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SEMIDISCRETE INTEGRABLE SYSTEMS INSPIRED BY THE DAVYDOV–KYSLUKHA MODEL

Intending to mimicry certain physical features of the Davydov–Kyslukha exciton-phonon system, we have suggested four distinct combinations of ansätze for matrix-valued Lax operators capable to generate a number of semidiscrete integrable nonlinear systems in the framework of the zero-curvature approach.

Dealing with Taylor-like ansätze for Lax operators, two types of general nonlinear integrable systems on infinite quasione-dimensional regular lattices are proposed. In accordance with the Mikhailov reduction group theory, both general systems turn out to be underdetermined, thereby permitting numerous reduced systems written in terms of true field variables. Each of the obtained reduced systems can be considered as an integrable version of two particular coupled subsystems and demonstrates the symmetry under the space and time reversal ($\mathcal{PT}$-symmetry). Thus, we have managed to unify the Toda-like vibrational subsystem and the self-trapping lattice subsystem into the single integrable system, thereby substantially extending the range of realistic physical problems that can be rigorously modeled. The several lowest conserved densities associated with either of the possible infinite hierarchies of local conservation laws are found explicitly in terms of prototype field functions.

When considering the Laurent-like ansätze for Lax operators, we have isolated four new semidiscrete nonlinear integrable systems interesting for physical applications. Thus, we have coupled the Toda-like subsystem with the induced-trapping subsystem of $\mathcal{PT}$-symmetric excitations. Another integrable system is set up as the subsystem of Frenkel-like excitons coupled with the subsystem of essentially nontrivial vibrations. We also have revealed the integrable system of two self-trapping subsystems coupled together by means of a mutually induced nonlinearity. At last, we have obtained the integrable system, where the Toda-like subsystem and the self-trapping subsystem are coupled akin to a charged particle with an electromagnetic field. In so doing, the vector-potential part of the Hamiltonian function is appeared as the density of excitations in the self-trapping subsystem. Each of the proposed systems admits the clear Hamiltonian representation characterized by the two pairs of canonical field variables with the standard (undeformed) Poisson structure. Several general local conserved densities having been found in the framework of a generalized direct procedure are presented explicitly. These conserved densities are readily adaptable to any integrable system under consideration.

Keywords: Davydov–Kyslukha model, Toda lattice, self-trapping system, integrable coupling, $\mathcal{PT}$-symmetry.

1. Introduction

Approximately forty years ago, Davydov and Kyslukha proposed the nonlinear model [1–3] suitable to describe the soliton-like waves supported by the interaction between the exciton (electron) and phonon (vibration) subsystems on quasione-dimensional regular lattices both of the synthetic [4, 5] and natural [6, 7] origin. One of the promising applications of the Davydov–Kyslukha model was claimed to be the transport of energy and charge in biological macromolecules [7, 8] invoked to resolve the so-called crisis in bioenergetics [9].

The Davydov–Kyslukha model is an essentially classical dynamical system characterized by the Hamiltonian function [8]

$$\mathscr{H} = -J \sum_{m=-\infty}^{\infty} [\psi^*(m)\psi(m+1) + \psi^*(m)\psi(m-1)] +$$
$$+\chi \sum_{m=-\infty}^{\infty} [\beta(m+1) - \beta(m-1)]\psi^*(m)\psi(m)+$$

$$+\sum_{m=-\infty}^{\infty}\left\{\pi^2(m)/2M+(w/2)[\beta(m)-\beta(m-1)]^2\right\} \quad (1.1)$$

with $\psi^*(n)$ and $\psi(n)$ serving as the complex conjugate probability amplitudes to find an exciton on the n-th site of the lattice while $\pi(n)$ and $\beta(n)$ to be, respectively, the momentum and the coordinate associated with a displacement of the n-th structural element (atom or molecule) from its equilibrium position. In so doing, each pair of the quantities $\psi^*(n)$, $\psi(n)$ and $\pi(n)$, $\beta(n)$ acquires the status of canonical field variables, inasmuch as having been governed by the respective pair of Hamiltonian equations

$$+i\hbar d\psi(n)/dt=\partial\mathscr{H}/\partial\psi^*(n), \quad (1.2)$$

$$-i\hbar d\psi^*(n)/dt=\partial\mathscr{H}/\partial\psi(n) \quad (1.3)$$

and

$$d\pi(n)/dt=-\partial\mathscr{H}/\partial\beta(n), \quad (1.4)$$

$$d\beta(n)/dt=+\partial\mathscr{H}/\partial\pi(n). \quad (1.5)$$

The Davydov–Kyslukha system is not integrable either in the Lax [10–12] or Liuoville [11, 12] sense and, as a consequence, does not permit a rigorous mathematical treatment. In this respect, it would be of interest to find out some integrable systems resembling the Davydov–Kyslukha one from the physical standpoint. Our recent numerous attempts of such findings are collected in the present paper.

As a matter of fact, all proposed integrable systems can be divided into two large groups depending on whether the particular system has been obtained in the framework of Taylor-like or Laurent-like ansätze for auxiliary spectral and evolution operators responsible for the system zero-curvature representation. Consequently, the bulk of the paper will be composed by two essentially autonomous parts (Sections 2–5 as the first part and Sections 6–9 as the second one) dealing with two distinct groups of integrable systems.

2. Two Types of Taylor-Like Ansätze for the Auxiliary Lax Operators

The main tool of our consideration will be a zero-curvature equation [11]

$$\dot{L}(n|\lambda)=A(n+1|\lambda)L(n|\lambda)-L(n|\lambda)A(n|\lambda), \quad (2.1)$$

which is able, in the case of properly chosen spectral $L(n|\lambda)$ and evolution $A(n|\lambda)$ operators, to provide the zero-curvature representation for some new integrable semidiscrete nonlinear system on a quasionedimensional lattice. Here, the integer n marks the ordinal number of a unit cell on the regular quasionedimensional infinite lattice, and λ stands for the time-independent spectral parameter. The overdot on the left-hand side of the zero-curvature equation (2.1) denotes the derivative with respect to the time τ.

The three forthcoming sections (Sections 3–5) will be devoted to the integrable systems arising from spectral and evolution operators taken as some 3×3 square matrices given by truncated Taylor series with respect to the spectral parameter. In due course of empirical searching, we have managed to discover two pairs of nontrivial ansätze for the Lax operators $L(n|\lambda)$ and $A(n|\lambda)$ capable to include additional degrees of freedom as compared with those typical of the integrable Toda [13–17] or integrable nonlinear self-trapping [18–23] lattice systems and thus to mimicry some features of the Davydov–Kyslukha one. These two pairs of ansätze are, respectively, as follows:

$$L(n|\lambda)=\begin{pmatrix} f_{11}(n)+\lambda^2h_{11}(n) & \lambda g_{12}(n) & f_{13}(n)\\ \lambda g_{21}(n) & f_{22}(n) & 0\\ f_{31}(n) & 0 & f_{33}(n)\end{pmatrix}, \quad (2.2)$$

$$A(n|\lambda)=\begin{pmatrix} a_{11}(n) & 0 & a_{13}(n)\\ 0 & a_{22}(n) & \lambda b_{23}(n)\\ a_{31}(n) & \lambda b_{32}(n) & a_{33}(n)+\lambda^2c_{33}(n)\end{pmatrix} \quad (2.3)$$

and

$$L(n|\lambda)=\begin{pmatrix} f_{11}(n)+\lambda^2h_{11}(n) & \lambda g_{12}(n) & f_{13}(n)\\ \lambda g_{21}(n) & f_{22}(n) & 0\\ f_{31}(n) & 0 & f_{33}(n)\end{pmatrix}, \quad (2.4)$$

$$A(n|\lambda)=\begin{pmatrix} a_{11}(n)+\lambda^2c_{11}(n) & \lambda b_{12}(n) & a_{13}(n)\\ \lambda b_{21}(n) & a_{22}(n) & 0\\ a_{31}(n) & 0 & a_{33}(n)\end{pmatrix}. \quad (2.5)$$

The systems linked with the first two ansätze (2.2) and (2.3) will be referred to as the primary systems of the first type, while the systems connected with the second two ansätze (2.4) and (2.5) will be called the primary systems of the second type. Here, we bear in mind the possibility to extend the truncated Taylor series for either evolutionary ansatz (2.3) or (2.5) in such a way as to recover any admissible integrable system from an infinite hierarchy initiated by the respective primary one.

3. Systems (with Taylor-Like Ansätze for Lax Operators) of the First Type

Addressing to systems of the first type, we substitute the pertinent ansätze (2.2) and (2.3) for matrices $L(n|\lambda)$ and $A(n|\lambda)$ into the zero-curvature equation (2.1). The obtained set of equations shows that the prototype field function $f_{33}(n)$ must be equalized to zero otherwise the theory acquires an essentially nonlocal character.

