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DEFORMED BOSE GAS MODELS AIMED AT TAKING INTO ACCOUNT BOTH COMPOSITENESS OF PARTICLES AND THEIR INTERACTION[1]

We consider the deformed Bose gas model with the deformation structure function that is the combination of a q-deformation and a quadratically polynomial deformation. Such a choice of the unifying deformation structure function enables us to describe the interacting gas of composite (two-fermionic or two-bosonic) bosons. Using the relevant generalization of the Jackson derivative, we derive a two-parametric expression for the total number of particles, from which the deformed virial expansion of the equation of state is obtained. The latter is interpreted as the virial expansion for the effective description of a gas of interacting composite bosons with some interaction potential.

K e y w o r d s: deformed oscillators, deformed Bose gas model, non-ideal Bose gas, virial expansion, modified Jackson derivative, virial coefficients, composite bosons.

1. Introduction

The treatment of real non-ideal gases in statistical physics or thermodynamics, which involves the interaction and the composite nature of particles, deals usually with special techniques or approximations. To take the interaction between particles into account, the virial expansions [1,2,3] of thermodynamic relations, the equation of state, *etc.* are analyzed at small densities. There also exist several approaches that account for the compositeness of particles and work in the second quantization scheme with exact transformations and results (i.e., without any simplifications), see, e.g., [4, 5, 6]. However, from the practical point of view, the direct account of the compositeness along with interaction between particles is quite complicated and hardly realizable.

The concept of deformed oscillators [7,8,9,10] and deformed Bose-(or Fermi-)gas [11, 12, 13] is one of the possibilities to consider the above-mentioned factors in an effective way (approximately, as a rule). This is achieved by means of the realization (see

some details in [14] and Section 2 below) of a gas of composite particles or a gas of pointlike interacting particles in terms of a specially constructed model of gas of deformed (quasi)particles. The systems of composite two-component Bose-type particles (called quasibosons) can be modeled by the corresponding systems (gases, *etc.*) of q-deformed bosons, as explicitly shown in [15]. There are also some other works [16,17,18,19,20,21] on the effective description of composite particles (excitons, nucleons, molecules, *etc.*) by the use of deformed ones. Recently, the two-fermionic (and also two-bosonic) composite bosons were considered [14, 22] from the viewpoint of their algebraic realization by deformed bosons. Namely, it was shown that the realization on the states is possible for a certain quadratic version of the deformation. Another version of the q-deformed Bose gas model was applied to the systems of interacting particles in [23]. Therein, the corresponding deformed virial expansion in powers of a deviation $\epsilon = q - 1$, where

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q is the deformation parameter, was analyzed. Moreover, for a p, q-deformed Bose gas, the virial expansion was treated in [13], and some interesting implications were drawn. In addition, the thermodynamics of diverse deformed Bose- and Fermi-gases was studied in detail (see, e.g., [12, 24]).

Combining the ideas of [23] and [14, 22], we propose a unified deformed model composed of the q-deformation and the quadratic deformation [14, 22] to effectively describe a gas of interacting composite (two-fermionic or two-bosonic) bosons. Here, the thermodynamical aspects, more specifically the virial expansion of the equation of state, are under study.

2. Preliminaries

Let us mention the earlier obtained results on the effective description of a gas of composite bosons with interaction between particles in terms of deformed oscillators, which serve as our starting point.

2.1. Compositeness aspects

Composite bosons constructed of two fermions (or two bosons) in the second quantization scheme have the creation and annihilation operators in the form [14, 15, 22]

$$A^\dagger_\alpha = \sum_{\mu\nu} \Phi^{\mu\nu}_\alpha a^\dagger_\mu b^\dagger_\nu, \quad A_\alpha = \sum_{\mu\nu} \overline{\Phi^{\mu\nu}_\alpha} b_\nu a_\mu, \qquad (1)$$

where $a^\dagger_\mu$, $b^\dagger_\nu$, and a_μ, b_ν are the creation and annihilation operators for the constituents. The matrices $\Phi^{\mu\nu}_\alpha$ are related to the constituents' wavefunctions. The commutator between A_α and $A^\dagger_\beta$ is of special interest for calculations and the analysis, and it is of the form

$$[A_\alpha, A^\dagger_\beta] = \delta_{\alpha\beta} - \Delta_{\alpha\beta},$$
$$\Delta_{\alpha\beta} = \sum_{\mu\mu'} (\Phi_\beta \Phi^\dagger_\alpha)^{\mu'\mu} a^\dagger_{\mu'} a_\mu + \sum_{\nu\nu'} (\Phi^\dagger_\alpha \Phi_\beta)^{\nu\nu'} b^\dagger_{\nu'} b_\nu.$$

