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EXTENDED STATE SPACE OF THE RATIONAL $sl(2)$ GAUDIN MODEL IN TERMS OF LAGUERRE POLYNOMIALS

We consider the rational Gaudin model with non-zero magnetic field, which physically corresponds to the central spin problem. The space of states is described in terms of separated variables. The states of a spin system are given by rational (up to an exponential factor) functions of these variables on the Lagrangian submanifold. We build a representation of the $sl(2)$ algebra of the model in terms of Laguerre polynomials and formulate the functional Bethe ansatz using it.

Keywords: Gaudin model, $sl(2)$ representation theory, Laguerre polynomials.

1. Introduction

The rational Gaudin model [1, 2] is an integrable model which appears as a limiting case of the integrable XXX spin chain (Gaudin magnet). M. Gaudin who introduced this model showed in [1] that it can be treated by means of the Bethe ansatz. In 1987, E. Sklyanin [3] applied the separation of variables technique for the Gaudin model and constructed Bethe vectors as the Schrödinger wave functions that depend on the generalized coordinates of underlying classical Hamiltonian system. The generalized coordinates used in [3] were the so-called “root variables” well known for a long time in the theory of finite-zone integration of classical Hamiltonian systems [4]. There are two ways to introduce those coordinates. The first way is to use the analog of variables used by V. Kotlyarov [6] adopted for the integration of a nonlinear Schrödinger equation (or, equivalently, classical Heisenberg magnet). The number of these variables is one less than that required for the parametrization of the complex Liouville torus of a classical integrable system, and they (together with canonically conjugated variables) parametrize only the reduced phase space. The additional variable which one needs to introduce corresponds to the dynamical variable $S^- = \sum_i S_i^-$ for a Gaudin magnet. It enters the set of separated variables non-symmetrically that causes the essential complication for a representation of the $sl(2)$ algebra in the state space of the Gaudin model.

The second way to introduce “root variables” is to use the analog of Previato’s variables [8]. Their number is enough for the parametrization of a complex Liouville torus which is holomorphically equivalent in this case to the generalized Jacobian of a singular hyperelliptic Riemann surface.

In this paper, the role of separated variables is played by Previato’s variables (and canonically conjugated to them), which are constructed using an algebraic curve. The latter serves as the generating function for a commuting family of integrals of motion and constraints. The quantum Hamiltonians of the Gaudin model and operators of the $sl(2)$ algebra are obtained as a result of the canonic quantization in terms of separated variables. The rules of ordering are established on the basis of the correspondence principle between the Poisson bracket and the commutators of elements of local $sl(2)$ algebras.

It is shown that the state space of the Gaudin model is (up to an exponential factor) a space of symmetric rational functions of the separated variables. A convenient basis in this space is formed by the eigenfunctions of the operator $S^z = \sum_i S_i^z$. We show that they can be expressed by Laguerre polynomials which depend on symmetric combinations of separated variables. This form of states is used for the formulation of the functional Bethe ansatz.

2. Formulation of a Model

2.1. Rational sl(2) Gaudin model as a central spin model

The Gaudin model is well known in numerous mathematical contexts and physical applications [2, 9–13]. It has the commutative set of integrals of motion:

$$\mathcal{H}_i = 2gS_i^z + \sum_{k=1,k\neq i}^{N} f(\varepsilon_i - \varepsilon_k)(\mathbf{S}_i, \mathbf{S}_k),$$

where ε_i, $i = 1, 2, ..., N$, are some complex parameters. The dependence on them can be rational $1/(\varepsilon_i - \varepsilon_k)$, trigonometric $1/\sin(\varepsilon_i - \varepsilon_k)$, or elliptic $1/\mathrm{sn}(\varepsilon_i - \varepsilon_k)$ (sn is the Jacobi elliptic function [16]).

If one chooses $f(\varepsilon_i - \varepsilon_k) = 1/(\varepsilon_i - \varepsilon_k)$, the corresponding model is referred to as a rational Gaudin model with Hamiltonians

$$\mathcal{H}_i = 2gS_i^z + \sum_{k=1,k\neq i}^{N} \frac{(\mathbf{S}_i, \mathbf{S}_k)}{\varepsilon_i - \varepsilon_k}. \qquad (1)$$

The presentation in such a form reveals immediately its physical nature of the so-called central spin model [12]. Indeed, it describes the system of spins $\mathbf{S}_i$, $i = 2, 3, ..., N$, which do not interact with each other, but every one interacts with the particular spin $\mathbf{S}_1$. If this "central" spin is subjected to an external static magnetic field B, then the Hamiltonian is nothing but:

$$\mathcal{H}_1 = -BS_1^z - \sum_{k=2}^{N} \gamma_k(\mathbf{S}_1, \mathbf{S}_k), \qquad (2)$$

where γ_k are constants that characterize the interaction of a distinguished spin $\mathbf{S}_1$ with some spin $\mathbf{S}_k$. This model can be regarded both as classical and quantum.

The comparison of Hamiltonians (1) and (2) gives evidence for a physical interpretation of the parameters in the Gaudin Hamiltonians: g corresponds to a magnetic field, and $\{\varepsilon_i\}$ are interaction constants.

2.2. Integrability of the classical model

The classical Gaudin model is a Hamiltonian system with Hamiltonians given by (1) and a non-canonical Poisson bracket

$$\{S_i^a, S_k^b\} = \delta_{ik}\varepsilon^{abc}S_i^c, \qquad (3)$$

where a, b, c denote spin projections. It leads to the following equations of motion for the dynamical variables

$$\frac{d}{dt_k}S_i^a = \{S_i^a, \mathcal{H}_k\}.$$

As the constraints $S_i^2 = \text{const}$, $i = 1, ..., N$, hold, the phase space is a product of N spheres $\mathcal{M}^{2N} = S^2 \times ... \times S^2$, which are called the Bloch spheres in the theory of magnetism. Therefore, the dimension of a phase state of the Gaudin model is $2N$.