Therefore, by assuming

$$f_{33}(n) \equiv 0, \tag{3.1}$$

the results of calculations are as follows:

$$\dot{h}_{11}(n) = a_{11}(n+1)h_{11}(n) - h_{11}(n)a_{11}(n), \tag{3.2}$$

$$\dot{f}_{11}(n) = a_{11}(n+1)f_{11}(n) + a_{13}(n+1)f_{31}(n) - f_{11}(n)a_{11}(n) - f_{13}(n)a_{31}(n), \tag{3.3}$$

$$\dot{f}_{22}(n) = a_{22}(n+1)f_{22}(n) - f_{22}(n)a_{22}(n), \tag{3.4}$$

$$\dot{g}_{12}(n) = a_{11}(n+1)g_{12}(n) - g_{12}(n)a_{22}(n) - f_{13}(n)b_{32}(n), \tag{3.5}$$

$$\dot{g}_{21}(n) = a_{22}(n+1)g_{21}(n) + b_{23}(n+1)f_{31}(n) - g_{21}(n)a_{11}(n), \tag{3.6}$$

$$\dot{f}_{13}(n) = a_{11}(n+1)f_{13}(n) - f_{11}(n)a_{13}(n) - f_{13}(n)a_{33}(n), \tag{3.7}$$

$$\dot{f}_{31}(n) = a_{31}(n+1)f_{11}(n) + a_{33}(n+1)f_{31}(n) - f_{31}(n)a_{11}(n), \tag{3.8}$$

where

$$c_{33}(n) = c_{33}, \tag{3.9}$$

$$a_{13}(n) = -f_{22}(n)f_{13}(n)c_{33}/d(n), \tag{3.10}$$

$$b_{23}(n) = +g_{21}(n)f_{13}(n)c_{33}/d(n), \tag{3.11}$$

$$a_{31}(n+1) = -c_{33}f_{31}(n)f_{22}(n)/d(n), \tag{3.12}$$

$$b_{32}(n+1) = +c_{33}f_{31}(n)g_{12}(n)/d(n), \tag{3.13}$$

$$d(n) \equiv h_{11}(n)f_{22}(n) - g_{12}(n)g_{21}(n), \tag{3.14}$$

while the functions $a_{11}(n)$, $a_{22}(n)$, and $a_{33}(n)$ referred to as the sampling ones remain arbitrary for the time being. The arbitrariness of the sampling functions is typical of other integrable systems [24–27] and appears to be in lines with the principles of the reduction group method developed by Mikhailov [28].

The admissible fixation of the sampling functions is not unique. The most suitable approach to this task is to rely on some local conservation laws [27] and on the requirement of physical advisability. In any event, we have to remember that any fixation of one sampling function inevitably imposes one constraint onto the prototype field functions. Thus, our general set of seven nonlinear evolution equations (3.2)–(3.8) must be reduced to four truly independent ones. Analyzing the general system (3.2)–(3.8), one can reveal at least two possibilities,

$$\dot{h}_{11}(n) = 0, \tag{3.15}$$

$$\dot{d}(n) = 0, \tag{3.16}$$

$$\frac{d}{d\tau}\left[f_{31}(n)f_{13}(n)\right] = 0 \tag{3.17}$$

and

$$\dot{h}_{11}(n) = 0, \tag{3.18}$$

$$\dot{f}_{22}(n) = 0, \tag{3.19}$$

$$\frac{d}{d\tau}\left[f_{31}(n)f_{13}(n)\right] = 0, \tag{3.20}$$

to handle such a reduction.

The former variant (3.15)–(3.17) looks as the more winning one, and we consider it in full details. First of all due to the specific structure of general evolution equations (3.2)–(3.8), the constraint $\dot{h}_{11}(n) = 0$ ensures that the function $h_{11}(n)$ is able to rescale the rest of prototype field functions, and, hence, it should be safely equalized to unity:

$$h_{11}(n) = 1. \tag{3.21}$$

The other two constraints (3.16) and (3.17) accompanied by the demand of space uniformity assume the parametrizations

$$g_{12}(n) = \sqrt{f_{22}}\, g_-(n), \tag{3.22}$$

$$g_{21}(n) = \sqrt{f_{22}}\, g_+(n), \tag{3.23}$$

$$f_{22}(n) = f_{22}\left[1 + g_+(n)g_-(n)\right], \tag{3.24}$$

$$f_{13}(n) = f_{13}\exp[+q(n)], \tag{3.25}$$

$$f_{31}(n) = f_{31}\exp[-q(n)], \tag{3.26}$$

where $\dot{f}_{22} = 0$, $\dot{f}_{13} = 0 = \dot{f}_{31}$, and the equality $h_{11}(n) = 1$ has been taken into account. On the other hand, the same three constraints (3.15)–(3.17) yield

$$a_{11}(n) = a_{11}, \tag{3.27}$$

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$$a_{22}(n) = a_{22} + (f_{13}c_{33}f_{31}/f_{22})g_+(n)g_-(n-1)\times$$
$$\times\exp[+q(n)-q(n-1)], \tag{3.28}$$
$$a_{33}(n) = a_{33}, \tag{3.29}$$

where the spatially independent quantities a_{11}, a_{22}, a_{33}, and c_{33} can be treated as some functions of the time τ. The particular choice of these parameters is dictated by the setting of a physical problem and by the adopted boundary conditions for the field variables $f_{11}(n)$, $q(n)$ and $g_+(n)$, $g_-(n)$.

The reduced semidiscrete nonlinear integrable system having been written in terms of the true field variables $f_{11}(n)$, $q(n)$ and $g_+(n)$, $g_-(n)$ looks as follows:

$$\dot{f}_{11}(n) = -f_{13}c_{33}f_{31}\,[1+g_+(n+1)g_-(n+1)]\times$$
$$\times\,\exp\,[+q(n+1)-q(n)]+$$
$$+\,f_{13}c_{33}f_{31}\,[1+g_+(n-1)g_-(n-1)]\times$$
$$\times\,\exp\,[+q(n)-q(n-1)], \tag{3.30}$$
$$\dot{q}(n) = c_{33}\,[1+g_+(n)g_-(n)]\,f_{11}(n)+a_{11}-a_{33}, \tag{3.31}$$
$$\dot{g}_+(n) = (a_{22}-a_{11})\,g_+(n)+$$
$$+\,(f_{13}c_{33}f_{31}/f_{22})\,[1+g_+(n)g_-(n)]\,g_+(n+1)\times$$
$$\times\,\exp\,[+q(n+1)-q(n)], \tag{3.32}$$
$$\dot{g}_-(n) = (a_{11}-a_{22})\,g_-(n)-$$
$$-\,(f_{13}c_{33}f_{31}/f_{22})\,[1+g_+(n)g_-(n)]\,g_-(n-1)\times$$
$$\times\,\exp\,[+q(n)-q(n-1)]. \tag{3.33}$$

Evidently, the parameter a_{11} plays no self-sufficient part in the structure of the equations and could be equalized to zero: $a_{11}=0$. Further, at $f_{13}f_{31}<0$ and $a_{33}-a_{11}=0$, the first two equations (3.30) and (3.31) of the obtained system describe the Toda-like subsystem relative to the immovable reference frame. As to the last two equations (3.32) and (3.33), they correspond to the self-trapping subsystem, whose integrable predecessors have been considered by a number of authors [18–23] in the wake of the Davydov–Kyslukha soliton model [1–3, 7, 8, 29, 30] and the Eilbeck–Lomdahl–Scott self-trapping model [31–33].

It is worth noting that the particular choice of the parameter $a_{22}-a_{11}$ in the last two equations (3.32) and (3.33) seems to be inessential for physical applications insofar as it does not appear in the concomitant local conservation law

$$\frac{d}{d\tau}\ln\big[1+g_+(n)g_-(n)\big] = (f_{13}c_{33}f_{31}/f_{22})\times$$
$$\times\,g_+(n+1)g_-(n)\exp\,[+q(n+1)-q(n)]-$$
$$-\,(f_{13}c_{33}f_{31}/f_{22})\times$$
$$\times\,g_+(n)g_-(n-1)\exp\,[+q(n)-q(n-1)]\,, \tag{3.34}$$

being the discrete-space analog of the continuity equation. In this respect, the quantity $\ln[1+g_+(n)\times\times g_-(n)]$ can be treated as the density of excitations in the self-trapping subsystem. As a matter of fact, the parameter $a_{22}-a_{11}$ can always be eliminated even from the basic equations (3.32) and (3.33) by an appropriate gauge transformation.

It is remarkable that, at $a_{11}-a_{22}$, $a_{11}-a_{33}$, and c_{33} being time-independent, the whole coupled system (3.30)–(3.33) clearly demonstrates the symmetry under the space and time reversal ($\mathcal{PT}$-symmetry) implying that the transformed field variables $f'_{11}(n|\tau)$, $q'(n|\tau)$ and $g'_+(n|\tau)$, $g'_-(n|\tau)$ defined as

$$f'_{11}(n|\tau) = +f_{11}(-n|-\tau), \tag{3.35}$$
$$q'(n|\tau) = -q(-n|-\tau) \tag{3.36}$$

and

$$g'_+(n|\tau) = g_-(-n|-\tau)\exp(+\alpha), \tag{3.37}$$
$$g'_-(n|\tau) = g_+(-n|-\tau)\exp(-\alpha) \tag{3.38}$$

are governed by the same set of equations (3.30)–(3.33) as the original variables $f_{11}(n|\tau)$, $q(n|\tau)$ and $g_+(n|\tau)$, $g_-(n|\tau)$.

Presently, the $\mathcal{PT}$-symmetric models become increasingly applicable in physical sciences [34], especially in nonlinear optics [35–37], inasmuch as they allow one to obtain physically meaningful results without invoking the more restrictive condition of Dirac Hermiticity [34].