Here, $\Delta_{\alpha\beta}$ gives a deviation from the purely bosonic relation. If it is described merely in terms of A_α and $A^\dagger_\alpha$, then we may speak about the realization of composite bosons' system. Such a situation is very useful from the viewpoint of simplification. The many-body system of composite (two-fermionic) bosons with certain components' wave functions can be realized, as shown in [14, 22], by a deformed Bose gas model with the quadratic structure function $\phi_{\tilde\mu}(n) = (1+\tilde\mu)n - \tilde\mu n^2$ of the deformation and the discrete deformation parameter $\tilde\mu = 1/m$, $m = 1, 2, \dots$. According to the

realization, under certain conditions, the gas of composite bosons can be treated on the states as the corresponding gas of deformed bosons. We remark that, for composite bosons realized by deformed oscillators, the characteristics of the intercomponent entanglement are remarkably expressed [25, 26] through the deformation parameter.

2.2. Account of the interaction between Bose particles

The q-deformed algebra with the structure function $\phi_q(n) = \frac{1-q^n}{1-q} \equiv [n]_q$ was used in [23] to describe an interacting gas of Bose particles. Therein, the effects of the interaction between particles of a Bose gas were incorporated, by using a deformed version of the model, and were explicitly demonstrated with the use of q-deformed thermodynamic relations. In particular, this concerns the expression for the specific volume given in terms of the fugacity, as well as the equation of state. We note that the deformed relations (e.g., for the total number of particles) can be obtained [23] by the use of the q-deformed or Jackson derivative instead of the non-deformed one. A somewhat more specific outline of the description of the systems of interacting particles using the q-deformation is given in Section 3.

In view of the already-mentioned fact of the realizability of composite bosons, we utilize a deformed Bose gas with (the quadratic) structure function $\phi_{\tilde\mu}(n)$ to find effective thermodynamic relations or functions for the ideal (non-interacting) *quantum gas of composite bosons*. Those include the deformed (thus, depending on $\tilde\mu$) virial expansion of the equation of state. To take the interaction between particles simultaneously with their compositeness into account, the two structure functions $\phi_{\tilde\mu}(n)$ and $\phi_q(n)$ can be combined to give the unifying deformation structure function $\phi_{\tilde\mu,q}(n) = (1+\tilde\mu)[n]_q - \tilde\mu([n]_q)^2$. The latter will play a central role in our treatment. An alternative version of the unifying structure function will be also discussed.

In Sections 3 and 4, these two separately treated situations are overviewed in more details.

3. Systems of Interacting Particles Described by a q-Deformed Algebra

Let us give a brief overview of the results of [23] concerning the interpretation of interacting many-boson systems using q-deformed oscillators (q-bosons) with-

out interaction. Each interpretation is based on the assumption that a suitably chosen q-deformed thermodynamic or statistical relation for non-interacting pointlike particles' system can be applied (at least within some approximation) to an interacting many-particle system with certain interaction.

In work [23], a system of interacting Bose particles described by the q-oscillator algebra is considered. As a starting point, the series expansion of the basic number $[N]_q$ in terms of the "deviation" $\epsilon \equiv q-1$ is taken:

$$[N]_q = \Big(1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{3} - \frac{\epsilon^3}{4} + ...\Big)N + \frac{\epsilon}{2!}\Big(1 - \epsilon + \frac{11}{12}\epsilon^2 - ...\Big) \times$$
$$\times N^2 + \frac{\epsilon^2}{3!}\Big(1 - \frac{3}{2}\epsilon + ...\Big)N^3 + O(N^4). \quad (2)$$

This expansion can be naturally interpreted as the one incorporating the interparticle interaction. According to [23, 27], the contributions from the interaction can be viewed either in terms of N, N^2, N^3, ... or in terms of ϵ, ϵ^2, ϵ^3, ... In the latter case (for which the contributions from the interaction are considered as those contained in the terms depending on the deformation parameter), expansion (2) is rewritten in the "perturbative" form

$$[N]_q = \sum_{i=0}^{\infty} \frac{\epsilon^i}{(i+1)!} \prod_{j=0}^{i}(N - j) =$$
$$= N + \frac{\epsilon}{2!}N(N-1) + \frac{\epsilon^2}{3!}N(N-1)(N-2) + O(\epsilon^3). \quad (3)$$

