}_{\mu} I_{\alpha}{}^{\mu} \equiv I_{\alpha}; & (22) \\
\left(U_{\alpha}{}^{\beta} - \frac{1}{3} N_{\lambda}{}^{\beta\rho\sigma} R^{\lambda}{}_{\alpha\rho\sigma} \right) + \\
+ \overset{*}{\nabla}_{\mu} \left(M_{\alpha}{}^{[\beta\mu]} - \frac{2}{3} \overset{*}{\nabla}_{\lambda} N_{\alpha}{}^{\lambda[\beta\mu]} + \frac{1}{3} N_{\alpha}{}^{[\beta|\rho\sigma} T^{|\mu]}{}_{\rho\sigma} \right) + \\
+ \frac{1}{2} \left(M_{\alpha}{}^{[\rho\sigma]} - \frac{2}{3} \overset{*}{\nabla}_{\lambda} N_{\alpha}{}^{\lambda[\rho\sigma]} + \frac{1}{3} N_{\alpha}{}^{[\rho|\kappa\lambda} T^{|\sigma]}{}_{\kappa\lambda} \right) \times \\
\times\ T^{\beta}{}_{\rho\sigma} \equiv -I_{\alpha}{}^{\beta}; & (23) \\
M_{\alpha}{}^{(\beta\gamma)} + \overset{*}{\nabla}_{\mu} N_{\alpha}{}^{\beta\gamma\mu} + N_{\alpha}{}^{(\beta|\mu\nu} T^{|\gamma)}{}_{\mu\nu} \equiv 0; & (24) \\
N_{\alpha}{}^{(\beta\gamma\delta)} \equiv 0. & (25)
\end{cases}
$$

2.3. The generalized Noether superpotential. The boundary Klein–Noether theorem

Substituting $\{\delta_{\xi} \Phi^{A}\}$ (6) and $\{I_{\alpha}\}$ from (22) into the general Noether identity (7), one obtains another identity

$$
\overset{*}{\nabla}_{\mu} \mathscr{J}^{\mu}[\delta\boldsymbol{\xi}] \equiv 0, \tag{26}
$$

which has a meaning of the continuity equation for the current defined as $\mathscr{J}[\delta\boldsymbol{\xi}] \overset{\text{def}}{=} \{\mathscr{J}^{\mu}[\delta\boldsymbol{\xi}]\}$, where

$$
\mathscr{J}[\delta\boldsymbol{\xi}] \overset{\text{def}}{=} (U_{\alpha}{}^{\mu} + I_{\alpha}{}^{\mu}) \delta\xi^{\alpha} +
$$
$$
+ M_{\alpha}{}^{\beta\mu} \nabla_{\beta}\, \delta\xi^{\alpha} + N_{\alpha}{}^{\beta\gamma\mu} \nabla_{(\gamma} \nabla_{\beta)} \delta\xi^{\alpha}. \tag{27}
$$

Note that identity (26) takes place *independently of whether the equations of motion hold*.

Statement 4 (Covariant Poincaré lemma). *In an arbitrary Riemann–Cartan space, an identically conserved current $\mathscr{J}[\delta\boldsymbol{\xi}]$ can be represented locally in the form*

$$
\mathscr{J}^{\mu}[\delta\boldsymbol{\xi}] = \overset{*}{\nabla}_{\nu} \theta^{\mu\nu}[\delta\boldsymbol{\xi}] + \frac{1}{2} \theta^{\rho\sigma}[\delta\boldsymbol{\xi}] T^{\mu}{}_{\rho\sigma}, \tag{28}
$$

where

$$
\boldsymbol{\theta}[\delta\boldsymbol{\xi}] \overset{\text{def}}{=} \{\theta^{\mu\nu}[\delta\boldsymbol{\xi}]\}; \quad \theta^{[\mu\nu]}[\delta\boldsymbol{\xi}] = \theta^{\mu\nu}[\delta\boldsymbol{\xi}] \tag{29}
$$

is an antisymmetric tensor — the generalized Noether superpotential.

Statement 5. *For current (27), the generalized Noether superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ (29) has the form*

$$
\theta^{\mu\nu}[\delta\boldsymbol{\xi}] =
$$

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$$
= \left\{ -M_{\alpha}{}^{[\mu\nu]} + \frac{2}{3} \left(\overset{*}{\nabla}_{\lambda} N_{\alpha}{}^{\lambda[\mu\nu]} + \frac{1}{2} T^{[\mu}{}_{\rho\sigma} N_{\alpha}{}^{\nu]\rho\sigma} \right) \right\} \delta\xi^{\alpha} +
$$
$$
+ \left\{ -\frac{4}{3} N_{\alpha}{}^{\beta[\mu\nu]} \right\} \nabla_{\beta} \delta\xi^{\alpha}. \tag{30}
$$

It is evident that the current $\mathscr{J}[\delta\boldsymbol{\xi}]$ is connected with the generalized Noether current $\mathbf{J}[\delta\boldsymbol{\xi}]$ (12) by the relation:

$$
J^{\mu}[\delta\boldsymbol{\xi}] = -I_{\alpha}{}^{\mu} \delta\xi^{\alpha} + \mathscr{J}^{\mu}[\delta\boldsymbol{\xi}]. \tag{31}
$$

Substituting expression (28) into formula (31), integrating it over Σ, and using the Stokes rule, we rewrite the generalized Noether charge (9) in the form

$$
Q[\delta\boldsymbol{\xi}; \Sigma] = -\int_{\Sigma} d\sigma_{\mu}\, I_{\alpha}{}^{\mu} \delta\xi^{\alpha} + \frac{1}{2!} \oint_{\partial\Sigma} ds_{\mu\nu}\, \theta^{\mu\nu}[\delta\boldsymbol{\xi}]. \tag{32}
$$

The above relation is a special case (in an integral form) of a more general

Statement 6 (The boundary Klein–Noether theorem). *In an arbitrary gauge-invariant theory, the Noether current is presented by a sum of two terms: the first vanishes on the equations of motion, the second is the divergence of a superpotential.*

2.4. The generalized symmetrized Noether current

From now, we call $\mathbf{J}[\delta\boldsymbol{\xi}]$ (12) and $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ (30) as the generalized *canonical* Noether current and the generalized *canonical* superpotential, respectively. Recall that $\mathbf{J}[\delta\boldsymbol{\xi}]$ contains the derivatives of a displacement vector $\nabla\delta\boldsymbol{\xi}$, $\nabla\nabla\delta\boldsymbol{\xi}$. We construct a new current $\overset{\text{sym}}{J}{}^{\mu}[\delta\boldsymbol{\xi}]$, instead of the canonical one $\mathbf{J}[\delta\boldsymbol{\xi}]$, with the property that it does not contain derivatives of $\delta\boldsymbol{\xi}$. In other words, we search for

$$
\overset{\text{sym}}{J}{}^{\mu}[\delta\boldsymbol{\xi}] = \overset{\text{sym}}{U}{}_{\alpha}{}^{\mu} \delta\xi^{\alpha}. \tag{33}
$$

Because a new current has to be also differentially conserved, we construct it by adding an antisymmetric tensor $\mathscr{B}[\delta\boldsymbol{\xi}] \overset{\text{def}}{=} \left\{ \mathscr{B}^{[\mu\nu]}[\delta\boldsymbol{\xi}] = \mathscr{B}^{\mu\nu}[\delta\boldsymbol{\xi}] \right\}$, similarly to (28):

$$
\overset{\text{sym}}{J}{}^{\mu}[\delta\boldsymbol{\xi}] \overset{\text{def}}{=} J^{\mu}[\delta\boldsymbol{\xi}] -
$$
$$
- \left(\overset{*}{\nabla}_{\nu} \mathscr{B}^{\mu\nu}[\delta\boldsymbol{\xi}] + \frac{1}{2} \mathscr{B}^{\rho\sigma}[\delta\boldsymbol{\xi}] T^{\mu}{}_{\rho\sigma} \right). \tag{34}
$$

We call this formula as a *generalized Belinfante relation*, and a tensor $\mathscr{B}[\delta\boldsymbol{\xi}]$ — as a *generalized Belinfante tensor*. Calculations show that