Finalizing this section, we briefly present the results arising from the second variant of admissible constraints (3.18)–(3.20). Thus, for the functions $h_{11}(n)$, $f_{22}(n)$ and $f_{13}(n)$, $f_{31}(n)$, we have

$$h_{11}(n) = 1, \tag{3.39}$$
$$f_{22}(n) = f_{22}, \tag{3.40}$$

$$f_{13}(n) = f_{13}\exp[+q(n)], \tag{3.41}$$

$$f_{31}(n) = f_{31}\exp[-q(n)], \tag{3.42}$$

where $\dot{f}_{22} = 0$, $\dot{f}_{13} = 0 = \dot{f}_{31}$, and the uniformity of space is implied. The same three constraints (3.18)–(3.20) yield

$$a_{11}(n) = a_{11}, \tag{3.43}$$

$$a_{22}(n) = a_{22}, \tag{3.44}$$

$$a_{33}(n) = a_{33}, \tag{3.45}$$

where the free parameters a_{11}, a_{22}, and a_{33} can be thought as arbitrary functions of the time. The reduced nonlinear evolution equations for the field variables $f_{11}(n)$, $q(n)$ and $g_{12}(n)$, $g_{21}(n)$ are easy reproducible from the general ones (3.2)–(3.8) by the use of the just obtained formulas (3.39)–(3.45) and the general formulas (3.9)–(3.14).

4. Systems (with Taylor-like Ansätze for Lax Operators) of the Second Type

In this section, we shall dwell on the integrable systems of the second kind, i.e., the systems associated with the second admissible collection of ansätze (2.4) and (2.5) for the spectral and evolution matrices $L(n|\lambda)$ and $A(n|\lambda)$. Having been inserted into the zero-curvature equation (2.1), these ansätze allow us to isolate the following set of evolution equations:

$$\dot{h}_{11}(n) = a_{11}(n+1)h_{11}(n) + b_{12}(n+1)g_{21}(n) - \\ - h_{11}(n)a_{11}(n) - g_{12}(n)b_{21}(n), \tag{4.1}$$

$$\dot{f}_{11}(n) = a_{11}(n+1)f_{11}(n) + a_{13}(n+1)f_{31}(n) - \\ - f_{11}(n)a_{11}(n) - f_{13}(n)a_{31}(n), \tag{4.2}$$

$$\dot{g}_{12}(n) = a_{11}(n+1)g_{12}(n) + b_{12}(n+1)f_{22}(n) - \\ - f_{11}(n)b_{12}(n) - g_{12}(n)a_{22}(n), \tag{4.3}$$

$$\dot{g}_{21}(n) = b_{21}(n+1)f_{11}(n) + a_{22}(n+1)g_{21}(n) - \\ - g_{21}(n)a_{11}(n) - f_{22}(n)b_{21}(n), \tag{4.4}$$

$$\dot{f}_{22}(n) = a_{22}(n+1)f_{22}(n) - f_{22}(n)a_{22}(n), \tag{4.5}$$

$$\dot{f}_{13}(n) = a_{11}(n+1)f_{13}(n) + a_{13}(n+1)f_{33}(n) - \\ - f_{11}(n)a_{13}(n) - f_{13}(n)a_{33}(n), \tag{4.6}$$

$$\dot{f}_{31}(n) = a_{31}(n+1)f_{11}(n) + a_{33}(n+1)f_{31}(n) - \\ - f_{31}(n)a_{11}(n) - f_{33}(n)a_{31}(n), \tag{4.7}$$

$$\dot{f}_{33}(n) = a_{33}(n+1)f_{33}(n) - f_{33}(n)a_{33}(n), \tag{4.8}$$

where

$$c_{11}(n) = c_{11}, \tag{4.9}$$

$$b_{12}(n) = c_{11}g_{12}(n)/h_{11}(n), \tag{4.10}$$

$$a_{13}(n) = c_{11}f_{13}(n)/h_{11}(n), \tag{4.11}$$

$$b_{21}(n+1) = g_{21}(n)c_{11}/h_{11}(n), \tag{4.12}$$

$$a_{31}(n+1) = f_{31}(n)c_{11}/h_{11}(n), \tag{4.13}$$

with the free parameter c_{11} being some arbitrary function of the time.

The sampling functions $a_{11}(n)$, $a_{22}(n)$, and $a_{33}(n)$ remain arbitrary for the time being. As we already know, any procedure of their fixation reduces the number of field variables. In accordance with the total number of sampling functions, we must impose three differential constraints linked with underdetermined local conservation laws.

However, in the case where the determinant of the spectral operator depends on several powers of the spectral parameter λ, there may appear at least one constraint of purely algebraic form. In order to prove this statement, let us take advantage of the universal local conservation law

$$\frac{d}{d\tau}\ln\det L(n|\lambda) = \mathrm{Sp}A(n+1|\lambda) - \mathrm{Sp}A(n|\lambda) \tag{4.14}$$

following directly from the zero-curvature equation (2.1) provided the matrix $L(n|\lambda)$ taken at arbitrary λ is not degenerate: $\det L(n|\lambda) \neq 0$. For the adopted ansatz (2.4) for the spectral matrix $L(n|\lambda)$, we have

$$\det L(n|\lambda) = \\ = f_{11}(n)f_{22}(n)f_{33}(n) - f_{31}(n)f_{22}(n)f_{13}(n) + \\ + \lambda^2[h_{11}(n)f_{22}(n)f_{33}(n) - g_{21}(n)f_{33}(n)g_{12}(n)]. \tag{4.15}$$

This expression when combined with the universal local conservation law (4.14) yields the equations

$$\frac{d}{d\tau}\big[f_{11}(n)f_{22}(n)f_{33}(n) - f_{31}(n)f_{22}(n)f_{13}(n)\big] = \\ = \big[a_{11}(n+1) + a_{22}(n+1) + a_{33}(n+1) - \\ - a_{11}(n) - a_{22}(n) - a_{33}(n)\big]\times \\ \times\big[f_{11}(n)f_{22}(n)f_{33}(n) - f_{31}(n)f_{22}(n)f_{13}(n)\big], \tag{4.16}$$

$$\frac{d}{d\tau}\big[h_{11}(n)f_{22}(n)f_{33}(n) - g_{21}(n)f_{33}(n)g_{12}(n)\big] = \\ = \big[a_{11}(n+1) + a_{22}(n+1) + a_{33}(n+1) - \\ - a_{11}(n) - a_{22}(n) - a_{33}(n)\big]\times \\ \times\big[h_{11}(n)f_{22}(n)f_{33}(n) - g_{21}(n)f_{33}(n)g_{12}(n)\big], \tag{4.17}$$

which supports the equality

$$A(n)\big[f_{11}(n)f_{22}(n)f_{33}(n) - f_{31}(n)f_{22}(n)f_{13}(n)\big] = \\ = B(n)\big[h_{11}(n)f_{22}(n)f_{33}(n) - g_{21}(n)f_{33}(n)g_{12}(n)\big]. \tag{4.18}$$

Here, the coefficients $A(n)$ and $B(n)$ are time-independent: $\dot{A}(n) = 0 = \dot{B}(n)$. We will ignore also the possibility of their spatial dependence and assume

$$A(n) = A, \tag{4.19}$$

$$B(n) = B, \tag{4.20}$$

thus asserting the uniformity of space. The obtained equality (4.18) is nothing but the natural constraint, which stays apart from the constraints necessary to fix the sampling functions.

Altogether, we will deal with four constraints and should come to one or another closed set of four nonlinear evolution equations.

As the first step common for either of the particular systems under forthcoming consideration, we adopt the constraint

$$\dot{h}_{11}(n) = 0. \tag{4.21}$$

In so doing, we immediately obtain

$$a_{11}(n) = a_{11} - c_{11}g_{12}(n)g_{21}(n-1)/h_{11}(n)h_{11}(n-1). \tag{4.22}$$

Then, by repeating the arguments of the previous section, we put

$$h_{11}(n) = 1 \tag{4.23}$$

without any loss of generality.

Appealing to the natural constraint (4.18) at arbitrary nonzero values of parameters A and B, it is conceivable to exclude any of the following five combinations $f_{11}(n)$, $f_{22}(n)$, $g_{21}(n)g_{12}(n)$, $f_{33}(n)$, and $f_{31}(n)f_{13}(n)$ from consideration and thereafter to impose two lacking differential constraints. However, all ramifications of such possibilities could produce a quite long list of integrable systems, and the task of their classification goes beyond the scope of our present report. Therefore, we will restrict ourselves with two limiting cases arising from the natural constraint (4.18) at $A \neq 0$, $B = 0$ and $A = 0$, $B \neq 0$, respectively.

Thus, at $A \neq 0$ and $B = 0$, we come to two variants. The first one is characterized by the constraints

$$f_{22}(n) = 0, \tag{4.24}$$

$$\frac{d}{d\tau}\left[g_{21}(n)g_{12}(n)\right] = 0, \tag{4.25}$$

$$\dot{f}_{33}(n) = 0 \tag{4.26}$$

yielding

$$a_{22}(n) = a_{22} - a_{11} + g_{21}c_{11}g_{12}\exp[+q(n) - q(n-1)], \tag{4.27}$$

$$a_{33}(n) = a_{33} \tag{4.28}$$

and

$$g_{12}(n) = g_{12}\exp[+q(n)], \tag{4.29}$$

$$g_{21}(n) = g_{21}\exp[-q(n)], \tag{4.30}$$

$$f_{33}(n) = f_{33} \tag{4.31}$$

with $\dot{g}_{12} = 0 = \dot{g}_{21}$ and $\dot{f}_{33} = 0$.