Another relevant quantity needed to be considered is the Hamiltonian $H_\epsilon = \frac{1}{2}\omega([N+1]_q + [N]_q)$ for a system of q-bosons (single-mode case). The expansion of H_ϵ in powers of ϵ linked with the deformation parameter q is given by [23]

$$H_\epsilon = H_0 + \omega \sum_{i=1}^{\infty} \epsilon^i \frac{(2N+1-i)}{2(i+1)!} \prod_{j=0}^{i-1}(N-j). \quad (4)$$

Similarly to (3), the terms of the first and higher orders in ϵ in series (4) are again interpreted as those arising from the interaction. To emphasize the physical meaning, they constitute nothing but the interaction Hamiltonian.

Remark that a direct relation of the deformation parameter q to the parameter(s) of an equivalent interaction Hamiltonian for many-particle systems was not explicitly derived in [23], and that constitutes a nontrivial, but important problem.

The idea of a q-deformed non-interacting (ideal) many-body system as a non-deformed, but interacting system is best illustrated by means of the virial expansion, at least at this stage. Basing on the deformed thermodynamic relations (particularly, the specific volume v as a function of the fugacity z and the deformed equation of state) derived in [11, 12] for a gas of q-bosons, the q-deformed virial expansion was obtained in the form

$$\frac{Pv}{k_B T} = \sum_{k=1}^{\infty} a_k(\epsilon)\Big(\frac{\lambda^3}{v}\Big)^{k-1}, \quad (5)$$

where $a_k(\epsilon)$ denote the corresponding virial coefficients. The first several coefficients given up to ϵ^3 are [23]:

$$a_1(\epsilon) = 1, \quad a_2(\epsilon) = -\frac{1}{4\sqrt{2}} - \frac{1}{48\sqrt{2}}\epsilon^2(1-\epsilon) + O(\epsilon^4),$$

$$a_3(\epsilon) = -\Big(\frac{2}{9\sqrt{3}} - \frac{1}{8}\Big) - \Big(\frac{1}{18\sqrt{3}} - \frac{1}{48}\Big)\epsilon^2(1-\epsilon) + O(\epsilon^4).$$

Similarly to the previous cases, $\epsilon \neq 0$ corrections to the standard virial coefficients of the ideal quantum Bose gas can be interpreted as those arising from some explicitly accounted interaction (certain interaction potential in the Hamiltonian). Note that, on the other hand, this interaction can be viewed in terms of q-bosons as such that arises from (the change of) their quantum statistics and, thus, is of the quantum statistical origin. This is in some analogy with the effective interaction due to the Pauli exclusion principle in the case of pure fermions. Thus, the many-particle system under study is effectively described (and interpreted) in terms of the deformed one, i.e., by using (quasi)particles with a non-standard statistics.

4. Deformed Bose Gas which Accounts for the Compositeness of Particles

In this section, we consider a $\tilde{\mu}$-deformed Bose gas with the quadratic structure function

$$\phi_{\tilde{\mu}}(n) \equiv [n]_{\tilde{\mu}} = (1 + \tilde{\mu})n - \tilde{\mu}n^2, \quad (6)$$

which was shown to realize (under certain conditions, see [14, 22]) the gas of two-fermion composite Bose-like particles. For such a deformation, we now derive the deformed virial expansion of the equation of state along with a few first virial coefficients, by using the concept of $\tilde{\mu}$-deformed derivative.

As a starting point, we take the logarithm of the Bose gas grand partition function:

$$\ln Z = -\sum_i \ln(1 - ze^{-\beta \varepsilon_i}). \tag{7}$$

Instead of the Jackson derivative $\mathcal{D}^q$ used in [23] or its p,q-extension $\mathcal{D}^{(p,q)}$ exploited in [13], we apply the following $\tilde{\mu}$-deformed extension of $\frac{d}{dz}$ defined through its action on monomials z^k:

$$\mathcal{D}_z^{(\tilde{\mu})} z^k = ((1 + \tilde{\mu})k - \tilde{\mu}k^2)z^{k-1}.$$

This can also be obtained by the action of the operator

$$z\mathcal{D}_z^{(\tilde{\mu})} = \left[z\frac{d}{dz}\right]_{\tilde{\mu}} = \left((1 + \tilde{\mu})\left(z\frac{d}{dz}\right) - \tilde{\mu}\left(z\frac{d}{dz}\right)^2\right)$$

on monomials. Moreover, the latter acts in an obvious way on an arbitrary smooth function $f(z) = \sum_{j=0}^{\infty} c_j z^j$.