Statement 7. *The generalized Belinfante tensor* $\mathscr{B}[\delta\boldsymbol{\xi}]$ *coincides with the generalized canonical superpotential* $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ (30)

$$\mathscr{B}^{\mu\nu}[\delta\boldsymbol{\xi}] = \theta^{\mu\nu}[\delta\boldsymbol{\xi}]; \qquad (35)$$

the tensor $\overset{\text{sym}}{\mathbf{U}} \overset{\text{def}}{=} \{\overset{\text{sym}}{U}{}_{\alpha}{}^{\mu}\}$ *has the form*

$$\overset{\text{sym}}{U}{}_{\alpha}{}^{\mu} = \left(U_{\alpha}{}^{\mu} - \frac{1}{3}N_{\lambda}{}^{\mu\rho\sigma}R^{\lambda}{}_{\alpha\rho\sigma}\right) +$$
$$+ \overset{*}{\nabla}_{\nu}\left[M_{\alpha}{}^{[\mu\nu]} - \frac{2}{3}\left(\overset{*}{\nabla}_{\lambda}N_{\alpha}{}^{\lambda[\mu\nu]} + \frac{1}{2}T^{[\mu}{}_{\rho\sigma}N_{\alpha}{}^{\nu]\rho\sigma}\right)\right] +$$
$$+ \frac{1}{2}\left[M_{\alpha}{}^{[\rho\sigma]} - \frac{2}{3}\left(\overset{*}{\nabla}_{\lambda}N_{\alpha}{}^{\lambda[\rho\sigma]} + \frac{1}{2}T^{[\rho}{}_{\varepsilon\kappa}N_{\alpha}{}^{\sigma]\varepsilon\kappa}\right)\right]T^{\mu}{}_{\rho\sigma}. \tag{36}$$

Note that the right-hand side of (36) exactly coincides with the left-hand side of the Klein identity (23). Therefore, one can write also

$$\overset{\text{sym}}{U}{}_{\alpha}{}^{\mu} = -I_{\alpha}{}^{\mu}. \tag{37}$$

It turns out that, in the manifestly generally covariant theories (see Sec. 3), the tensor $\overset{\text{sym}}{\mathbf{U}}$ coincides with the symmetrized energy-momentum tensor $\overset{\text{sym}}{\mathbf{t}} \overset{\text{def}}{=} \{\overset{\text{sym}}{t}{}^{\mu}{}_{\alpha}\}$:

$$\overset{\text{sym}}{U}{}_{\alpha}{}^{\mu} = \overset{\text{sym}}{t}{}^{\mu}{}_{\alpha}. \tag{38}$$

Then relations (37) and (38) are a proof of the

Statement 8. *The symmetrized energy-momentum tensor* $\overset{\text{sym}}{\mathbf{t}}$ *does not depend on the divergences in the Lagrangian.*

3. Manifestly Generally Covariant Theories

3.1. Structure of Lagrangians

In Sec. 2 the quantities and relations in the theories of the most general type have been constructed. In the present section, we will specify them. We apply the developed formalism to the study of manifestly generally covariant theories, and all the statements, here, are related to them. We call a theory as *manifestly generally covariant* if its Lagrangian $\mathscr{L}$ is a generally covariant scalar constructed as an

algebraic scalar function of the manifestly covariant objects, which are transformed following the linear *homogeneous* representations of the diffeomorphism group. This means that $\mathscr{L}$ is an algebraic function of the scalar contractions of tensor (and/or spinor) field functions and their covariant derivatives; in addition to the manifested dependence on the field variables, $\mathscr{L}$ can also depend on the curvature and torsion tensors independently. It seems that almost all the physically interesting theories are manifestly generally covariant or can be presented in such a form.

Consider the field theories presented by action (1), with Lagrangians in a manifestly covariant form:

$$\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R}; \; \mathbf{T}, \boldsymbol{\nabla}\mathbf{T}, \boldsymbol{\nabla}\boldsymbol{\nabla}\mathbf{T}; \; \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varphi}). \tag{39}$$

Here, the total set of fields $\boldsymbol{\Phi}$ is presented by the metric tensor $\mathbf{g}$, by the torsion tensor $\mathbf{T}$, and by a set of matter fields $\boldsymbol{\varphi} \overset{\text{def}}{=} \{\varphi^a(x); \; a = \overline{1,n}\}$, which are considered as tensorial ones as well. Lagrangians of the type (39) include, together with the minimal coupling, the non-minimal coupling related both to the curvature and to the torsion. In this section, we present relations and conserved quantities (currents and superpotentials) constructed in the Sec. 2 in a maximally specific form that follows from the specific structure of Lagrangian (39).

Because the fields $\mathbf{T}$ and $\boldsymbol{\varphi}$ are included in the Lagrangian in a similar way, we unite them for simplification of notations into the unique set $\boldsymbol{\phi}$:

$$\mathbf{T}, \boldsymbol{\varphi} \to \boldsymbol{\phi} \overset{\text{def}}{=} \{\phi^a\} \overset{\text{def}}{=} \{\mathbf{T}, \boldsymbol{\varphi}\}. \tag{40}$$

Now, Lagrangian (39) is presented as

$$\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R}; \; \boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\phi}). \tag{41}$$

One has to keep in mind that the torsion $\mathbf{T}$ is included in Lagrangian (39) not only *explicitly* as arguments $\mathbf{T}$, $\boldsymbol{\nabla}\mathbf{T}$, and $\boldsymbol{\nabla}\boldsymbol{\nabla}\mathbf{T}$, but *not explicitly* also over the connection $\boldsymbol{\Gamma}$, which is used for constructing the covariant derivative $\boldsymbol{\nabla}$ and the curvature tensor $\mathbf{R}$.

3.2. Structure of the tensors U, M, N. Canonical EMT t and ST s

Direct calculations show that, in the case of a Lagrangian of the type (41), the tensors $\mathbf{U}$ (13), $\mathbf{M}$ (14),

and **N** (15) have the form

$$
\begin{cases}
U_\alpha{}^\mu = t^\mu{}_\alpha + \left(\Delta^{\overline{\mu\beta\gamma}}_{\pi\rho\sigma} s^{\pi,\,\rho\sigma}\right) T_{\gamma,\,\beta\alpha} + \\[4pt]
\quad + \dfrac{1}{2} G^{\beta\gamma\varepsilon\mu} R_{\alpha\beta\gamma\varepsilon} + \left[\overset{*}{\nabla}_\nu \left(G^{\beta\gamma\mu\nu} T_{\beta,\,\gamma\alpha}\right) + \right. \\[4pt]
\quad \left. + \dfrac{1}{2}\left(G^{\beta\gamma\rho\sigma} T_{\beta,\,\gamma\alpha}\right) T^\mu{}_{\rho\sigma}\right]; \hfill (42) \\[8pt]
M_\kappa{}^{\lambda\mu} = -\left(\Delta^{\overline{\mu\lambda\alpha}}_{\pi\rho\sigma} s^{\pi,\,\rho\sigma}\right) g_{\alpha\kappa} - \\[4pt]
\quad - \left(\overset{*}{\nabla}_\nu G_\kappa{}^{\lambda\mu\nu} + \dfrac{1}{2} G_\kappa{}^{\lambda\rho\sigma} T^\mu{}_{\rho\sigma}\right) + \\[4pt]
\quad + \dfrac{1}{2} G_\kappa{}^{\alpha\beta\mu} T^\lambda{}_{\alpha\beta} - \left(G_\alpha{}^{\beta\lambda\mu} T^\alpha{}_{\beta\kappa}\right); \hfill (43) \\[8pt]
N_\alpha{}^{\beta\gamma\mu} = G_\alpha{}^{(\beta\gamma)\mu}. \hfill (44)
\end{cases}
$$

Here,

$$
G_\kappa{}^{\lambda\mu\nu} \overset{\text{def}}{=} 2\frac{\partial\mathscr{L}}{\partial R^\kappa{}_{\lambda\mu\nu}}; \tag{45}
$$

$$
\Delta^{\overline{\alpha\beta\gamma}}_{\lambda\mu\nu} \overset{\text{def}}{=} \frac{1}{2}\left(\delta^\beta_\lambda \delta^\alpha_\mu \delta^\gamma_\nu + \delta^\gamma_\lambda \delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\lambda \delta^\beta_\mu \delta^\gamma_\nu\right); \tag{46}
$$

$$
\begin{aligned}
t^\mu{}_\alpha \overset{\text{def}}{=}{} & \mathscr{L}\delta^\mu_\alpha - \frac{\Delta I}{\Delta(\nabla_\mu\phi^a)}\nabla_\alpha\phi^a - \\
& - \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu\phi^a)}\nabla_\alpha\nabla_\nu\phi^a - G^{\beta\gamma\varepsilon\mu} R_{\beta\gamma\varepsilon\alpha};
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
s^\pi{}_{\rho\sigma} \overset{\text{def}}{=}{} & -2\left(\overset{*}{\nabla}_\eta G_{\rho\sigma}{}^{\pi\eta} + \frac{1}{2} G_{\rho\sigma}{}^{\alpha\beta} T^\pi{}_{\alpha\beta}\right) + \\
& + 2\left(\frac{\Delta I}{\Delta(\nabla_\pi\phi^a)}(\Delta_{[\rho\sigma]})^a\big|_b\,\phi^b + \right. \\
& \left. + \frac{\partial\mathscr{L}}{\partial(\nabla_\pi\nabla_\alpha\phi^a)}\left[(\Delta_{[\rho\sigma]})^a\big|_b\,\nabla_\alpha\phi^b - g_{\alpha[\rho}\nabla_{\sigma]}\phi^a\right]\right);
\end{aligned} \tag{48}
$$

$$
\frac{\Delta I}{\Delta(\nabla_\mu\phi^a)} \overset{\text{def}}{=} \frac{\partial\mathscr{L}}{\partial(\nabla_\mu\phi^a)} - \overset{*}{\nabla}_\nu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\nu\nabla_\mu\phi^a)}\right) \tag{49}
$$

and $\{(\Delta_{\rho\sigma})^a\big|_b\}$ are *the Belinfante–Rosenfeld symbols* – some combinations of the Kronecker deltas (explicit expressions are given in Appendix C.1 of Ref. [2]).