The second variant is characterized by the constraints

$$f_{11}(n) = f_{13}(n)f_{31}(n)/f_{33}(n), \tag{4.32}$$

$$\dot{f}_{22}(n) = 0, \tag{4.33}$$

$$\dot{f}_{33}(n) = 0 \tag{4.34}$$

yielding

$$a_{22}(n) = a_{22}, \tag{4.35}$$

$$a_{33}(n) = a_{33}, \tag{4.36}$$

and

$$f_{22}(n) = f_{22}, \tag{4.37}$$

$$f_{33}(n) = f_{33} \tag{4.38}$$

with $\dot{f}_{22} = 0 = \dot{f}_{33}$.

At $A = 0$ and $B \neq 0$, we also come to two variants. The first one is determined by the constraints

$$f_{33}(n) = 0, \tag{4.39}$$

$$\dot{f}_{22}(n) = 0, \tag{4.40}$$

$$\frac{d}{d\tau}\left[f_{13}(n)f_{31}(n)\right] = 0 \tag{4.41}$$

yielding

$$a_{22}(n) = a_{22}, \tag{4.42}$$

$$a_{33}(n) = a_{33} - a_{11} + c_{11} g_{12}(n) g_{21}(n-1) \tag{4.43}$$

and

$$f_{22}(n) = f_{22}, \tag{4.44}$$

$$f_{13}(n) = f_{13} \exp[+q(n)], \tag{4.45}$$

$$f_{31}(n) = f_{31} \exp[-q(n)], \tag{4.46}$$

with $\dot{f}_{22} = 0$ and $\dot{f}_{13} = 0 = \dot{f}_{31}$.

The second variant is determined by the constraints

$$g_{12}(n) g_{21}(n) = f_{22}(n), \tag{4.47}$$

$$\dot{f}_{22}(n) = 0, \tag{4.48}$$

$$\dot{f}_{33}(n) = 0 \tag{4.49}$$

yielding

$$a_{22}(n) = a_{22}, \tag{4.50}$$

$$a_{33}(n) = a_{33} \tag{4.51}$$

and

$$g_{12}(n) = \sqrt{f_{22}} \exp[+q(n)], \tag{4.52}$$

$$g_{21}(n) = \sqrt{f_{22}} \exp[-q(n)], \tag{4.53}$$

$$f_{22}(n) = f_{22}, \tag{4.54}$$

$$f_{33}(n) = f_{33} \tag{4.55}$$

with $\dot{f}_{22} = 0 = \dot{f}_{33}$.

In all variants considered in this section, the adopted parametrizations have been performed in such a way as to preserve the uniformity of space. We do not write down explicitly either of four particular integrable systems linked with the just listed reductions, by assuming that the interested reader can readily fill in this gap relying upon the already prepared formulas.

One can always fit all these nonlinear integrable systems to be invariant under the space and time reversal.

5. Local Conserved Densities for the Systems with Taylor-Like Lax Operators

By definition, any local conservation law linked with some semidiscrete integrable system given on infinite quasione-dimensional lattice can be written in the form

$$\dot{\rho}(n) = J(n|n-1) - J(n+1|n), \tag{5.1}$$

where the quantities $\rho(n)$ and $J(n+1/2|n-1/2)$ are referred to as the local density and the local current, respectively. According to the previous section, some of the lowest local conservation laws are obtainable directly from the universal local conservation law (4.14)

However, there exists the generalized procedure [26] allowing one to develop an infinite set of local conservation laws recursively without any reference on the scattering data of an auxiliary spectral problem. The approach [26] generalizes the ideas suggested by Konno, Sanuki, Ichikawa, and Wadati [38, 39] to the case of multicomponent integrable systems linked with a spectral operator of arbitrary order R.

Omitting rather tedious calculations having been performed in the framework of the generalized procedure [26], we will present several lowest local densities written in terms of prototype field functions. They look as follows:

$$\rho(n|0) = \ln h_{11}(n), \tag{5.2}$$

$$\rho^{+}(n|1) = \frac{f_{11}(n)}{h_{11}(n)} + \frac{g_{12}(n+1) g_{21}(n)}{h_{11}(n+1) h_{11}(n)}, \tag{5.3}$$

$$\rho^{-}(n|1) = \frac{f_{11}(n)}{h_{11}(n)} + \frac{g_{12}(n) g_{21}(n-1)}{h_{11}(n) h_{11}(n-1)}, \tag{5.4}$$

$$\begin{aligned} \rho^{+}(n|2) = {} & \frac{f_{13}(n+1) f_{31}(n)}{h_{11}(n+1) h_{11}(n)} - \\ & - \frac{1}{2} \left[\frac{f_{11}(n)}{h_{11}(n)} + \frac{g_{12}(n+1) g_{21}(n)}{h_{11}(n+1) h_{11}(n)} \right]^2 - \\ & - \frac{f_{11}(n+1) g_{12}(n+1) g_{21}(n)}{h_{11}^2(n+1) h_{11}(n)} + \\ & + [f_{22}(n+1) h_{11}(n+1) - g_{21}(n+1) g_{12}(n+1)] \times \\ & \times \frac{g_{21}(n) g_{12}(n+2)}{h_{11}(n+2) h^2(n+1) h_{11}(n)}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \rho^{-}(n|2) = {} & \frac{f_{13}(n) f_{31}(n-1)}{h_{11}(n) h_{11}(n-1)} - \\ & - \frac{1}{2} \left[\frac{f_{11}(n)}{h_{11}(n)} + \frac{g_{12}(n) g_{21}(n-1)}{h_{11}(n) h_{11}(n-1)} \right]^2 - \end{aligned}$$

$$-\frac{g_{12}(n)g_{21}(n-1)f_{11}(n-1)}{h_{11}(n)h_{11}^2(n-1)}+$$
$$+\left[f_{22}(n-1)h_{11}(n-1)-g_{21}(n-1)g_{12}(n-1)\right]\times$$
$$\times\frac{g_{21}(n-2)g_{12}(n)}{h_{11}(n)h_{11}^2(n-1)h_{11}(n-2)}. \qquad (5.6)$$

Having been rewritten in terms of appropriate true field variables, these formulas are applicable equally well to any relevant integrable systems taken among the two types of systems considered in two previous sections (Sections 3 and 4). This statement is based on the fact that the general form of a local conserved density is dictated exclusively by the general form of a spectral operator, which turns out to be common for both types of the suggested systems (see formulas (2.2) and (2.4) for comparison).

As for the local currents, they are essentially dependent on the type of a system dictated by the type of an ansatz for the evolution operator. The formulas for the local currents turn out to be very cumbersome, and we do not write them down.

6. Two Types of Laurent-Like Ansätze for Auxiliary Lax Operators

This section opens the second part of the paper distinguished from the first one by the more constructive choice of ansätze for the Lax operators $L(n|z)$ and $A(n|z)$ in the zero-curvature equation

$$\dot{L}(n|z)=A(n+1|z)L(n|z)-L(n|z)A(n|z). \qquad (6.1)$$

Here, for the sake of convenience, we adopted the new notation z for the time-independent spectral parameter.

The three forthcoming sections (Sections 7–9) will be devoted to the integrable systems arising from spectral and evolution operators taken as some 3×3 square matrices given by the truncated Laurent series with respect to the spectral parameter. Our consideration will be based upon two pairs of nontrivial ansätze for the Lax operators $L(n|z)$ and $A(n|z)$, which have been sorted out as

$$L(n|z)=\begin{pmatrix} f_{11}(n)+h_{11}(n)(z^2+z^{-2}) & g_{12}(n)z & f_{13}(n)\\ g_{21}(n)z^{-1} & f_{22}(n) & 0\\ f_{31}(n) & 0 & f_{33}(n)\end{pmatrix}, \qquad (6.2)$$

$$A(n|z)=\begin{pmatrix} a_{11}(n) & 0 & a_{13}(n)\\ 0 & a_{22}(n) & b_{23}(n)z^{-1}\\ a_{31}(n) & b_{32}(n)z & a_{33}(n)+c_{33}(n)(z^2+z^{-2})\end{pmatrix} \qquad (6.3)$$

and

$$L(n|z)=\begin{pmatrix} f_{11}(n)+h_{11}(n)(z^2+z^{-2}) & g_{12}(n)z & f_{13}(n)\\ g_{21}(n)z^{-1} & f_{22}(n) & 0\\ f_{31}(n) & 0 & f_{33}(n)\end{pmatrix}, \qquad (6.4)$$

$$A(n|z)=\begin{pmatrix} a_{11}(n)+c_{11}(n)(z^2+z^{-2}) & b_{12}(n)z & a_{13}(n)\\ b_{21}(n)z^{-1} & a_{22}(n) & 0\\ a_{31}(n) & 0 & a_{33}(n)\end{pmatrix} \qquad (6.5)$$

respectively.