For the grand partition function logarithm, we have the following expansion in z:

$$\ln Z = \frac{\pi\sqrt{\pi}V}{(2\pi)^3}\left(\frac{2m}{\beta\hbar^2}\right)^{3/2} \sum_{n=1}^{\infty} \frac{z^n}{n^{5/2}} =$$

$$= \frac{V}{\lambda^3} \sum_{n=1}^{\infty} \frac{z^n}{n^{5/2}}$$

where $\lambda = h/(2\pi m k_B T)^{1/2}$ is the thermal wavelength. Then, in case of the deformed picture, we obtain that the number of particles $N^{(\tilde{\mu})}$ according to the above consideration is

$$N^{(\tilde{\mu})} = \left[z\frac{d}{dz}\right]_{\tilde{\mu}} \ln Z \equiv z\mathcal{D}_z^{(\tilde{\mu})} \ln Z = \frac{V}{\lambda^3} \sum_{n=1}^{\infty} [n]_{\tilde{\mu}} \frac{z^n}{n^{5/2}} \tag{8}$$

or, equivalently,

$$\frac{\lambda^3}{v} = \sum_{n=1}^{\infty} [n]_{\tilde{\mu}} \frac{z^n}{n^{5/2}}, \tag{9}$$

where the specific volume $v = \frac{V}{N^{(\tilde{\mu})}}$ is introduced. The equation of state for a quantum gas of non-interacting, but composite particles

$$\frac{PV}{k_B T} = \ln Z = \frac{V}{\lambda^3}\left(z + \frac{z^2}{2^{5/2}} + \frac{z^3}{3^{5/2}} + \frac{z^4}{4^{5/2}} + \frac{z^5}{5^{5/2}} + ...\right) \tag{10}$$

is now deformed in the following way:

$$\frac{PV}{k_B T} = \ln Z^{(\tilde{\mu})}. \tag{11}$$

The desired $\tilde{\mu}$-deformed partition function $Z^{(\tilde{\mu})}$ is defined from the deformed analog of the relation $N = \left(z\frac{d}{dz}\right)\ln Z$, i.e., from $N^{(\tilde{\mu})} = \left(z\frac{d}{dz}\right)\ln Z^{(\tilde{\mu})}$ or

$$\ln Z^{(\tilde{\mu})} = \left(z\frac{d}{dz}\right)^{-1} N^{(\tilde{\mu})}. \tag{12}$$

As a result, we obtain the following expansion for the deformed equation of state:

$$\frac{PV}{k_B T} = \ln Z^{(\tilde{\mu})} =$$

$$= \frac{V}{\lambda^3}\left(z + \frac{[2]_{\tilde{\mu}}}{2^{7/2}}z^2 + \frac{[3]_{\tilde{\mu}}}{3^{7/2}}z^3 + \frac{[4]_{\tilde{\mu}}}{4^{7/2}}z^4 + \frac{[5]_{\tilde{\mu}}}{5^{7/2}}z^5 + ...\right). \tag{13}$$

In order to deduce the corresponding virial expansion, we have to find the function $z(\lambda^3/v)$ in the form of a series, through inverting (9). For this purpose, we expand $z(\lambda^3/v) = z(\lambda^3 N/V)$ in a Taylor series as

$$z(\lambda^3/v) = z'_N|_{N=0}N + \frac{z''_N}{2!}\Big|_{N=0}N^2 + \frac{z'''_N}{3!}\Big|_{N=0}N^3 +$$

$$+ \frac{z_N^{(IV)}}{4!}\Big|_{N=0}N^4 + \frac{z_N^{(V)}}{5!}\Big|_{N=0}N^5 + ... \tag{14}$$