Note also that the tensors $\mathbf{t} \overset{\text{def}}{=} \{t^\mu{}_\nu\}$ (47) and $\mathbf{s} \overset{\text{def}}{=} \{s^\pi{}_{\rho\sigma}\}$ (48) are just the *generalized canonical energy-momentum tensor* (EMT) and *generalized canonical spin tensor* (ST), corresponding to Lagrangian (39). The reason for this statement is that, basing on the above definitions of the EMT and ST, one obtains the standard equations of balance for the EMT. In addition, as we will see in the Sec. 4, the

gravitational field equations acquire the form naturally generalizing the gravitational field equations of the Einstein–Cartan theory.

It is worth to note that the sequence of the second derivative in the multiplier $\{\nabla_\alpha\nabla_\nu\phi^a\}$ in (47) is *reverse* to the sequence that follows from the construction of the canonical EMT by the direct application of the 1-st Noether theorem. The last term in (47), as well as the items in the first parentheses on the right-hand side of (48), has appeared due to the non-minimal coupling with the metric field. These items cannot be obtained *in principle* with the use of the 1-st Noether theorem in the Minkowski space and the covariantization of expressions.

Relations (44) and (45) yield directly

Statement 9. *The tensor* **N** *(44) is not equal to zero only if the Lagrangian contains explicitly the curvature tensor* **R***.*

Next, the substitution of expressions (42)–(44) into formula (12) leads to

Statement 10. *The canonical current* $\mathbf{J}[\delta\boldsymbol{\xi}]$ *(12) is essentially constructed, basing on the canonical dynamic quantities* $\mathbf{t}$ *(47),* $\mathbf{s}$ *(48), and the tensor* $\mathbf{G}$ *(45).*

We have also:

Statement 11. *The symmetrized Noether current* $\overset{\text{sym}}{\mathbf{J}}[\delta\boldsymbol{\xi}]$ *is expressed through only the symmetrized EMT* $\overset{\text{sym}}{\mathbf{t}}$ *even in the case of a Lagrangian of the most general type (39):*

$$
\overset{\text{sym}}{J}{}^\mu[\delta\boldsymbol{\xi}] \overset{\text{def}}{=} \overset{\text{sym}}{t}{}^\mu{}_\alpha \delta\xi^\alpha. \tag{50}
$$

3.3. Structure of the variational derivatives. Modified canonical ST $\overset{\text{mod}}{\mathbf{s}}$

In work [2], we have shown the general

Statement 12 (Structure of the $\Delta I/\Delta\mathbf{T}$). *The variational derivative $\Delta I/\Delta\mathbf{T}$ of the action functional I with respect to the torsion tensor $\mathbf{T}$ is equal to*

$$
\frac{\Delta I}{\Delta T^\varepsilon{}_{\beta\gamma}} = \frac{\Delta^* I}{\Delta T^\varepsilon{}_{\beta\gamma}} + \frac{1}{2} b^{\gamma\beta}{}_\varepsilon, \tag{51}
$$

where

$$
\begin{aligned}
\frac{\Delta^* I}{\Delta T^\varepsilon{}_{\beta\gamma}} \overset{\text{def}}{=}{} & \frac{\partial^*\mathscr{L}}{\partial T^\varepsilon{}_{\beta\gamma}} - \overset{*}{\nabla}_\mu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\mu T^\varepsilon{}_{\beta\gamma})}\right) + \\
& + \overset{*}{\nabla}_\nu\overset{*}{\nabla}_\mu\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\nabla_\nu T^\varepsilon{}_{\beta\gamma})}\right);
\end{aligned} \tag{52}
$$

$$b^{\gamma\beta\alpha} \overset{\text{def}}{=} \Delta^{\overline{\gamma\beta\alpha}}_{\pi\rho\sigma} s^{\pi,\rho\sigma}; \quad b^{[\gamma\beta]\alpha} = b^{\gamma\beta\alpha}.$$

The Belinfante tensor: $\mathbf{b} \overset{\text{def}}{=} \{b^{\gamma\beta\alpha}\}$ *is induced by the ST* $\mathbf{s}$.

Consequence 1. *In the case of only minimal* $\mathbf{T}$*-coupling (when the Lagrangian* $\mathscr{L}$ *does not contain the torsion tensor* $\mathbf{T}$ *explicitly), one has*

$$\frac{\Delta I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} = \frac{1}{2} b^{\gamma\beta}{}_{\varepsilon}. \tag{53}$$

Earlier, the same result (53) has been proved only for Lagrangians of the type $\mathscr{L} = \mathscr{L}(\mathbf{g}; \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi})$ with a more simple presentations both of the ST and the Belinfante tensor (see Refs. [10–16]). We have proved a more general claim: *formula (53) is left valid for Lagrangians of the more general type* $\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R}; \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varphi})$.

Formula (51) shows that the presence of a non-minimal coupling with torsion changes (53). The requirement (the desire) to conserve a sense of the variational derivative (53) even in the presence of a non-minimal $\mathbf{T}$-coupling leads to the necessity to modify both the initial Belinfante tensor and the initial ST. Let us demonstrate the modification step-by-step. We rewrite formula (51) in the form of (53):

$$\frac{\Delta I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} = \frac{1}{2} \overset{\text{mod}}{b}{}^{\gamma\beta}{}_{\varepsilon}. \tag{54}$$

Here, the modified Belinfante tensor $\overset{\text{mod}}{\mathbf{b}} = \{\overset{\text{mod}}{b}{}^{\gamma\beta\alpha}\}$ is defined analogously to the initial one (i.e., with the use of *any* ST):

$$\overset{\text{mod}}{b}{}^{\gamma\beta\alpha} \overset{\text{def}}{=} \Delta^{\overline{\gamma\beta\alpha}}_{\pi\rho\sigma} \overset{\text{mod}}{s}{}^{\pi,\rho\sigma}. \tag{55}$$

The *modified* Belinfante tensor and the canonical ST can be represented as initial ones with corresponding additions:

$$\overset{\text{mod}}{b}{}^{\gamma\beta\alpha} \overset{\text{def}}{=} b^{\gamma\beta\alpha} + \overset{\text{add}}{b}{}^{\gamma\beta\alpha}; \tag{56}$$

$$\overset{\text{mod}}{s}{}^{\pi}{}_{\rho\sigma} \overset{\text{def}}{=} s^{\pi}{}_{\rho\sigma} + \overset{\text{add}}{s}{}^{\pi}{}_{\rho\sigma}. \tag{57}$$

Finally, combining (51)–(55), one obtains the definitions for the additional Belinfante tensor and ST:

$$\overset{\text{add}}{b}{}^{\gamma\beta}{}_{\varepsilon} = 2 \frac{\Delta^* I}{\Delta T^{\varepsilon}{}_{\beta\gamma}}; \tag{58}$$

$$\overset{\text{add}}{s}{}^{\pi,\rho\sigma} = -4 g^{[\sigma|\varepsilon} \frac{\Delta^* I}{\Delta T^{\varepsilon}{}_{|\rho]\pi}}. \tag{59}$$

The *metric EMT* $\overset{\text{met}}{\mathbf{t}} \overset{\text{def}}{=} \{\overset{\text{met}}{t}{}^{\beta\gamma}\}$ is defined by a standard way:

$$\frac{1}{2} \overset{\text{met}}{t}{}^{\beta\gamma} \overset{\text{def}}{=} \frac{\Delta I}{\Delta g_{\beta\gamma}}. \tag{60}$$

3.4. Generalization of the Belinfante symmetrization procedure. Symmetrized EMT $\overset{\text{sym}}{\mathbf{t}}$

In the Riemann–Cartan space, the Belinfante symmetrization procedure is generalized in a non-trivial way.