The systems linked with the first two ansätze (6.2) and (6.3) will be referred to as primary systems of the first type, while the systems connected with the second two ansätze (6.4) and (6.5) will be called as primary systems of the second type. Here, of course, we should remember that now we classify the systems originated by the Lax operators written as the truncated Laurent series in contrast to the early considered systems originated by the Lax operators written as the truncated Taylor series.

7. Systems (with Laurent-Like Ansätze for Lax Operators) of the First Type

In order to generate the systems of the first type, we substitute the pertinent ansätze (6.2) and (6.3) for the Lax matrices $L(n|z)$ and $A(n|z)$ into the zero-curvature equation (6.1). The straightforward calculations yield the following general equations:

$$\dot{h}_{11}(n)=a_{11}(n+1)h_{11}(n)-h_{11}(n)a_{11}(n), \qquad (7.1)$$

$$\dot{f}_{11}(n)=a_{11}(n+1)f_{11}(n)+a_{13}(n+1)f_{31}(n)-$$
$$-f_{11}(n)a_{11}(n)-f_{13}(n)a_{31}(n), \qquad (7.2)$$

$$\dot{g}_{12}(n)=a_{11}(n+1)g_{12}(n)-$$
$$-g_{12}(n)a_{22}(n)-f_{13}(n)b_{32}(n), \qquad (7.3)$$

$$\dot{g}_{21}(n)=a_{22}(n+1)g_{21}(n)+b_{23}(n+1)f_{31}(n)-$$
$$-g_{21}(n)a_{11}(n), \qquad (7.4)$$

$$\dot{f}_{22}(n)=a_{22}(n+1)f_{22}(n)-f_{22}(n)a_{22}(n), \qquad (7.5)$$

$$\dot{f}_{13}(n) = a_{11}(n+1)f_{13}(n) + a_{13}(n+1)f_{33}(n) - - f_{11}(n)a_{13}(n) - g_{12}(n)b_{23}(n) - f_{13}(n)a_{33}(n), \quad (7.6)$$

$$\dot{f}_{31}(n) = a_{31}(n+1)f_{11}(n) + b_{32}(n+1)g_{21}(n) + + a_{33}(n+1)f_{31}(n) - f_{31}(n)a_{11}(n) - f_{33}(n)a_{31}(n), \quad (7.7)$$

$$\dot{f}_{33}(n) = a_{33}(n+1)f_{23}(n) - f_{33}(n)a_{33}(n), \quad (7.8)$$

where

$$c_{33}(n) = c_{33}, \quad (7.9)$$

$$a_{13}(n) = -f_{13}(n)c_{33}/h_{11}(n), \quad (7.10)$$

$$a_{31}(n+1) = -c_{33}f_{31}(n)/h_{11}(n), \quad (7.11)$$

while $b_{23}(n)$ and $b_{32}(n)$ must be determined from the equations

$$b_{23}(n+1)f_{33}(n) - f_{22}(n)b_{23}(n) = = -g_{21}(n)f_{13}(n)c_{33}/h_{11}(n), \quad (7.12)$$

$$b_{32}(n+1)f_{22}(n) - f_{33}(n)b_{32}(n) = = +c_{33}f_{31}(n)g_{12}(n)/h_{11}(n). \quad (7.13)$$

Here, the free parameter c_{33} can be thought as an arbitrary function of the time. The sampling functions $a_{11}(n)$, $a_{22}(n)$, and $a_{33}(n)$ must be found from three suitable differential constraints linked with underdetermined local conservation laws. On the other hand, the locality of theory so desirable for physical applications can be achieved by imposing one more but purely algebraic constraint (namely, $f_{22}(n) = 0$ or $f_{33}(n) = 0$).

We begin with the differential constraint

$$\dot{h}_{11}(n) = 0, \quad (7.14)$$

which can be safely replaced by the equality

$$h_{11}(n) = 1 \quad (7.15)$$

due to a specific structure of the general equations (7.1)–(7.13). As a consequence, we obtain

$$a_{11}(n) = a_{11}, \quad (7.16)$$

where a_{11} is some arbitrary function of the time. The particular choice of other constraints is unable to change the formulas of this paragraph. Therefore, they (formulas (7.14)–(7.16)) will be common for either of two admissible reduced systems arising from the general equations (7.1)–(7.13).

The first reduced system is based on the constraints

$$f_{33}(n) = 0, \quad (7.17)$$

$$\dot{f}_{22}(n) = 0, \quad (7.18)$$

$$\frac{d}{d\tau}\left[f_{13}(n)f_{31}(n)\right] = 0. \quad (7.19)$$

These constraints yield

$$f_{22}(n) = f_{22}, \quad (7.20)$$

$$f_{13}(n) = f_{13}\exp[+q(n)], \quad (7.21)$$

$$f_{31}(n) = f_{31}\exp[-q(n)] \quad (7.22)$$

and

$$b_{32}(n+1) = (c_{33}f_{31}/f_{22})\, g_{12}(n)\exp[-q(n)], \quad (7.23)$$

$$b_{23}(n) = (f_{13}c_{33}/f_{22})\, g_{21}(n)\exp[+q(n)], \quad (7.24)$$

$$a_{22}(n) = a_{22}, \quad (7.25)$$

$$a_{33}(n) = a_{33} \quad (7.26)$$

with $\dot{f}_{22} = 0$ and $\dot{f}_{13} = 0 = \dot{f}_{31}$, while a_{22} and a_{33} being some arbitrary functions of the time.

One can readily verify that the reduced system of our interest admits the concise Hamiltonian representation

$$\dot{p}(n) = -\partial H/\partial q(n), \quad (7.27)$$

$$\dot{q}(n) = +\partial H/\partial p(n), \quad (7.28)$$

$$\dot{g}_{+}(n) = -\partial H/\partial g_{-}(n), \quad (7.29)$$

$$\dot{g}_{-}(n) = +\partial H/\partial g_{+}(n) \quad (7.30)$$

in terms of the canonical field variables $p(n)$, $q(n)$ and $g_{+}(n)$, $g_{-}(n)$ with the Hamiltonian function H defined by the expression

$$H = \sum_{m=-\infty}^{\infty}(a_{11}-a_{33})p(m) + \sum_{m=-\infty}^{\infty}c_{33}p^2(m)/2 - - \sum_{m=-\infty}^{\infty}f_{13}c_{33}f_{31}\left[1 + g_{+}(m)g_{-}(m-1)/f_{22}\right] \times \times \exp[+q(n) - q(n-1)] + + \sum_{m=-\infty}^{\infty}(a_{11}-a_{22})[f_{22} + g_{+}(m)g_{-}(m)]. \quad (7.31)$$

Here, the relationships

$$p(n) = f_{11}(n) - g_{12}(n)g_{21}(n)/f_{22}, \quad (7.32)$$

$$g_+(n) = g_{21}(n)/\sqrt{f_{22}}, \tag{7.33}$$

$$g_-(n) = g_{12}(n)/\sqrt{f_{22}}, \tag{7.34}$$

between the old and new field variables $f_{11}(n)$, $g_{21}(n)$, $g_{12}(n)$ and $p(n)$, $g_+(n)$, $g_-(n)$ have been adopted. From the physical point of view, the most reasonable choice of the free parameters $a_{11} - a_{33}$ and $a_{11} - a_{22}$ is given by the formulas

$$a_{11} - a_{33} = 0, \tag{7.35}$$

$$a_{11} - a_{22} = f_{13}c_{33}f_{31}/f_{22}. \tag{7.36}$$

The obtained evolution equations (7.27)–(7.30) (with H given by (7.31)) describe the unified nonlinear dynamics of two coupled subsystems. Thus, the first two equations (7.27) and (7.28) correspond to the Toda-like vibrational subsystem. The second two equations (7.29) and (7.30) can be treated as equations for the induced-trapping subsystem, inasmuch as their nonlinearities are entirely induced by the Toda-like subsystem. In so doing, the conserved quantity

$$N = \sum_{m=-\infty}^{\infty} g_+(m)g_-(m) \tag{7.37}$$

should be understood as the total number of excitations in this induced-trapping subsystem.

Similarly to the integrable nonlinear system consisting of the coupled Toda-like and self-trapping subsystems (3.30)–(3.33), the present nonlinear system (7.27)–(7.31) taken for the time-independent free parameters $a_{11} - a_{22}$, $a_{11} - a_{33}$, and c_{33} is also $\mathcal{PT}$-symmetric. In other words, it is invariant under the space and time reversal. This fact opens the broad possibilities for the system applications in various branches of physics [34–37]. For example, we expect that the proposed semidiscrete integrable system (7.27)–(7.31) could support some physical features of the Davydov–Kyslukha exciton-phonon nonlinear model [1–3, 7, 8, 29, 30], in particular, the formation of stable solitary waves so valuable for the transport of energy and charge through the low-dimensional regular lattice structures [7, 8].