The derivatives $z_N^{(l)}|_{N=0}$, $l = 1, 2, ...$, can be expressed through analogous derivatives $N_z^{(r)}|_{z=0}$, $r = 1, 2, ...$. From (8), we find the r-th order derivative of $N^{(\tilde{\mu})}$ with respect to z at $z = 0$:

$$N_z^{(r)}\Big|_{z=0} = \frac{\pi\sqrt{\pi}V}{(2\pi)^3}\left(\frac{2m}{\beta\hbar^2}\right)^{3/2} r! \frac{[r]_{\tilde{\mu}}}{r^{5/2}} = \frac{V}{\lambda^3} r! \frac{[r]_{\tilde{\mu}}}{r^{5/2}}. \tag{15}$$

For $z'_N|_{N=0}, ..., z_N^{(V)}|_{N=0}$, we infer

$$z'_N|_{N=0} = \frac{1}{N'_z}\Big|_{N=0} = \frac{\lambda^3}{V}\frac{1}{[1]_{\tilde{\mu}}},$$

$$z''_N|_{N=0} = -\frac{N''_z}{(N'_z)^3} = -\left(\frac{\lambda^3}{V}\right)^2 \frac{2!}{2^{5/2}}\frac{[2]_{\tilde{\mu}}}{[1]_{\tilde{\mu}}^3},$$

$$z'''_N|_{N=0} = \left(-\frac{N'''_z}{(N'_z)^5} + 3\frac{(N''_z)^2}{(N'_z)^5}\right)\Big|_{z=0} =$$

$$= \left(\frac{\lambda^3}{V}\right)\left(3\frac{(2!)^2}{2^5}\frac{[2]_{\tilde{\mu}}}{[1]_{\tilde{\mu}}^5} - \frac{3!}{3^{5/2}}\frac{[3]_{\tilde{\mu}}}{[1]_{\tilde{\mu}}^4}\right),$$

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Substituting these derivatives into (14), we find

$$z(\lambda^3/v) = \frac{1}{[1]_{\tilde\mu}}\frac{\lambda^3}{v} - \frac{1}{2^{5/2}}\frac{[2]_{\tilde\mu}}{[1]_{\tilde\mu}^3}\left(\frac{\lambda^3}{v}\right)^2 +$$
$$+\left(\frac{1}{2^4}\frac{[2]_{\tilde\mu}^3}{[1]_{\tilde\mu}^5} - \frac{1}{3^{5/2}}\frac{[3]_{\tilde\mu}}{[1]_{\tilde\mu}^4}\right)\left(\frac{\lambda^3}{v}\right)^3 +$$
$$+\left(-\frac{1}{4^{5/2}}\frac{[4]_{\tilde\mu}}{[1]_{\tilde\mu}^5} + \frac{5}{2^{5/2}3^{5/2}}\frac{[2]_{\tilde\mu}[3]_{\tilde\mu}}{[1]_{\tilde\mu}^6} - \frac{5}{2^{15/2}}\frac{[2]_{\tilde\mu}^3}{[1]_{\tilde\mu}^7}\right)\left(\frac{\lambda^3}{v}\right)^4 +$$
$$+\left(-\frac{1}{5^{5/2}}\frac{[5]_{\tilde\mu}}{[1]_{\tilde\mu}^6} + \frac{3}{2^{13/2}}\frac{[2]_{\tilde\mu}[4]_{\tilde\mu}}{[1]_{\tilde\mu}^7} + \frac{1}{3^4}\frac{[3]_{\tilde\mu}^2}{[1]_{\tilde\mu}^7} - \right.$$
$$\left.-\frac{7}{2^5 3^{3/2}}\frac{[2]_{\tilde\mu}^2[3]_{\tilde\mu}}{[1]_{\tilde\mu}^8} + \frac{7}{2^9}\frac{[2]_{\tilde\mu}^4}{[1]_{\tilde\mu}^9}\right)\left(\frac{\lambda^3}{v}\right)^5 + ... \quad (16)$$