Statement 13. *In an arbitrary Riemann–Cartan space, the Belinfante symmetrized EMT* $\overset{\text{sym}}{\mathbf{t}} \overset{\text{def}}{=} \{\overset{\text{sym}}{t}{}^{\mu}{}_{\nu}\}$ *must be constructed by the rule*

$$\overset{\text{sym}}{t}{}^{\mu}{}_{\nu} \overset{\text{def}}{=} t^{\mu}{}_{\nu} +$$
$$+ \left[\overset{*}{\nabla}_{\lambda} b^{\mu\lambda}{}_{\nu} + \frac{1}{2} b^{\rho\sigma}{}_{\nu} T^{\mu}{}_{\rho\sigma} + b^{\mu\beta}{}_{\alpha} T^{\alpha}{}_{\beta\nu} \right]. \tag{61}$$

3.5. The physical sense of the Klein and Noether identities. Equations of balance. Modified canonical EMT $\overset{\text{mod}}{\mathbf{t}}$

Let us examine a physical sense of the identities by Noether and Klein. Consequent substitutions of the expressions for the tensors $\mathbf{U}$ (42), $\mathbf{M}$ (43), and $\mathbf{N}$ (44) into identities (22), (24), (25), (23), and (16) allow us to prove the next statements:

Statement 14 (The physical sense of the Noether identity). *The Noether identity (22) has the explicit form*

$$\overset{*}{\nabla}_{\mu} \overset{\text{met}}{t}{}^{\mu}{}_{\nu} \equiv - \overset{\text{met}}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} +$$
$$+ \left\{ \overset{*}{\nabla}_{\mu} \left[\frac{\Delta I}{\Delta\phi^a} (\Delta^{\mu}{}_{\nu}) \,^a|_b \phi^b \right] + \right.$$
$$+ \frac{\Delta I}{\Delta\phi^a} \left[\nabla_{\nu}\phi^a + (\Delta^{\mu}{}_{\lambda}) \,^a|_b \phi^b \, T^{\lambda}{}_{\mu\nu} \right] \right\} \tag{62}$$

or (in an expanded presentation)

$$\overset{*}{\nabla}_{\mu} \left(\overset{\text{met}}{t}{}^{\mu}{}_{\nu} + \overset{*}{\nabla}_{\eta} \overset{\text{mod}}{b}{}^{\eta\mu}{}_{\nu} \right) \equiv$$
$$\equiv - \left(\overset{\text{met}}{t}{}^{\mu}{}_{\lambda} + \overset{*}{\nabla}_{\eta} \overset{\text{mod}}{b}{}^{\eta\mu}{}_{\lambda} \right) T^{\lambda}{}_{\mu\nu} +$$
$$+ \frac{1}{2} \overset{\text{mod}}{s}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} + \left\{ \overset{*}{\nabla}_{\mu} \left[\frac{\Delta I}{\Delta\varphi^a} (\Delta^{\mu}{}_{\nu}) \,^a|_b \varphi^b \right] + \right.$$
$$+ \frac{\Delta I}{\Delta\varphi^a} \left[\nabla_{\nu}\varphi^a + (\Delta^{\mu}{}_{\lambda}) \,^a|_b \varphi^b \, T^{\lambda}{}_{\mu\nu} \right] \right\} \tag{63}$$

and is the basis for defining the equations of balance for the metric EMT:

$$\overset{*}{\nabla}_\mu \overset{\text{met}}{t}{}^\mu{}_\nu = - \overset{\text{met}}{t}{}^\mu{}_\lambda T^\lambda{}_{\mu\nu} \ (\text{on the } \phi\text{-equations}) \quad (64)$$

or

$$\overset{*}{\nabla}_\mu \left(\overset{\text{met}}{t}{}^\mu{}_\nu + \overset{*}{\nabla}_\eta \overset{\text{mod}}{b}{}^{\eta\mu}{}_\nu \right) =$$

$$= - \left(\overset{\text{met}}{t}{}^\mu{}_\lambda + \overset{*}{\nabla}_\eta \overset{\text{mod}}{b}{}^{\eta\mu}{}_\lambda \right) T^\lambda{}_{\mu\nu} +$$

$$+ \frac{1}{2} \overset{\text{mod}}{s}{}^\pi{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} \ (\text{on the } \varphi\text{-equations}). \quad (65)$$

Statement 15 (On the 4-th and 3-rd Klein identities). *The 4-th and 3-rd Klein identities (25) and (24) are satisfied automatically.*

Statement 16 (The physical sense of the 2-nd Klein identity). *The 2-nd Klein identity (23) has the explicit form*

$$\overset{\text{sym}}{t}{}^\mu{}_\nu \equiv \overset{\text{met}}{t}{}^\mu{}_\nu - \frac{\Delta I}{\Delta \phi^a}(\Delta^\mu{}_\nu)^a|_b \phi^b \quad (66)$$

or

$$t^\mu{}_\nu - \overset{*}{\nabla}_\lambda b^{\lambda\mu}{}_\nu \equiv \overset{\text{met}}{t}{}^\mu{}_\nu +$$

$$+ \left(\frac{1}{2} \overset{\text{add}}{b}{}^{\rho\sigma}{}_\nu T^\mu{}_{\rho\sigma} + \overset{\text{add}}{b}{}^{\mu\beta}{}_\alpha T^\alpha{}_{\beta\nu} \right) -$$

$$- \frac{\Delta I}{\Delta \varphi^a}(\Delta^\mu{}_\nu)^a|_b \varphi^b \quad (67)$$

and claims that, on the ϕ-equations, the symmetrized EMT $\overset{\text{sym}}{\mathbf{t}}$ (61) is equal to the metric EMT $\overset{\text{met}}{\mathbf{t}}$ (60):

$$\overset{\text{sym}}{t}{}^\mu{}_\nu = \overset{\text{met}}{t}{}^\mu{}_\nu \ (\text{on the } \phi\text{-equations}) \quad (68)$$

or

$$t^\mu{}_\nu - \overset{*}{\nabla}_\lambda b^{\lambda\mu}{}_\nu = \overset{\text{met}}{t}{}^\mu{}_\nu +$$

$$+ \left(\frac{1}{2} \overset{\text{add}}{b}{}^{\rho\sigma}{}_\nu T^\mu{}_{\rho\sigma} + \overset{\text{add}}{b}{}^{\mu\beta}{}_\alpha T^\alpha{}_{\beta\nu} \right)$$

$$(\text{on the } \varphi\text{-equations}). \quad (69)$$

Statement 17 (The physical sense of the 1-st Klein identity). *The 1-st Klein identity (16) has the explicit form*

$$\overset{*}{\nabla}_\mu t^\mu{}_\nu \equiv -t^\mu{}_\lambda T^\lambda{}_{\mu\nu} + \frac{1}{2} s^\pi{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} -$$

$$- \frac{1}{2} b^{\gamma\beta}{}_\alpha \nabla_\nu T^\alpha{}_{\beta\gamma} + \frac{\Delta I}{\Delta \phi^a} \nabla_\nu \phi^a \quad (70)$$

or

$$\overset{*}{\nabla}_\mu t^\mu{}_\nu \equiv -t^\mu{}_\lambda T^\lambda{}_{\mu\nu} + \frac{1}{2} s^\pi{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} +$$

$$+ \frac{1}{2} \overset{\text{add}}{b}{}^{\gamma\beta}{}_\alpha \nabla_\nu T^\alpha{}_{\beta\gamma} + \frac{\Delta I}{\Delta \varphi^a} \nabla_\nu \varphi^a \quad (71)$$

and is the basis for constructing the equations of balance for the canonical EMT $\mathbf{t}$

$$\overset{*}{\nabla}_\mu t^\mu{}_\nu = -t^\mu{}_\lambda T^\lambda{}_{\mu\nu} + \frac{1}{2} s^\pi{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} -$$

$$- \frac{1}{2} b^{\gamma\beta}{}_\alpha \nabla_\nu T^\alpha{}_{\beta\gamma} \ (\text{on the } \phi\text{-equations}) \quad (72)$$

or

$$\overset{*}{\nabla}_\mu t^\mu{}_\nu = -t^\mu{}_\lambda T^\lambda{}_{\mu\nu} + \frac{1}{2} s^\pi{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} +$$

$$+ \frac{1}{2} \overset{\text{add}}{b}{}^{\gamma\beta}{}_\alpha \nabla_\nu T^\alpha{}_{\beta\gamma} \ (\text{on the } \varphi\text{-equations}). \quad (73)$$