Let us now formulate the second reduced system. It is based on the constraints

$$f_{22}(n) = 0, \tag{7.38}$$

$$\frac{d}{d\tau}[g_{21}(n)g_{12}(n)] = 0, \tag{7.39}$$

$$\dot{f}_{33}(n) = 0. \tag{7.40}$$

These constraints yield

$$f_{33}(n) = f_{33}, \tag{7.41}$$

$$g_{12}(n) = g_{12}\exp[+q(n)], \tag{7.42}$$

$$g_{21}(n) = g_{21}\exp[-q(n)] \tag{7.43}$$

and

$$b_{23}(n+1) = -(g_{21}c_{33}/f_{33})f_{13}(n)\exp[-q(n)], \tag{7.44}$$

$$b_{32}(n) = -(c_{33}g_{12}/f_{33})f_{31}(n)\exp[+q(n)], \tag{7.45}$$

$$a_{22}(n) = a_{22}, \tag{7.46}$$

$$a_{33}(n) = a_{33} \tag{7.47}$$

with $\dot{f}_{33} = 0$ and $\dot{g}_{12} = 0 = \dot{g}_{21}$, while a_{22} and a_{33} being some arbitrary functions of the time.

Then, introducing the substitutions

$$p(n) = f_{11}(n) - f_{13}(n)f_{31}(n)/f_{33} \tag{7.48}$$

and

$$f_+(n) = f_{31}(n)/\sqrt{f_{33}}, \tag{7.49}$$

$$f_-(n) = f_{13}(n)/\sqrt{f_{33}}, \tag{7.50}$$

we reveal that the required reduced system acquires the standard Hamiltonian form

$$\dot{p}(n) = -\partial H/\partial q(n), \tag{7.51}$$

$$\dot{q}(n) = +\partial H/\partial p(n), \tag{7.52}$$

$$\dot{f}_+(n) = -\partial H/\partial f_-(n), \tag{7.53}$$

$$\dot{f}_-(n) = +\partial H/\partial f_+(n) \tag{7.54}$$

with the Hamiltonian function H given by the expression

$$\begin{aligned} H = &\sum_{m=-\infty}^{\infty} \big[a_{11} - a_{22} + c_{33}f_+(m)f_-(m)\big]p(m) + \\ &+ \sum_{m=-\infty}^{\infty} \big(g_{21}g_{12}c_{33}/f_{33}\big)f_+(m)f_-(m-1)\times \\ &\times \exp[+q(m) - q(m-1)] + \\ &+ \sum_{m=-\infty}^{\infty} (a_{11} - a_{33})f_+(m)f_-(m) - \\ &- \sum_{m=-\infty}^{\infty} c_{33}f_{33}f_+(m)f_-(m+1) + \\ &+ \sum_{m=-\infty}^{\infty} (c_{33}/2)f_+^2(m)f_-^2(m) \end{aligned} \tag{7.55}$$

and the quantities $p(n)$, $q(n)$ and $f_+(n)$, $f_-(n)$ serving as two pairs of canonical field variables.

To maintain the $\mathcal{PT}$-symmetry of this integrable system (7.51)–(7.55), we must assume the time independence of the free parameters $a_{11}-a_{22}$, $a_{11}-a_{33}$, and c_{33}.

Considering the second and fourth terms in the obtained Hamiltonian function (7.55), the question arises whether there exists some sort of a canonic transformation converting them into the quantities of equal status. We will try to resolve this problem by addressing once again directly to the early adopted constraints (7.38)–(7.40), but inventing a new parametrization of the prototype field functions. In order to avoid the unnecessary misunderstanding, we shall carefully list all new formulas, although some of them will formally be found among the old ones (7.41)–(7.54).

First of all, we introduce the intermediate functions $F_{13}(n)$ and $F_{31}(n)$ by the substitutions

$$f_{13}(n) = F_{13}(n)\,(\mathrm{i}\eta\sqrt{g_{12}g_{21}}/f_{33})^n \exp[+q(n)], \quad (7.56)$$

$$f_{31}(n) = F_{31}(n)\,(f_{33}/\mathrm{i}\eta\sqrt{g_{12}g_{21}})^n \exp[-q(n)], \quad (7.57)$$

where $\eta^2 \equiv 1$. This step in combination with the reducing constraints (7.38)–(7.40) predetermines the following list of formulas:

$$f_{33}(n) = f_{33}, \quad (7.58)$$

$$g_{12}(n) = g_{12}\exp[+2q(n)], \quad (7.59)$$

$$g_{21}(n) = g_{21}\exp[-2q(n)] \quad (7.60)$$

and

$$b_{23}(n+1) = -(g_{21}c_{33}/f_{33})F_{13}(n)\,(\mathrm{i}\eta\sqrt{g_{12}g_{21}}/f_{33})^n \times \times\exp[-q(n)], \quad (7.61)$$

$$b_{32}(n) = -(c_{33}g_{12}/f_{33})F_{31}(n)\,(f_{33}/\mathrm{i}\eta\sqrt{g_{12}g_{21}})^n \times \times\exp[+q(n)], \quad (7.62)$$

$$a_{22}(n) = a_{22}, \quad (7.63)$$

$$a_{33}(n) = a_{33}. \quad (7.64)$$

Here as previously, $\dot{f}_{33} = 0$ and $\dot{g}_{12} = 0 = \dot{g}_{21}$, while a_{22} and a_{33} are time-dependent free parameters.

The proper analysis of the basic evolution equations (7.2)–(7.8) rewritten in terms of the functions $f_{11}(n)$, $q(n)$ and $F_{13}(n)$, $F_{31}(n)$ allows us to grope the most justified rearrangements

$$f_{11}(n) - F_{13}(n)F_{31}(n)/2f_{33} = \mathrm{i}p(n)/2, \quad (7.65)$$

$$F_{31}(n) = \sqrt{f_{33}}\,f_+(n), \quad (7.66)$$

$$F_{13}(n) = \sqrt{f_{33}}\,f_-(n). \quad (7.67)$$

As a result, the desired reduced system is convertible into the standard Hamiltonian form

$$\dot{p}(n) = -\partial H/\partial q(n), \quad (7.68)$$

$$\dot{q}(n) = +\partial H/\partial p(n), \quad (7.69)$$

$$+\mathrm{i}\dot{f}_+(n) = \partial H/\partial f_-(n), \quad (7.70)$$

$$-\mathrm{i}\dot{f}_-(n) = \partial H/\partial f_+(n) \quad (7.71)$$

with the Hamiltonian function H given by the formula

$$H = \sum_{m=-\infty}^{\infty}(c_{33}/2)f_+(m)f_-(m)p(m) - - \sum_{m=-\infty}^{\infty}\eta c_{33}\sqrt{g_{12}g_{21}}\,\exp[+q(m)-q(m-1)]\times \times[f_+(m)f_-(m-1)+f_+(m-1)f_-(m)] + + \sum_{m=-\infty}^{\infty}(a_{11}/2-a_{22}/2)p(m) - - \sum_{m=-\infty}^{\infty}\mathrm{i}(a_{11}/2+a_{22}/2-a_{33})f_+(m)f_-(m), \quad (7.72)$$

which is seen to be symmetric against the permutation $f_+(n)\rightleftarrows f_-(n)$.The quantities $p(n)$, $q(n)$ and $f_+(n)$, $-\mathrm{i}f_-(n)$ serve as two pairs of canonical field variables.

For $\mathrm{Im}\,c_{33}=0$, $\mathrm{Im}\sqrt{g_{12}g_{21}}=0$ and $\mathrm{Im}(a_{11}-a_{22}) = 0$, $\mathrm{Re}(a_{11}+a_{22}-2a_{33})=0$ the Hamiltonian function (7.72) becomes a purely real one, $H^*=H$. In so doing, the functions $f_+(n)$ and $f_-(n)$ are obliged to be complex conjugate, $f_-^*(n)\equiv f_+(n)$, whereas the functions $p(n)$ and $q(n)$ must be the real ones. Under these circumstances, the obtained integrable system (7.68)–(7.72) can be associated with the subsystem of Frenkel-like excitons [40] interacting with the pseudovibrational subsystem of a pretty unusual origin. The density of excitations $f_+(n)f_-(n)$ in the exciton subsystem is governed by the discrete-space analog of continuity equation

$$\frac{d}{d\tau}[f_+(n)f_-(n)] = = \mathrm{i}\eta c_{33}\sqrt{g_{12}g_{21}}\,[f_+(n+1)f_-(n)-f_+(n)f_-(n+1)]\times \times\exp[+q(n+1)-q(n)] - - \mathrm{i}\eta c_{33}\sqrt{g_{12}g_{21}}\,[f_+(n)f_-(n-1)-f_+(n-1)f_-(n)]\times \times\exp[+q(n)-q(n-1)] \quad (7.73)$$

and has the meaning of a conserved density.

The latter equation (7.73) is valid irrespective whether or not the requirements of Hamiltonian reality are satisfied.