Plugging this expression into (13) we arrive at the desired virial expansion depending on the parameter $\tilde\mu$, which corresponds to a *non-interacting gas of composite bosons*:

$$\frac{P}{k_{\rm B}T} = v^{-1}\left\{\sum_{k=1}^{\infty} V_k(\tilde\mu)\left(\frac{\lambda^3}{v}\right)^{k-1}\right\} =$$
$$= v^{-1}\left\{1 - \frac{[2]_{\tilde\mu}}{2^{7/2}}\frac{\lambda^3}{v} + \left(\frac{[2]_{\tilde\mu}^2}{2^5} - \frac{2[3]_{\tilde\mu}}{3^{7/2}}\right)\left(\frac{\lambda^3}{v}\right)^2 +\right.$$
$$+\left(-\frac{3[4]_{\tilde\mu}}{4^{7/2}} + \frac{[2]_{\tilde\mu}[3]_{\tilde\mu}}{2^{5/2}3^{3/2}} - \frac{5[2]_{\tilde\mu}^3}{2^{17/2}}\right)\left(\frac{\lambda^3}{v}\right)^3 +$$
$$+\left(-\frac{4[5]_{\tilde\mu}}{5^{7/2}} + \frac{[2]_{\tilde\mu}[4]_{\tilde\mu}}{2^{11/2}} - \frac{2[3]_{\tilde\mu}^3}{3^5} - \frac{[2]_{\tilde\mu}^2[3]_{\tilde\mu}}{2^3 3^{3/2}} + \frac{7[2]_{\tilde\mu}^4}{2^{10}}\right)\times$$
$$\times\left(\frac{\lambda^3}{v}\right)^4 + ...\right\}. \quad (17)$$

Here, $V_k(\tilde\mu)$, $k = 1, 2, ...$, are the virial coefficients. Note that, unlike (5) and [23], the obtained virial coefficients are found exactly.

Let us make the observation that the second virial coefficient $V_2 = -\frac{[2]_{\tilde\mu}}{2^{7/2}} = -\frac{1-\tilde\mu}{2^{5/2}}$ vanishes, when $\tilde\mu$ reaches 1. This can be interpreted as a compensation, at $\tilde\mu = 1$, of the compositeness effects, against the quantum-statistical many-particle effects, so that the quantum gas of composite bosons behaves itself like a classical gas of pointlike particles, at least to λ^3/v-terms. An analogous statement may be deduced also for the next virial coefficients V_3, V_4, ...

Like before, the modified virial coefficients, in the absence of the explicit interaction between composite bosons in the Hamiltonian, can be viewed, in the current interpretation, as those reflecting an effective interaction of the quantum statistical origin between bosons (like an effective attraction (respectively, a repulsion) of the quantum origin between bosons (respectively, fermions)).

5. Account for Both Compositeness and Interaction of Particles

To consider the interaction jointly with the compositeness of the particles of a gas, we use the function

$$\phi_{\tilde\mu,q}(n) = (1+\tilde\mu)[n]_q - \tilde\mu([n]_q)^2 \quad (18)$$

and the respective $(\tilde\mu, q)$-extension of the Jackson derivative

$$z\mathcal{D}_z^{(\tilde\mu,q)} = \left((1+\tilde\mu)\left[z\frac{d}{dz}\right]_q - \tilde\mu\left[z\frac{d}{dz}\right]_q^2\right).$$

This latter acts in an obvious way on monomials and on an arbitrary smooth function $f(z) = \sum_{j=0}^{\infty} c_j z^j$.

We have to remark that, similarly to the Hamiltonian H_ϵ in (4), the two-parameter Hamiltonian $H_{\tilde\mu,\epsilon} = \frac{1}{2}\omega\left([N+1]_{\tilde\mu,q} + [N]_{\tilde\mu,q}\right)$ for the system of $\tilde\mu, q$-bosons (single-mode case) can be split, in the present case, into the Hamiltonian H_0 (with no interaction and no compositeness) and the double-interaction Hamiltonian $H_1(\epsilon, \tilde\mu; N)$ that depends on N and is the double series in $\tilde\mu$ and $\epsilon = q - 1$. The compositeness "causes" the extra amount of an effective interaction of the quantum origin, on the equal footing with that encoded in $q = 1 + \epsilon$.

Proceeding along lines similar to the previous sections, we can obtain the virial expansion in the case under consideration, using the deformation structure function $\phi_{\tilde\mu,q}(n)$ and the deformed derivative $\mathcal{D}_z^{(\tilde\mu,q)}$. We note that $\mathcal{D}_z^{(\tilde\mu)}$ and $\mathcal{D}_z^{(\tilde\mu,q)}$ have similar definitions in terms of the respective structure functions, and $[1]_{\tilde\mu} = [1]_{\tilde\mu,q} = 1$. So, we infer the desired formulas by making the replacement $[k]_{\tilde\mu} \to [k]_{\tilde\mu,q}$, $k = 2, 3, 4, 5, ...$ in (17). The resulting virial expansion of the equation of state takes the form