In Refs. [10–16], for a Lagrangian of the type $\mathscr{L} = \mathscr{L}(\mathbf{g}; \ \boldsymbol{\varphi}, \nabla\boldsymbol{\varphi})$, the equation of balance for the canonical EMT

$$\overset{*}{\nabla}_\mu t^\mu{}_\nu = -t^\mu{}_\lambda T^\lambda{}_{\mu\nu} + \frac{1}{2} s^{\pi, \rho\sigma} R_{\rho\sigma\pi\nu}$$

$$(\text{on the } \varphi\text{-equations}) \quad (74)$$

has been obtained. Result (74) is left valid also in a more general case, when the Lagrangian has the form: $\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R}; \ \boldsymbol{\varphi}, \nabla\boldsymbol{\varphi}, \nabla\nabla\boldsymbol{\varphi})$ because the last term in (73) does not appear. In the case of non-minimal $\mathbf{T}$-coupling, the right-hand side of (73) contains the *additional* term $\left(\frac{1}{2} \overset{\text{add}}{b}{}^{\gamma\beta}{}_\alpha \nabla_\nu T^\alpha{}_{\beta\gamma} \right)$. However, the *new* equation (73) can be also transformed to the form (74) (see [2] for details). Only, the EMT and ST in (74) are changed to modified ones. The modification of the EMT is analogous to the modification of the canonical ST in (57) and (59). The *modified canonical EMT* $\overset{\text{mod}}{\mathbf{t}} \overset{\text{def}}{=} \{ \overset{\text{mod}}{t}{}^\mu{}_\nu \}$:

$$\overset{\text{mod}}{t}{}^\mu{}_\nu \overset{\text{def}}{=} t^\mu{}_\nu + \overset{\text{add}}{t}{}^\mu{}_\nu, \quad (75)$$

where the *additional EMT* $\overset{\text{add}}{\mathbf{t}} \overset{\text{def}}{=} \{ \overset{\text{add}}{t}{}^\mu{}_\nu \}$ is defined as

$$- \overset{\text{add}}{t}{}^\mu{}_\nu \overset{\text{def}}{=} \overset{*}{\nabla}_\lambda \overset{\text{add}}{b}{}^{\mu\lambda}{}_\nu +$$

$$+ \frac{1}{2} \overset{\text{add}}{b}{}^{\kappa\lambda}{}_\nu T^\mu{}_{\kappa\lambda} + \overset{\text{add}}{b}{}^{\mu\lambda}{}_\kappa T^\kappa{}_{\lambda\nu}. \quad (76)$$

It is evident that EMT $\overset{\text{mod}}{\mathbf{t}}$ (75) in the case of minimal **T**-coupling only transforms to the usual canonical EMT **t**. By definition (76), the Belinfante symmetrization of the type (61) applied to $\overset{\text{add}}{\mathbf{t}}$ leads to zero identically. Therefore,

Statement 18. *The symmetrized EMT $\overset{\text{sym}}{\mathbf{t}}$ constructed by the symmetrization of $\overset{\text{mod}}{\mathbf{t}}$ (75) with the use of $\overset{\text{mod}}{\mathbf{b}}$ (55) by the rule*

$$\overset{\text{sym}}{t}{}^{\mu}{}_{\nu} = \overset{\text{mod}}{t}{}^{\mu}{}_{\nu} +$$
$$+ \left[\overset{*}{\nabla}_{\lambda} \overset{\text{mod}}{b}{}^{\mu\lambda}{}_{\nu} + \frac{1}{2} \overset{\text{mod}}{b}{}^{\kappa\lambda}{}_{\nu} T^{\mu}{}_{\kappa\lambda} + \overset{\text{mod}}{b}{}^{\mu\lambda}{}_{\kappa} T^{\kappa}{}_{\lambda\nu} \right] \quad (77)$$

exactly coincides with the (usual) symmetrized EMT $\overset{\text{sym}}{\mathbf{t}}$ (61).

Statement 19. *In the terms of the modified canonical EMT $\overset{\text{mod}}{\mathbf{t}}$, the 1-st Klein identities (70) or (71) can be rewritten as*

$$\overset{*}{\nabla}_{\mu} \overset{\text{mod}}{t}{}^{\mu}{}_{\nu} \equiv - \overset{\text{mod}}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} \overset{\text{mod}}{s}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} -$$
$$- \frac{1}{2} \overset{\text{mod}}{b}{}^{\gamma\beta}{}_{\alpha} \nabla_{\nu} T^{\alpha}{}_{\beta\gamma} + \frac{\Delta I}{\Delta \phi^a} \nabla_{\nu} \phi^a \quad (78)$$

or

$$\overset{*}{\nabla}_{\mu} \overset{\text{mod}}{t}{}^{\mu}{}_{\nu} \equiv - \overset{\text{mod}}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} +$$
$$+ \frac{1}{2} \overset{\text{mod}}{s}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} + \frac{\Delta I}{\Delta \varphi^a} \nabla_{\nu} \varphi^a. \quad (79)$$

They are the basis for the equations of balance for the modified canonical EMT $\overset{\text{mod}}{\mathbf{t}}$:

$$\overset{*}{\nabla}_{\mu} \overset{\text{mod}}{t}{}^{\mu}{}_{\nu} = - \overset{\text{mod}}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} \overset{\text{mod}}{s}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} -$$
$$- \frac{1}{2} \overset{\text{mod}}{b}{}^{\gamma\beta}{}_{\alpha} \nabla_{\nu} T^{\alpha}{}_{\beta\gamma} \; (\text{on the } \phi\text{-equations}) \quad (80)$$

or

$$\overset{*}{\nabla}_{\mu} \overset{\text{mod}}{t}{}^{\mu}{}_{\nu} = - \overset{\text{mod}}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} \overset{\text{mod}}{s}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu}$$

$$(\text{on the } \varphi\text{-equations}). \quad (81)$$

Now, Eq. (81) has the same structure as Eq. (74). We note also that if the equations for the torsion field $\Delta I / \Delta \mathbf{T} = 0$ hold, then

$$\overset{\text{mod}}{t}{}^{\mu}{}_{\nu} = \overset{\text{sym}}{t}{}^{\mu}{}_{\nu} \; (\text{on the } \mathbf{T}\text{-equations}), \quad (82)$$

as it follows from (77) and (54).

At last, let us find the identities and the equations of balance for the symmetrized EMT $\overset{\text{sym}}{\mathbf{t}}$. Use (61) for rewriting **t** as a function of $\overset{\text{sym}}{\mathbf{t}}$ and **b**, substitute the result into (70) and (71), and find, respectively,

Statement 20. *The 1-st Klein identity, as well as the 2-nd one, leads to the identities*

$$\overset{*}{\nabla}_{\mu} \overset{\text{sym}}{t}{}^{\mu}{}_{\nu} \equiv - \overset{\text{sym}}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{\Delta I}{\Delta \phi^a} \nabla_{\nu} \phi^a \quad (83)$$
or
$$\overset{*}{\nabla}_{\mu} \overset{\text{sym}}{t}{}^{\mu}{}_{\nu} \equiv - \overset{\text{sym}}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} +$$
$$+ \frac{1}{2} \overset{\text{mod}}{b}{}^{\gamma\beta}{}_{\alpha} \nabla_{\nu} T^{\alpha}{}_{\beta\gamma} + \frac{\Delta I}{\Delta \varphi^a} \nabla_{\nu} \varphi^a, \quad (84)$$

which are the basis for constructing the equations of balance for the symmetrized EMT $\overset{\text{sym}}{\mathbf{t}}$:

$$\overset{*}{\nabla}_{\mu} \overset{\text{sym}}{t}{}^{\mu}{}_{\nu} = - \overset{\text{sym}}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu}$$
$$(\text{on the } \phi\text{-equations}) \quad (85)$$
or
$$\overset{*}{\nabla}_{\mu} \overset{\text{sym}}{t}{}^{\mu}{}_{\nu} = - \overset{\text{sym}}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} +$$
$$+ \frac{1}{2} \overset{\text{mod}}{b}{}^{\gamma\beta}{}_{\alpha} \nabla_{\nu} T^{\alpha}{}_{\beta\gamma} \; (\text{on the } \varphi\text{-equations}). \quad (86)$$

3.6. Explicit form of the superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$

Let us turn to the superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$. Substituting the explicit expressions for the tensors **U** (42), **M** (43), and **N** (44) into formula (30), we obtain

Statement 21. *In manifestly generally covariant field theories with Lagrangians of the type (39), the generalized superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ has the explicit form*

$$\theta^{\mu\nu}[\delta\boldsymbol{\xi}] \overset{\text{def}}{=} \left[-b^{\mu\nu}{}_{\alpha} + G_{\kappa}{}^{\lambda\mu\nu} T^{\kappa}{}_{\lambda\alpha} \right] \delta\xi^{\alpha} +$$
$$+ \left[-G_{\alpha}{}^{\beta\mu\nu} \right] \nabla_{\beta} \delta\xi^{\alpha}, \quad (87)$$

*i.e., it is expressed through only the Belinfante tensor **b** induced by the canonical ST **s** and the tensor **G**.*

4. Structure and Interpretation of the Equations of Gravitational Field

4.1. Splitting of the total Lagrangian

Represent the total Lagrangian (39) as a sum of the pure gravitational $\mathscr{L}^G$ and matter $\mathscr{L}^M$ parts:

$$\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R}; \; \mathbf{T}, \boldsymbol{\nabla}\mathbf{T}, \boldsymbol{\nabla}\boldsymbol{\nabla}\mathbf{T}; \; \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varphi}) \overset{\text{def}}{=}$$
$$\overset{\text{def}}{=} \mathscr{L}(\mathbf{g}, \mathbf{R}; \; \boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\phi}) = \mathscr{L}^G + \mathscr{L}^M, \quad (88)$$

where

$$\mathscr{L}^G = \mathscr{L}^G(\mathbf{g}, \mathbf{R}) \overset{\text{def}}{=} \mathscr{L}(\mathbf{g}, \mathbf{R}; 0, 0, 0); \qquad (89)$$

$$\mathscr{L}^M = \mathscr{L}^M(\mathbf{g}, \mathbf{R}; \ \phi, \boldsymbol{\nabla}\phi, \boldsymbol{\nabla}\boldsymbol{\nabla}\phi) \overset{\text{def}}{=} \mathscr{L} - \mathscr{L}^G. \quad (90)$$

Remark that, in spite of the Lagrangian $\mathscr{L}^G$ does not contain the torsion $\mathbf{T}$ *explicitly*, it contains the torsion *not explicitly* over the connection $\boldsymbol{\Gamma}$, which is used for constructing the curvature tensor $\mathbf{R}$. Therefore, the proposed splitting (88)–(90) is non-trivial.