8. Systems (with Laurent-Like Ansätze for Lax Operators) of the Second Type

According to our agreement, the integrable systems of the second type arise from the zero-curvature equation (6.1), when being inserted by the second pertinent combination (6.4) and (6.5) of ansätze for the Lax matrices $L(n|z)$ and $A(n|z)$. The general form of such equations (i.e., the form with the unfixed sampling functions $a_{11}(n)$, $a_{22}(n)$, and $a_{33}(n)$) looks as follows:

$$\dot{h}_{11}(n) = a_{11}(n+1)h_{11}(n) - h_{11}(n)a_{11}(n), \tag{8.1}$$

$$\dot{f}_{11}(n) = a_{11}(n+1)f_{11}(n) + b_{12}(n+1)g_{21}(n) + +a_{13}(n+1)f_{31}(n) - f_{11}(n)a_{11}(n) - -g_{12}(n)b_{21}(n) - f_{13}(n)a_{31}(n), \tag{8.2}$$

$$\dot{g}_{12}(n) = a_{11}(n+1)g_{12}(n) + b_{12}(n+1)f_{22}(n) - -f_{11}(n)b_{12}(n) - g_{12}(n)a_{22}(n), \tag{8.3}$$

$$\dot{g}_{21}(n) = b_{21}(n+1)f_{11}(n) + a_{22}(n+1)g_{21}(n) - -g_{21}(n)a_{11}(n) - f_{22}(n)b_{21}(n), \tag{8.4}$$

$$\dot{f}_{22}(n) = a_{22}(n+1)f_{22}(n) - f_{22}(n)a_{22}(n), \tag{8.5}$$

$$\dot{f}_{13}(n) = a_{11}(n+1)f_{13}(n) + a_{13}(n+1)f_{33}(n) - -f_{11}(n)a_{13}(n) - f_{13}(n)a_{33}(n), \tag{8.6}$$

$$\dot{f}_{31}(n) = a_{31}(n+1)f_{11}(n) + a_{33}(n+1)f_{31}(n) - -f_{31}(n)a_{11}(n) - f_{33}(n)a_{31}(n), \tag{8.7}$$

$$\dot{f}_{33}(n) = a_{33}(n+1)f_{33}(n) - f_{33}(n)a_{33}(n), \tag{8.8}$$

where

$$c_{11}(n) = c_{11}, \tag{8.9}$$

$$b_{12}(n) = c_{11}g_{12}(n)/h_{11}(n), \tag{8.10}$$

$$a_{13}(n) = c_{11}f_{13}(n)/h_{11}(n), \tag{8.11}$$

$$b_{21}(n+1) = g_{21}(n)c_{11}/h_{11}(n), \tag{8.12}$$

$$a_{31}(n+1) = f_{31}(n)c_{11}/h_{11}(n), \tag{8.13}$$

while the free parameter c_{11} can be understood as an arbitrary function of the time.

Noticing that the determinant of the matrix $L(n|z)$ depends on several powers of the spectral parameter z and assuming that the matrix $L(n|z)$ taken at arbitrary z is nonsingular, we are capable to analyze the universal conservation law

$$\frac{d}{d\tau} \ln \det L(n|z) = \mathrm{Sp}A(n+1|z) - \mathrm{Sp}A(n|z) \tag{8.14}$$

and to reveal the following natural constraint:

$$A(n)\big[f_{11}(n)f_{22}(n)f_{33}(n) - -g_{21}(n)g_{12}(n)f_{33}(n) - f_{31}(n)f_{13}(n)f_{22}(n)\big] = = B(n)h_{11}(n)f_{22}(n)f_{33}(n). \tag{8.15}$$

Here, the free coefficients $A(n)$ and $B(n)$ must be time-independent: $\dot{A}(n) = 0 = \dot{B}(n)$. To preserve the uniformity of space, we should eliminate also the possibility of their spatial dependence: $A(n) = A$ and $B(n) = B$. In what follows, we will consider the most representative variants of reduced integrable systems ignited by the natural constraint (8.15) at two simplest choices of its coefficients: $A \neq 0$, $B = 0$ and $A = 0$, $B \neq 0$, respectively.

We start with the differential constraint

$$\dot{h}_{11}(n) = 0 \tag{8.16}$$

and adopt it to be common for all feasible variants of reductions considered later. Then we immediately obtain

$$a_{11}(n) = a_{11}. \tag{8.17}$$

As we might have already got accustomed, the structure of the remained general equations (8.2)–(8.13) must tolerate the equality

$$h_{11}(n) = 1. \tag{8.18}$$

Now, let us concentrate on the case where $A \neq 0$ and $B = 0$. Due to the symmetry of the general equations (8.1)–(8.8) with respect to the permutations $g_{12}(n) \rightleftarrows f_{13}(n)$, $g_{21}(n) \rightleftarrows f_{31}(n)$, $f_{22}(n) \rightleftarrows f_{33}(n)$, and $a_{22}(n) \rightleftarrows a_{33}(n)$, the only reasonable use of the natural constraint (8.15) is to exclude the function $f_{11}(n)$ from the further consideration. As a consequence, we come to the variant characterized by the constraints

$$f_{11}(n) = g_{12}(n)g_{21}(n)/f_{22}(n) + f_{13}(n)f_{31}(n)/f_{33}(n), \tag{8.19}$$

$$\dot{f}_{22}(n) = 0, \tag{8.20}$$

$$\dot{f}_{33}(n) = 0. \tag{8.21}$$

These constraints yield

$$f_{22}(n) = f_{22}, \tag{8.22}$$

$$f_{33}(n) = f_{33} \tag{8.23}$$

and

$$a_{22}(n) = a_{22}, \tag{8.24}$$
$$a_{33}(n) = a_{33} \tag{8.25}$$

with $\dot{f}_{22} = 0 = \dot{f}_{33}$, while a_{22} and a_{33} being some arbitrary functions of the time.

Then, introducing the definitions

$$g_+(n) = g_{12}(n)/\sqrt{f_{22}}, \tag{8.26}$$
$$g_-(n) = g_{21}(n)/\sqrt{f_{22}}, \tag{8.27}$$

and

$$f_+(n) = f_{13}(n)/\sqrt{f_{33}}, \tag{8.28}$$
$$f_-(n) = f_{31}(n)/\sqrt{f_{33}}, \tag{8.29}$$

we reveal that the reduced integrable system of our interest can be written in the standard Hamiltonian form

$$\dot{f}_+(n) = -\partial H/\partial f_-(n), \tag{8.30}$$
$$\dot{f}_-(n) = +\partial H/\partial f_+(n), \tag{8.31}$$
$$\dot{g}_+(n) = -\partial H/\partial g_-(n), \tag{8.32}$$
$$\dot{g}_-(n) = +\partial H/\partial g_+(n) \tag{8.33}$$

with the Hamiltonian function H given by the formula

$$H = \sum_{m=-\infty}^{\infty} (a_{33} - a_{11}) f_+(m) f_-(m) + \\ + \sum_{m=-\infty}^{\infty} (a_{22} - a_{11}) g_+(m) g_-(m) - \\ - \sum_{m=-\infty}^{\infty} c_{11} f_{33} f_+(m) f_-(m-1) - \\ - \sum_{m=-\infty}^{\infty} c_{11} f_{22} g_+(m) g_-(m-1) + \\ + \sum_{m=-\infty}^{\infty} (c_{11}/2)[f_+(m) f_-(m) + g_+(m) g_-(m)]^2 \tag{8.34}$$

and the quantities $f_+(n)$, $f_-(n)$ and $g_+(n)$, $g_-(n)$ serving as two pairs of canonical field variables.

This integrable system (8.30)–(8.34) can be treated as a sort of two self-trapping subsystems coupled together by an additional mutual-trapping nonlinearity. In so doing, the conserved quantities

$$N_f = \sum_{m=-\infty}^{\infty} f_+(m) f_-(m) \tag{8.35}$$

and

$$N_g = \sum_{m=-\infty}^{\infty} g_+(m) g_-(m) \tag{8.36}$$

should be understood as the total numbers of excitations in the f-th and g-th subsystems, respectively.

As usual, the time independence of the free parameters $a_{11} - a_{22}$, $a_{11} - a_{33}$, and c_{33} ensures the obtained nonlinear system (8.30)–(8.34) to be $\mathcal{PT}$-symmetric.

At last, it is the time to switch-over our attention onto the case where $A = 0$ and $B \neq 0$. Analyzing the natural constraint (8.15) and the general equations (8.1)–(8.13), we may specify the two seemingly distinct variants of reduced systems. They are characterized by the two following sets of constraints:

$$f_{33}(n) = 0, \tag{8.37}$$
$$\frac{d}{d\tau}[f_{31}(n) f_{13}(n)] = 0, \tag{8.38}$$
$$\dot{f}_{22}(n) = 0 \tag{8.39}$$

and

$$f_{22}(n) = 0, \tag{8.40}$$
$$\frac{d}{d\tau}[g_{21}(n) g_{12}(n)] = 0, \tag{8.41}$$
$$\dot{f}_{33}(n) = 0, \tag{8.42}$$

respectively. However, owing to the permutation symmetry, $g_{12}(n) \leftrightarrows f_{13}(n)$, $g_{21}(n) \leftrightarrows f_{31}(n)$, $f_{22}(n) \leftrightarrows f_{33}(n)$, $a_{22}(n) \leftrightarrows a_{33}(n)$, of the general equations (8.1)–(8.13), these two reduced systems turn out to be physically indistinguishable. For this reason, we will isolate only one of them. To be definite, we will rely upon the first set (8.37)–(8.39) of constraints. These constraints yield

$$f_{13}(n) = f_{13} \exp[+q(n)], \tag{8.43}$$
$$f_{31}(n) = f_{31} \exp[-q(n)], \tag{8.44}$$
$$f_{22}(n) = f_{22} \tag{8.45}$$

and

$$a_{33}(n) = a_{33}, \tag{8.46}$$
$$a_{22}(n) = a_{22} \tag{8.47}$$

with $\dot{f}_{13} = 0 = \dot{f}_{31}$ and $\dot{f}_{22} = 0$, while a_{33} and a_{22} being some arbitrary functions of the time.