$$\frac{Pv}{k_{\rm B}T} = \left\{\sum_{k=1}^{\infty} V_k(\tilde\mu,q)\left(\frac{\lambda^3}{v}\right)^{k-1}\right\} = \left\{1 - \frac{[2]_{\tilde\mu,q}}{2^{7/2}}\frac{\lambda^3}{v} +\right.$$
$$+\left(\frac{[2]_{\tilde\mu,q}^2}{2^5} - \frac{2[3]_{\tilde\mu,q}}{3^{7/2}}\right)\left(\frac{\lambda^3}{v}\right)^2 + \left(-\frac{3[4]_{\tilde\mu,q}}{4^{7/2}} + \frac{[2]_{\tilde\mu,q}[3]_{\tilde\mu,q}}{2^{5/2}3^{3/2}} -\right.$$
$$\left.-\frac{5[2]_{\tilde\mu,q}^3}{2^{17/2}}\right)\left(\frac{\lambda^3}{v}\right)^3 + \left(-\frac{4[5]_{\tilde\mu,q}}{5^{7/2}} + \frac{[2]_{\tilde\mu,q}[4]_{\tilde\mu,q}}{2^{11/2}} -\right.$$
$$\left.-\frac{2[3]_{\tilde\mu,q}^3}{3^5} - \frac{[2]_{\tilde\mu,q}^2[3]_{\tilde\mu,q}}{2^3 3^{3/2}} + \frac{7[2]_{\tilde\mu,q}^4}{2^{10}}\right)\left(\frac{\lambda^3}{v}\right)^4 + ...\right\}. \quad (19)$$

It is now natural to interpret the virial expansion (19) as the effective one corresponding to the interacting gas of composite (quasi)particles. The information about the interaction and the composite structure is merely encoded in the two deformation parameters q and $\tilde\mu$, respectively. In the limiting case

$\tilde{\mu} = 0$, expansion (19) accounts only for the interaction between the particles; likewise, when $q = 1$, formula (19) should be interpreted as accounting only for the compositeness of particles. When $q \neq 1$ and $\tilde{\mu} \neq 0$, expression (19) incorporates the both mentioned factors of non-ideality of a Bose gas jointly.

Consider a deviation of the second virial coefficient V_2 from its non-deformed value in the limiting cases where $\tilde{\mu} = 0$ or respectively $q = 1$, namely,

$$\Delta V_2 \xrightarrow{\tilde{\mu}=0} \frac{1-q}{2^{7/2}} \quad \text{resp.} \quad \Delta V_2 \xrightarrow{q=1} \frac{\tilde{\mu}}{2^{5/2}}. \tag{20}$$

Remark that, for these deviations, the explicit formulas through interaction potentials could be obtained (this is, however, a rather non-trivial problem being beyond the scope of this paper) from the corresponding expressions for the virial coefficients (see, e.g., [1, 2, 3]). By comparing those explicit formulas with (20), it is possible to relate the deformation parameters q and $\tilde{\mu}$ with the parameters in the Hamiltonian of interaction between composite bosons and inside them (i.e., between the constituents).

6. An Alternative Description Using Other Structure Functions

The deformation structure function $\phi_{\tilde{\mu},q}(N) \equiv \phi_{\tilde{\mu}}(\phi_q(N))$ from (18) is not the only possible one. Indeed, some other structure functions can be used to take the compositeness along with the interaction into account. For instance, the function

$$\phi_{q,\tilde{\mu}}(N) \equiv \phi_q(\phi_{\tilde{\mu}}(N)) = \frac{1-q^{[N]_{\tilde{\mu}}}}{1-q} \equiv \frac{1-q^{(1+\tilde{\mu})N-\tilde{\mu}N^2}}{1-q} \tag{21}$$

also possesses the limiting cases $\phi_{q,\tilde{\mu}}|_{q=1}(N) = \phi_{\tilde{\mu}}(N)$, $\phi_{q,\tilde{\mu}}|_{\tilde{\mu}=0}(N) = \phi_q(N)$. Thus, it can be used as another admissible structure function instead of $\phi_{\tilde{\mu},q}(N)$. Moreover, from the structure functions $\phi_{\tilde{\mu},q}(N)$ and $\phi_{q,\tilde{\mu}}(N)$, we can form the whole one-parameter family of structure functions