It is evident that the splitting of Lagrangian (88) leads to a correspondent splitting of the action functional:

$$I = \int\limits_{\Sigma_1}^{\Sigma_2} dx \sqrt{-g}\mathscr{L} = \int\limits_{\Sigma_1}^{\Sigma_2} dx \sqrt{-g}\mathscr{L}^G +$$

$$+ \int\limits_{\Sigma_1}^{\Sigma_2} dx \sqrt{-g}\mathscr{L}^M \overset{\text{def}}{=} I^G + I^M.$$

Of course, the Lagrangian of the vacuum system $\mathscr{L}^G$ has to be *generally covariant scalar*, and then the matter Lagrangian $\mathscr{L}^M$ is, like this, also. Therefore, all the above results and conclusions related to the total Lagrangian $\mathscr{L}$ are left valid for each of the Lagrangians $\mathscr{L}^G$ and $\mathscr{L}^M$.

4.2. The material and geometric tensors

Define the following *matter tensors*.

$$S^{\pi}{}_{\rho\sigma} \overset{\text{def}}{=} s^{\pi}{}_{\rho\sigma}\big|_{\mathscr{L} = \mathscr{L}^M} \qquad (91)$$

(the canonical ST of matter);

$$B^{\gamma\beta\alpha} \overset{\text{def}}{=} b^{\gamma\beta\alpha}\big|_{\mathscr{L} = \mathscr{L}^M} \qquad (92)$$

(the Belinfante tensor for ST $\mathbf{S}$);

$$T^{\mu}{}_{\nu} \overset{\text{def}}{=} t^{\mu}{}_{\nu}\big|_{\mathscr{L} = \mathscr{L}^M} \qquad (93)$$

(the canonical EMT of matter)

and analogously for the tensors $\{\overset{\text{add}}{S}{}^{\pi}{}_{\rho\sigma}\}$, $\{\overset{\text{mod}}{S}{}^{\pi}{}_{\rho\sigma}\}$; $\{\overset{\text{add}}{B}{}^{\gamma\beta\alpha}\}$, $\{\overset{\text{mod}}{B}{}^{\gamma\beta\alpha}\}$; $\{\overset{\text{add}}{T}{}^{\mu}{}_{\nu}\}$, $\{\overset{\text{mod}}{T}{}^{\mu}{}_{\nu}\}$, $\{\overset{\text{sym}}{T}{}^{\mu\nu}\}$ and $\{\overset{\text{met}}{T}{}^{\mu\nu}\}$.

For the above-defined matter tensors, the relations analogous to those between the total tensors take place. In particular,

$$\frac{\Delta I^M}{\Delta T^{\alpha}{}_{\beta\gamma}} = \frac{1}{2}\overset{\text{mod}}{B}{}^{\gamma\beta}{}_{\alpha}; \qquad (94)$$

$$2\frac{\Delta I^M}{\Delta g_{\mu\nu}} = \overset{\text{met}}{T}{}^{\mu\nu} \equiv$$

$$\equiv \overset{\text{mod}}{T}{}^{\mu\nu} - \overset{*}{\nabla}_{\lambda}\overset{\text{mod}}{B}{}^{\lambda\mu\nu} + \frac{\Delta I^M}{\Delta\varphi^a}(\Delta^{\mu\nu})\,{}^a\big|_b\,\varphi^b. \qquad (95)$$

Now, we define the *Cartan tensor* $\mathscr{C} \overset{\text{def}}{=} \{\mathscr{C}^{[\gamma\beta]}{}_{\alpha} = \mathscr{C}^{\gamma\beta}{}_{\alpha}\}$, the *(generalized) Einstein tensor* $\mathscr{E} \overset{\text{def}}{=} \{\mathscr{E}^{(\mu\nu)} \neq \mathscr{E}^{\mu\nu}\}$ and its symmetric part:

$$\frac{-1}{2k}\mathscr{C}^{\gamma\beta}{}_{\alpha} \overset{\text{def}}{=} \frac{\Delta I^G}{\Delta T^{\alpha}{}_{\beta\gamma}} = \frac{1}{2}\,b^{\gamma\beta}{}_{\alpha}\big|_{\mathscr{L} = \mathscr{L}^G}, \qquad (96)$$

$$\frac{-1}{2k}\mathscr{E}^{\mu}{}_{\nu} \overset{\text{def}}{=} \frac{1}{2}\,t^{\mu}{}_{\nu}\big|_{\mathscr{L} = \mathscr{L}^G} =$$

$$= \frac{1}{2}\left(\mathscr{L}^G\delta^{\mu}_{\nu} - {}^{(G)}G^{\alpha\beta\gamma\mu}R_{\alpha\beta\gamma\nu}\right), \qquad (97)$$

$$\frac{-1}{2k}\left(\mathscr{E}^{(\mu\nu)} - \overset{*}{\nabla}_{\lambda}\mathscr{C}^{\lambda(\mu\nu)}\right) \overset{\text{def}}{=}$$

$$\overset{\text{def}}{=} \frac{\Delta I^G}{\Delta g_{\mu\nu}} = \frac{1}{2}\overset{\text{met}}{t}{}^{\mu\nu}\bigg|_{\mathscr{L} = \mathscr{L}^G}. \qquad (98)$$

Here,

$$^{(G)}G_{\alpha}{}^{\beta\gamma\delta} \overset{\text{def}}{=} G_{\alpha}{}^{\beta\gamma\delta}\big|_{\mathscr{L} = \mathscr{L}^G} = 2\frac{\partial\mathscr{L}^G}{\partial R^{\alpha}{}_{\beta\gamma\delta}}; \qquad (99)$$

$k \overset{\text{def}}{=} (D - 1)\Omega_{(D-1)}\ \varkappa$; $\Omega_{(D-1)}$ is an area of the $(D-1)$-dimensional unit sphere, and $\varkappa$ is the Newtonian gravitational constant in the $(D+1)$-dimensional space-time.

A restriction of the 2-nd Klein identity (66) and definition (61) to the case of the Lagrangian $\mathscr{L} = \mathscr{L}^G$ gives, with regard for definitions (96)–(97), the

Statement 22. *The antisymmetric part of the generalized Einstein tensor is the divergence of the antisymmetric part of the Cartan tensor:*

$$\mathscr{E}^{[\mu\nu]} \equiv \overset{*}{\nabla}_{\lambda}\mathscr{C}^{\lambda[\mu\nu]}. \qquad (100)$$

Using identity (100), one can represent (98) in the form

$$2\frac{\Delta I^G}{\Delta g_{\mu\nu}} \equiv \frac{-1}{k}\left(\mathscr{E}^{\mu\nu} - \overset{*}{\nabla}_{\lambda}\mathscr{C}^{\lambda\mu\nu}\right). \qquad (101)$$

Formulae (79) and (75) rewritten for the Lagrangian $\mathscr{L}^G$ after using definitions (97) and (96) lead to the

Statement 23 (The generalized twice contracted Bianchi identity). *The generalized Einstein tensor $\mathscr{E}$ and the Cartan tensor $\mathscr{C}$ obey the identity*

$$\overset{*}{\nabla}_{\mu}\mathscr{E}^{\mu}{}_{\nu} \equiv -\mathscr{E}^{\mu}{}_{\lambda}T^{\lambda}{}_{\mu\nu} - \mathscr{C}^{\pi}{}_{\rho\sigma}R^{\rho\sigma}{}_{\pi\nu}. \qquad (102)$$

4.3. Gravitational field equations in the split form

By Eqs. (101) and the definition of the tensor $\{\overset{\text{met}}{T}{}^{\mu\nu}\}$, the equations of motion of the metric field, $\Delta(I^G + I^M)/\Delta g_{\mu\nu} = 0$, can be rewritten as

$$\mathscr{E}^{\mu\nu} - \overset{*}{\nabla}_\lambda \mathscr{C}^{\lambda\mu\nu} = k \overset{\text{met}}{T}{}^{\mu\nu}. \tag{103}$$