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The next step consists in the proper adjustment of field variables. Thus, introducing the substitutions

$$p(n) = g_{12}(n)g_{21}(n)/f_{22} - f_{11}(n) \tag{8.48}$$

and

$$g_+(n) = g_{12}(n)/\sqrt{f_{22}}, \tag{8.49}$$

$$g_-(n) = g_{21}(n)/\sqrt{f_{22}}, \tag{8.50}$$

we achieve that the reduced integrable system under study can be embedded into the standard Hamiltonian form

$$\dot{p}(n) = -\partial H/\partial q(n), \tag{8.51}$$

$$\dot{q}(n) = +\partial H/\partial p(n), \tag{8.52}$$

$$\dot{g}_+(n) = -\partial H/\partial g_-(n), \tag{8.53}$$

$$\dot{g}_-(n) = +\partial H/\partial g_+(n) \tag{8.54}$$

with the Hamiltonian function H defined by the expression

$$\begin{aligned} H = &\sum_{m=-\infty}^{\infty} (a_{11} - a_{33})p(m) + \\ &+ \sum_{m=-\infty}^{\infty} (a_{22} - a_{11})g_+(m)g_-(m) + \\ &+ \sum_{m=-\infty}^{\infty} (c_{11}/2)[p(m) - g_+(m)g_-(m)]^2 - \\ &- \sum_{m=-\infty}^{\infty} f_{31}c_{11}f_{13}\exp[+q(m) - q(m-1)] - \\ &- \sum_{m=-\infty}^{\infty} c_{11}f_{22}g_+(m)g_-(m-1) \end{aligned} \tag{8.55}$$

and the quantities $p(n)$, $q(n)$ and $g_+(n)$, $g_-(n)$ making sense of the canonical field variables for two interacting subsystems.

The subsystem described by the variables $p(n)$ and $q(n)$ can be treated as some Toda-like subsystem, while the subsystem described by the variables $g_+(n)$ and $g_-(n)$ can be understood as some self-trapping subsystem with the total number of excitations

$$N = \sum_{m=-\infty}^{\infty} g_+(m)g_-(m) \tag{8.56}$$

being a conserved quantity. As to the highly nontrivial interaction between the subsystems, it appears to find some remote likeness with the interaction between a charged particle with the electromagnetic radiation [41, 42]. Another interesting reminiscence concerning the nontrivial term $\sum_{m=-\infty}^{\infty}(c_{11}/2) \times [p(m) - g_+(m)g_-(m)]^2$ in the Hamiltonian function (8.55) can be awaked by the famous Lee–Low–Pines Hamiltonian function [43–45] in the theory of polarons.

The integrable system (8.51)–(8.55) as a whole is proved to be $\mathcal{PT}$-symmetric provided the free parameters $a_{11} - a_{22}$, $a_{11} - a_{33}$, and c_{33} are time-independent.

9. Local Conserved Densities for the Systems with Laurent-Like Lax Operators

Inasmuch as each semidiscrete nonlinear system obtained in this (second) part admits the zero-curvature representation, it is possible to generate the respective hierarchy of local conservation laws by the generalized recursive procedure [26].

Several first conserved densities found in the framework of the generalized recursive scheme [26] for the general (unreduced) systems (systems (7.1)–(7.13) and(8.1)–(8.13)) isolated in the second part of the present work are as follows:

$$\rho(n|0) = \ln h_{11}(n), \tag{9.1}$$

$$\rho(n|1) = f_{11}(n)/h_{11}(n), \tag{9.2}$$

$$\begin{aligned} \rho^+(n|2) = &\frac{g_{12}(n+1)g_{21}(n)}{h_{11}(n+1)h_{11}(n)} + \\ &+ \frac{f_{13}(n+1)f_{31}(n)}{h_{11}(n+1)h_{11}(n)} - \frac{f_{11}^2(n)}{2h_{11}^2(n)} \end{aligned} \tag{9.3}$$

$$\begin{aligned} \rho^-(n|2) = &\frac{g_{12}(n)g_{21}(n-1)}{h_{11}(n)h_{11}(n-1)} + \\ &+ \frac{f_{13}(n)f_{31}(n-1)}{h_{11}(n)h_{11}(n-1)} - \frac{f_{11}^2(n)}{2h_{11}^2(n)}. \end{aligned} \tag{9.4}$$

In order to apply these general formulas (9.1)–(9.4) for local densities to any particular reduced system taken among the listed ones in two previous sections, they must be rewritten in terms of the pertinent true field variables.

Moreover, except for the unessential linear terms, the general form (9.3) or (9.4) of the second local conserved density $\rho^+(n|2)$ or $\rho^-(n|2)$ is convertible into the Hamiltonian density for any system of the second type (see formulas (8.34) and (8.55) in Section 8). On the other hand, such a conversion turns out to be impossible for the systems of the first type listed in Section 7 (see formulas (7.31), (7.55), and (7.72) for the respective Hamiltonians).

The latter fact indicates strictly that at least the first-type integrable systems have a good chance to become the bi-Hamiltonian ones. This conjecture is in lines with the fundamental property of the bi-Hamiltonian or multi-Hamiltonian presentability demonstrated by a majority of already known semidiscrete integrable systems [46–55].

10. Conclusion

In this paper, we have presented two large classes of integrable nonlinear dynamical systems on regular quasione-dimensional lattices obtained in the framework of a matrix-valued semidiscrete zero-curvature equation. The first class is characterized by the Taylor-like ansatz for a spectral operator of the third order and consists of two subclasses distinguished by the ansatz for the evolution operator. The second class is characterized by the Laurent-like ansatz for a spectral operator of the third order and consists of two subclasses distinguished by the ansatz for the evolution operator.

Each system from the second class demonstrates a clear Hamiltonian structure with the standard Poisson brackets and includes two interacting subsystems of the sufficiently understandable physical origin. Each system from the first class also includes two interacting subsystems, but their physical interpretation is not simple, inasmuch as the respective Hamiltonian structure must be supposedly linked with the essentially nonstandard Poisson brackets.

Due to the interaction between the constituent subsystems, each particular system as a whole must exhibit a richer dynamical behavior as compared with the dynamics of its noninteracting constituents. Such an enrichment of the integrable dynamics is expected to find its natural applicability to the rigorous modeling of interesting physical problems feeded by the complex phenomena on quasione-dimensional lattice structures of a widely diversified origin.

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НАПІВДИСКРЕТНІ ІНТЕГРОВНІ СИСТЕМИ, НАВІЯНІ МОДЕЛЛЮ ДАВИДОВА–КИСЛУХИ

Р е з ю м е

У спробі відтворити деякі фізичні риси екситон-фононної системи Давидова–Кислухи ми виявили чотири різні комбінації анзаців для матричнозначних операторів Лакса, здатних в рамках представлення нульової кривини згенерувати цілу низку напівдискретних інтегровних нелінійних систем.

Спираючись на тейлорівську форму анзаців для операторів Лакса, запропоновано два типи загальних нелінійних інтегровних систем на безмежних квазіодновимірних регулярних ґратках. Відповідно до теорії редукційних груп Михайлова обидві загальні системи виявилися недовизначеними, що дозволяє започаткувати численні редуковані системи в термінах справжніх польових змінних. Кожну з одержаних редукованих систем слід вважати інтегровною версією певних двох підсистем, причому системі в цілому властива симетрія інверсії простору та часу ($\mathcal{PT}$-симетрія). Так, вдалося об'єднати коливну підсистему, подібну до тодівської, з ґратчастою підсистемою самозахоплення в єдину інтегровну систему, тим самим суттєво розширивши перелік реалістичних фізичних систем, придатних для строгого моделювання. В термінах прототипних польових функцій явно знайдено декілька перших густин, пов'язаних з будь-якою з можливих ієрархій локальних законів збереження.

Звернувшись до лоранівської форми анзаців для операторів Лакса, знайдено чотири нові напівдискретні нелінійні інтегровні системи, цікаві для фізичних застосувань. Поперше, підсистему, подібну до тодівської, вдалося пов'язати з підсистемою $\mathcal{PT}$-симетричних екситонів з наведеною нелінійністю. Інша інтегровна система виникла як підсистема екситонів типу френкелівських, пов'язаних з суттєво нетривіальною коливною підсистемою. Виявлено також інтегровну систему, що складається з двох самозахопних підсистем, поєднаних за допомогою взаємно-індукованої нелінійности. Нарешті, одержано інтегровну систему, де Тода-подібна підсистема та самозахопна підсистема взаємодіють на кшталт зарядженої частинки з електромагнітним полем. При цьому, частина гамільтоніна з вектор-потенціалом виявилася пропорційною густині збуджень в самозахопній підсистемі. Кожна з запропонованих інтегровних систем допускає чітке гамільтонівське представлення, що характеризується двома парами канонічних польових змінних зі стандартною (недеформованою) пуасонівською структурою. В рамках узагальненої прямої процедури явно знайдено декілька густин із загальних локальних законів збереження. Ці густини легко адаптувати до будь-якої інтегровної системи, пов'язаної з операторами Лакса лоранівської форми.