$$\phi_t(N) \equiv \phi_{t;\tilde{\mu},q}(N) = t\phi_{\tilde{\mu},q}(N) + (1-t)\phi_{q,\tilde{\mu}}, \tag{22}$$

which are related to the above-mentioned one-parameter limits (20). The corresponding virial coefficients, by exploiting the analogy to (19), are the following:

$$V_1 = 1, \ V_2(t;\tilde{\mu},q) = -\frac{\phi_{t;\tilde{\mu},q}(2)}{2^{7/2}},$$

$$V_3(t;\tilde{\mu},q) = \frac{(\phi_{t;\tilde{\mu},q}(2))^2}{2^5} - \frac{2\phi_{t;\tilde{\mu},q}(3)}{3^{7/2}},$$

$$V_4(t;\tilde{\mu},q) =$$
$$-\frac{3\phi_{t;\tilde{\mu},q}(4)}{4^{7/2}} + \frac{\phi_{t;\tilde{\mu},q}(2)\phi_{t;\tilde{\mu},q}(3)}{2^{5/2}3^{3/2}} - \frac{5(\phi_{t;\tilde{\mu},q}(2))^3}{2^{17/2}}, \tag{23}$$

$$V_5(t;\tilde{\mu},q) =$$
$$-\frac{4\phi_{t;\tilde{\mu},q}(5)}{5^{7/2}} + \frac{\phi_{t;\tilde{\mu},q}(2)\phi_{t;\tilde{\mu},q}(4)}{2^{11/2}} - \frac{2(\phi_{t;\tilde{\mu},q}(3))^3}{3^5} -$$
$$-\frac{(\phi_{t;\tilde{\mu},q}(2))^2\phi_{t;\tilde{\mu},q}(3)}{2^3 3^{3/2}} + \frac{7(\phi_{t;\tilde{\mu},q}(2))^4}{2^{10}}.$$

Of course, the further study on more physical (phenomenological) grounds is needed in order that the preference could be made for one deformed model, see (19), with respect to those contained in (22).

7. Conclusions

Basing on [14, 22] and [23], the specially designed two-parameter deformed Bose gas model capable to (effectively) describe the interacting gas of composite bosons is constructed. The specific deformation structure function, which characterizes the deformed bosons (oscillators) of this model, is constructed by combining the previously studied ones from [14, 22] and [23]. The "building block" structure functions are the quadratic polynomial structure function of deformation and the q-deformed structure function of the Arik–Coon type which provides the effective description of an interacting gas of elementary bosons.

For the proposed deformed Bose gas model, the corresponding deformed virial expansion is found along with the first five virial coefficients and interpreted as the virial expansion accounting for both the interaction of composite bosons and their composite structure. The thermodynamic relations for the deformed Bose gas model (including the equation of state), which were utilized in the process of derivation of its virial expansion, are obtained, by using an appropriate generalization of the Jackson derivative for the concerned unified deformation.

Some alternative deformation structure functions for the deformed Bose gas model needed for the effective description are proposed. Those include the composition of the q-deformed structure function and the quadratic one, taken in the opposite order, see (21), and the t-parameter family interpolating between the both, see (22)–(23).

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Note that it is also of interest to examine the correlation function intercepts for the considered deformed Bose gas model in conjunction with the above-mentioned interpretation and to make comparison with experimental data (e.g., those on the π^--mesons, the known composites). For some other deformed Bose gas models, the comparison of π-mesonic correlation functions intercepts of the second (and third) order with experimental data is already done in a few previous papers, see, e.g., [28, 29, 30].

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МОДЕЛІ ДЕФОРМОВАНОГО БОЗЕ-ГАЗУ,
ЗДАТНІ ОДНОЧАСНО ВРАХУВАТИ СКЛАДЕНУ
СТРУКТУРУ ЧАСТИНОК ТА ЇХ ВЗАЄМОДІЮ

Р е з ю м е

В даній роботі ми розглядаємо модель деформованого бозе-газу зі структурною функцією, яка є комбінацією q-деформації та квадратичної поліноміальної деформації. Такий вибір уніфікованої структурної функції деформації дає можливість описати взаємодіючий газ складених (двоферміонних чи двобозонних) бозонів. Використовуючи відповідне узагальнення похідної Джексона, отримано деформований віріальний розклад. Останній інтерпретується як віріальний розклад для газу взаємодіючих складених бозонів із деяким потенціалом взаємодії.