If we account for the equations of motion for the torsion field

$$\frac{\Delta(I^G + I^M)}{\Delta T^\nu{}_{\mu\lambda}} = 0 \Leftrightarrow \mathscr{C}^{\lambda\mu\nu} = k \overset{\text{mod}}{B}{}^{\lambda\mu\nu} \tag{104}$$

and the equations of motion for the φ-fields: $\Delta I^M / \Delta \varphi^a = 0$, one can transform (103) to the equation for the metric field only:

$$\mathscr{E}^{\mu\nu} = k \overset{\text{mod}}{T}{}^{\mu\nu} \qquad \text{(on the } \phi\text{-equations).} \tag{105}$$

Now, let us turn to (104). After the antisymmetrization in indices μ and ν, using formula (55), the definitions of the tensors $\{\overset{\text{mod}}{B}{}^{\gamma\beta\alpha}\}$, $\{\overset{\text{mod}}{S}{}^{\gamma,\beta\alpha}\}$, and the identity

$$\Delta_{\lambda\mu\nu}^{\overline{\gamma[\beta\alpha]}} = -\frac{1}{4}\delta_\lambda^\gamma \delta_{\mu\nu}^{\beta\alpha}, \tag{106}$$

Eq. (104) acquires an equivalent form:

$$-2\mathscr{C}^{\lambda[\mu\nu]} = k \overset{\text{mod}}{S}{}^{\lambda,\mu\nu}. \tag{107}$$

Thus, the next is valid:

Statement 24. *In arbitrary metric-torsion field theories, the total system of field equations*

$$\begin{cases} \Delta I/\Delta g_{\mu\nu} = 0; & (108) \\ \Delta I/\Delta T^\lambda{}_{\mu\nu} = 0; & (109) \\ \Delta I/\Delta \varphi^a = 0 & (110) \end{cases}$$

can be always represented in the Einstein–Cartan-theory-like form

$$\begin{cases} \mathscr{E}^{\mu\nu} = k \overset{\text{mod}}{T}{}^{\mu\nu} \text{ (the } \mathbf{g}\text{-equations)}; & (111) \\ -2\mathscr{C}^\lambda{}_{[\mu\nu]} = k \overset{\text{mod}}{S}{}^\lambda{}_{\mu\nu} \text{ (the } \mathbf{T}\text{-equations)}; & (112) \\ \Delta I^M/\Delta \varphi^a = 0 \text{ (the } \boldsymbol{\varphi}\text{-equations)}. & (113) \end{cases}$$

The interpretation of the gravitational equations of the system is as follows. The source of the metric field **g** *is the modified canonical EMT of matter* $\overset{\text{mod}}{\mathbf{T}}$, *whereas the source of the torsion field* **T** *is the modified canonical ST of matter* $\overset{\text{mod}}{\mathbf{S}}$.

5. Boundary Terms and Killing Vectors

5.1. Corrections induced by boundary terms

In Section 2 2.4, it has been shown that the symmetrized EMT $\overset{\text{sym}}{\mathbf{t}}$, consequently the *symmetrized* current $\overset{\text{sym}}{\mathbf{J}}[\delta\boldsymbol{\xi}]$ also, does not depend on the divergences in a Lagrangian. Now, let us analyze the problem how the total divergence in the Lagrangian influences the *canonical* current $\mathbf{J}[\delta\boldsymbol{\xi}]$ (12) and superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ (30).

Let $\{\mathscr{L}^\mu\}$ be an arbitrary vector constructed from field variables and their derivatives, and let

$$\mathscr{L}' = \mathscr{L} + \Delta\mathscr{L}, \quad \Delta\mathscr{L} \overset{\text{def}}{=} \overset{*}{\nabla}_\mu \mathscr{L}^\mu \tag{114}$$

be a new Lagrangian with the corresponding new current and superpotential:

$$\mathbf{J}'[\delta\boldsymbol{\xi}] = \mathbf{J}[\delta\boldsymbol{\xi}] + \Delta\mathbf{J}[\delta\boldsymbol{\xi}], \tag{115}$$
$$\boldsymbol{\theta}'[\delta\boldsymbol{\xi}] = \boldsymbol{\theta}[\delta\boldsymbol{\xi}] + \Delta\boldsymbol{\theta}[\delta\boldsymbol{\xi}]. \tag{116}$$

Of course, both new $\mathbf{J}'[\delta\boldsymbol{\xi}]$ and old $\mathbf{J}[\delta\boldsymbol{\xi}]$ satisfy the Klein–Noether boundary theorem. Therefore, formulae (31) and (28) rewritten for new quantities yield

Statement 25. *The total divergence in the Lagrangian induces changes of the canonical current* $\Delta\mathbf{J}[\delta\boldsymbol{\xi}]$ *and the superpotential* $\Delta\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$, *which are connected as*

$$\Delta J^\mu[\delta\boldsymbol{\xi}] = \overset{*}{\nabla}_\nu \left(\Delta\theta^{\mu\nu}[\delta\boldsymbol{\xi}]\right) + \frac{1}{2}\left(\Delta\theta^{\rho\sigma}[\delta\boldsymbol{\xi}]\right) T^\mu{}_{\rho\sigma}. \tag{117}$$

Define an explicit expression for $\Delta\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$. Repeating the logic presented in Sections 2 2.1–2 2.3 for a Lagrangian of the type $\mathscr{L} = \overset{*}{\nabla}_\mu \mathscr{L}^\mu$, one obtains

Statement 26. *The addition of the item* $\Delta\mathscr{L} = \overset{*}{\nabla}_\mu \mathscr{L}^\mu$ *to the Lagrangian induces adding the term* $\Delta\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ *to the canonical superpotential, which does not depend on the structure of the vector* $\{\mathscr{L}^\mu\}$ *and is defined in a unique way:*

$$\Delta\theta^{\mu\nu}[\delta\boldsymbol{\xi}] = -2\mathscr{L}^{[\mu}\delta\xi^{\nu]}. \tag{118}$$

This generalizes the statement in [17] to the case of the Riemann–Cartan space.

5.2. Currents on the Killing vectors

The generalized charges $Q[\boldsymbol{\chi}]$, currents $\mathbf{J}[\boldsymbol{\chi}]$, $\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}]$ and superpotentials $\boldsymbol{\theta}[\boldsymbol{\chi}]$ constructed with the use of

the Killing vectors $\boldsymbol{\chi}$ of the Riemann–Cartan space are extremely important in applications.

A definition of the Killing vector field $\boldsymbol{\chi} \overset{\text{def}}{=} \{\chi^\mu\}$ in the Riemann–Cartan space, of course, is more complicated, than in the Riemannian geometry, although is defined also by the Lie transport. Thus,

Definition 1. *The vector field $\boldsymbol{\chi}$ is called the Killing one in the Riemann–Cartan space $\mathscr{C}(1, D)$, if, under the Lie translation along the vector $\boldsymbol{\chi}$, both the metric tensor $\mathbf{g}$ and the torsion tensor $\mathbf{T}$ are invariant, i.e.,*

$$
\begin{cases}
\mathcal{L}_{\boldsymbol{\chi}} g_{\mu\nu} = 0; & (119) \\
\mathcal{L}_{\boldsymbol{\chi}} T^\lambda{}_{\mu\nu} = 0. & (120)
\end{cases}
$$

Let us define an explicit expression for $\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}]$, its physical sense, and a condition for its conservation. Under the decomposition of Lagrangian (88)–(90), the current $\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}]$ is decomposed also as

$$
\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}] = {}^{(G)}\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}] + {}^{(M)}\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}]; \tag{121}
$$

$$
{}^{(G)}\overset{\text{sym}}{J}{}^\mu[\boldsymbol{\chi}] = -\frac{1}{k}\left[\mathscr{E}^\mu{}_\alpha + \overset{*}{\nabla}_\nu \mathscr{C}^{\mu\nu}{}_\alpha + \right.
$$
$$
\left. + \frac{1}{2}\mathscr{C}^{\rho\sigma}{}_\alpha T^\mu{}_{\rho\sigma} + \mathscr{C}^{\mu\gamma}{}_\beta T^\beta{}_{\gamma\alpha} \right]\chi^\alpha; \tag{122}
$$

$$
{}^{(M)}\overset{\text{sym}}{J}{}^\mu[\boldsymbol{\chi}] = \overset{\text{sym}}{T}{}^\mu{}_\alpha \chi^\alpha. \tag{123}
$$

The last current has a clear physical sense:

Statement 27. *The charges related to the Killing vectors $\boldsymbol{\chi}_{(a)}$; $a = 1, 2, \dots$*

$$
Q_{(a)} \overset{\text{def}}{=} Q[\boldsymbol{\chi}_{(a)}] = \int_\Sigma d\sigma_\mu \, {}^{(M)}\overset{\text{sym}}{J}{}^\mu[\boldsymbol{\chi}_{(a)}] \tag{124}
$$

define the corresponding conserved quantities in the domain Σ, those could be energy, momentum, etc.

Next, as a consequence of identity (66), one has

$$
\overset{\text{sym}}{t}{}^{[\mu\nu]} \equiv -\frac{\Delta I}{\Delta \phi^a}(\Delta^{[\mu\nu]})^a{}_b \, \phi^b. \tag{125}
$$

In the case of the Killing vector $\boldsymbol{\chi}$ as a displacement vector, using identities (83) and (125), one easily obtains

$$
\overset{*}{\nabla}_\mu \overset{\text{sym}}{J}{}^\mu[\boldsymbol{\chi}] = \frac{\Delta I}{\Delta \varphi^a} \mathcal{L}_{\boldsymbol{\chi}} \varphi^a. \tag{126}
$$

Decomposing this equality onto the pure gravitational and pure matter parts, one proves the next statements.

Statement 28. *The pure gravitational part ${}^{(G)}\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}]$ of the generalized symmetrized current $\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}]$ is conserved identically, independently on holding the equations of motion at all:*

$$
\overset{*}{\nabla}_\mu {}^{(G)}\overset{\text{sym}}{J}{}^\mu[\boldsymbol{\chi}] \equiv 0 \text{ (off-shell).} \tag{127}
$$

Statement 29. *The pure matter part ${}^{(M)}\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}]$ is conserved identically, if the matter fields $\boldsymbol{\varphi}$ are Lie-dragged:*

$$
\overset{*}{\nabla}_\mu {}^{(M)}\overset{\text{sym}}{J}{}^\mu[\boldsymbol{\chi}] \equiv 0 \text{ (off-shell)} \iff \mathcal{L}_{\boldsymbol{\chi}}\varphi^a = 0. \tag{128}
$$

Statement 30. *The current ${}^{(M)}\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}]$ is conserved if only the $\boldsymbol{\varphi}$-equations hold:*

$$
\overset{*}{\nabla}_\mu {}^{(M)}\overset{\text{sym}}{J}{}^\mu[\boldsymbol{\chi}] = 0 \qquad \text{(on the $\boldsymbol{\varphi}$-equations).} \tag{129}
$$

Consequence 2. *Because the current ${}^{(M)}\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}]$ is conserved independently on holding both the $\mathbf{g}$- and the $\mathbf{T}$-equations, the matter fields can be classified as* external *ones with respect to the $\mathbf{g}$- and $\mathbf{T}$-fields. This gives a possibility to construct and study dynamic variables for the matter fields propagating in a given/fixed Riemann–Cartan space.*

Turn to the generalized canonical current

$$
\mathbf{J}[\boldsymbol{\chi}] = {}^{(G)}\mathbf{J}[\boldsymbol{\chi}] + {}^{(M)}\mathbf{J}[\boldsymbol{\chi}]. \tag{130}
$$

The pure matter part of this ${}^{(M)}\mathbf{J}[\boldsymbol{\chi}]$ is very cumbersome, see [3]. The expression for the pure gravitational part ${}^{(G)}\mathbf{J}[\boldsymbol{\chi}]$ is, conversely, very simple:

$$
{}^{(G)}\mathbf{J}[\boldsymbol{\chi}] = \mathscr{L}^G \chi^\mu. \tag{131}
$$

Conditions for the conservation of these currents can be found in [3].

5.3. A sense of the generalized superpotential

Consider a pure gravitational part of the canonical charge

$$
{}^{(G)}Q[\boldsymbol{\chi}] \overset{\text{def}}{=} \int_\Sigma d\sigma_\mu \, {}^{(G)}J^\mu[\boldsymbol{\chi}]. \tag{132}
$$

Using the Klein–Noether boundary theorem, one can write

$$^{(G)}Q[\boldsymbol{\chi}] = \int\limits_{\Sigma} d\sigma_\mu \, {}^{(G)}\overset{\text{sym}}{J}{}^{\mu}[\boldsymbol{\chi}] +$$

$$+\frac{1}{2!} \oint\limits_{\partial\Sigma} ds_{\mu\nu} \, {}^{(G)}\theta^{\mu\nu}[\boldsymbol{\chi}]. \tag{133}$$

Because $^{(G)}\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}] = -^{(M)}\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}]$ on-shell, the last term takes the form

$$\frac{1}{2!} \oint\limits_{\partial\Sigma} ds_{\mu\nu} \, {}^{(G)}\theta^{\mu\nu}[\boldsymbol{\chi}] \text{ (on-shell) } =$$

$$= \int\limits_{\Sigma} d\sigma_\mu \left({}^{(G)}J^\mu[\boldsymbol{\chi}] + {}^{(M)}\overset{\text{sym}}{J}{}^{\mu}[\boldsymbol{\chi}] \right). \tag{134}$$

Substituting into the right-hand side the expressions $^{(G)}\mathbf{J}[\boldsymbol{\chi}]$ (131) and $^{(M)}\overset{\text{sym}}{\mathbf{J}}[\boldsymbol{\chi}]$ (123), one obtains

$$\frac{1}{2!} \oint\limits_{\partial\Sigma} ds_{\mu\nu} \, {}^{(G)}\theta^{\mu\nu}[\boldsymbol{\chi}] = \int\limits_{\Sigma} d\sigma_\mu \left[\mathscr{L}^G \chi^\mu + \overset{\text{sym}}{T}{}^{\mu}{}_{\nu}\chi^\nu \right]$$

(on-shell) \hfill (135)

This shows what the *matter* source structure is presented specifically by the surface integral of the *gravitational* superpotential $^{(G)}\boldsymbol{\theta}[\boldsymbol{\chi}]$.

The gravitational theories with Lagrangians of the type

$$\mathscr{L}^G = \mathscr{L}_{(n)}(\mathbf{g}, \mathbf{R}) - \Lambda \tag{136}$$

are very important now. Here, Λ means the cosmological constant, whereas $\mathscr{L}_{(n)}$ is homogeneous in the curvature tensor $\mathbf{R}$ function of the degree n. For example, the Einstein–Cartan theory, curvature squared theories, the pure Lovelock–Cartan theories are of this type. It is easy to show that

Statement 31. *In the theories with Lagrangians of the type* (136), *the generalized charge $^{(G)}Q[\boldsymbol{\chi}]$ is expressed only through the cosmological constant Λ and the EMT of the matter fields $\overset{\text{met}}{\mathbf{T}}$:*

$$^{(G)}Q[\boldsymbol{\chi}] = \frac{1}{2!} \oint\limits_{\partial\Sigma} ds_{\mu\nu} \, {}^{(G)}\theta^{\mu\nu}[\boldsymbol{\chi}] =$$

$$= \int\limits_{\Sigma} d\sigma_\mu \frac{2n}{(D+1)-2n}\Lambda\chi^\mu +$$

$$+ \int\limits_{\Sigma} d\sigma_\mu \left[\overset{\text{sym}}{T}{}^{\mu}{}_{\nu} - \frac{1}{(D+1)-2n} \overset{\text{mod}}{T} \delta^\mu_\nu \right] \chi^\nu \overset{\text{def}}{=}$$

$$\overset{\text{def}}{=} Q_{\text{Vac}}[\boldsymbol{\chi}] + Q_{\text{Sour}}[\boldsymbol{\chi}]. \tag{137}$$

The term $Q_{\text{Vac}}[\boldsymbol{\chi}]$ describes the dynamic characteristics (energy, momentum, ...) of the gravitational vacuum polarized by the matter sources (ϕ-fields). The second term $Q_{\text{Sour}}[\boldsymbol{\chi}]$ describes the dynamic characteristics of disturbances induced by the matter sources under vacuum. One has

Statement 32. *Every of the charges $Q_{\text{Vac}}[\boldsymbol{\chi}]$ and $Q_{\text{Sour}}[\boldsymbol{\chi}]$ is conserved independently on one another; thus, there is no exchange of dynamic characteristics between the gravitational vacuum and the matter.*

Formula (137) allows us to conclude the following.

Consequence 3. *If the cosmological constant is equal to zero, $\Lambda = 0$, then $^{(G)}Q[\boldsymbol{\chi}] \neq 0$ only if $\overset{\text{sym}}{\mathbf{T}} \neq 0$ or $\overset{\text{mod}}{\mathbf{T}} \neq 0$. In such a theory, an arbitrary dynamic characteristic of the gravitational vacuum (free gravitational $\mathbf{g}$- and $\mathbf{T}$-fields without matter) automatically is equal to zero.*

Consequence 4. *The matter influences the gravitational fields by the way that a result of this action brings the information related to the matter dynamic characteristics, which can be identified.*

Consequence 5. *To define the dynamic characteristics of matter sources, which fill the domain Σ, it is enough to describe the gravitational fields only at the boundary S of this domain.*

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КОВАРІАНТНІ ДИФЕРЕНЦІАЛЬНІ
ТОТОЖНОСТІ ТА ЗАКОНИ ЗБЕРЕЖЕННЯ
В МЕТРИЧНИХ ТЕОРІЯХ ГРАВІТАЦІЇ З КРУЧЕННЯМ

Р е з ю м е

В роботі описано нещодавно розроблений авторами загальний явно загальноковаріантний формалізм для побудови законів збереження та величин, що зберігаються, в довільних метричних теоріях гравітації з крученням.