It is easy to check that Hamiltonians (1) are in involution $\{\mathcal{H}_i, \mathcal{H}_k\} = 0$ with respect to the bracket (3). Thus, the classical Gaudin model is integrable due to the Liouville theorem. The latter implies that, as the number of Gaudin Hamiltonians is equal to a half of the phase space dimension, and they are in involution, then there exists the Liouville torus as a joint level surface for the integrals of motion

$$\{\mathcal{H}_i = h_i = \text{const} \simeq T^N = S^1 \times S^1 ... \times S^1.$$

But, for the further quantization of the system, it is much more important that the Gaudin model is algebraically integrable. Particularly, this means that one can construct the Lax matrix that corresponds to the Gaudin model

$$L(\lambda) = \begin{bmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & -\alpha(\lambda) \end{bmatrix} = \begin{bmatrix} g + \sum_{i=1}^{N} \frac{S_i^z}{\lambda - \varepsilon_i} & \sum_{i=1}^{N} \frac{S_i^-}{\lambda - \varepsilon_i} \\ \sum_{i=1}^{N} \frac{S_i^+}{\lambda - \varepsilon_i} & -g - \sum_{i=1}^{N} \frac{S_i^z}{\lambda - \varepsilon_i} \end{bmatrix}, \qquad (4)$$

where λ is called a spectral parameter.

The dynamics of the Lax operator is isospectral, so the quantities like $Tr(L^n)$ conserve. Since $L(\lambda)$ is traceless, $Tr(L^2)$ presents the first nontrivial conserved combination. Evidently, it is a generating function for the integrals of motion

$$\mu^2(\lambda) \equiv \frac{1}{2}\mathrm{Tr}(L^2) = \alpha^2(\lambda) + \beta(\lambda)\gamma(\lambda) = g^2 + \sum_{k=1}^{N} \frac{S_k^2}{(\lambda - \varepsilon_k)^2} + \sum_{k=1}^{N} \frac{\mathcal{H}_k}{(\lambda - \varepsilon_k)}, \qquad (5)$$

and

$$\mathcal{H}_k = 2gS_k^z + \sum_{i=1,i\neq k}^{N} \frac{2S_k^z S_i^z + S_k^+ S_i^- + S_k^- S_i^+}{\varepsilon_k - \varepsilon_i}$$

coincide with the Gaudin Hamiltonians (1).

Let us multiply both sides of (5) by $\prod_k(\lambda - \varepsilon_k)^2$ and introduce a function $\tilde{\mu}^2(\lambda) \equiv \prod_k(\lambda - \varepsilon_k)^2\mu^2(\lambda)$.

The r.h.s. of (5) becomes a polynomial in λ of degree $2N$ which we will denote $P_{2N}(\lambda)$. As a result, we have equation

$$\tilde{\mu}^2(\lambda) - P_{2N}(\lambda) = 0. \tag{6}$$

It corresponds to the algebraic curve that is a hyperelliptic Riemann surface of genus $N-1$.

The algebraic integrability of the model implies that there exists some complexification of its Liouville torus $T^N \to T^N_{\mathbb{C}}$ such that $T^N_{\mathbb{C}}$ is the Jacobian of a Riemann surface corresponding to the algebraic curve (6). Due to the general theory of finite-zone integration, one can choose a set of variables on the algebraic curve (separated variables) and formulate the Jacobi inversion problem [4, 5]. There exists a constructive procedure which allows one to solve it and to construct explicit expressions for the dynamics of the variables S_i^a generated by the Hamiltonian $\mathcal{H}_k$. For the classical Gaudin magnet, which consists of N spins, the solutions are given in terms of the Riemann theta-functions of genus $N-1$ or Kleinian σ- and ζ-functions of the same genus [12].

2.3. Separation of variables

We need to introduce separated variables for the further quantization. Due to the standard procedure, it is possible to obtain $N-1$ variables as the zeros of a polynomial $\gamma(\lambda)$ (or, equivalently, $\beta(\lambda)$). The canonically conjugated variables are computed as the values of $\tilde{\mu} \equiv \sqrt{\tilde{\mu}^2(\lambda)}$ at these points. Therefore, we have $2N-2$ new variables. But, as the initial problem has $2N$ dynamical variables, we lost the one-to-one correspondence between these two sets. In the theory of classical integration, this results in the necessity of an additional integration, as the remaining variable (and conjugated to it) satisfies a separate equation depending on all other variables [12]. In the case of quantization, it causes that the remaining variable becomes distinguished, and the representation is not symmetric in all variables [3].

Therefore, we will use the so-called Previato separated variables, whose number coincides with that of initial dynamical variables. For this goal, we make a gauge transformation with the Lax matrix (4) which does not change its invariants, but increases the degree of off-diagonal polynomials:

$$\widetilde{L}(\lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & -a(\lambda) \end{bmatrix} = \begin{bmatrix} \frac{\beta+\gamma}{2} & \frac{\beta-\gamma}{2} - \alpha \\ \frac{\gamma-\beta}{2} - \alpha & -\frac{\beta+\gamma}{2} \end{bmatrix}.$$

If we now choose λ_i as zeros of the rational function $c(\lambda)$, their number will be equal to N:

$$c(\lambda_i) = 0 \quad \Leftrightarrow \quad \sum_k \frac{\imath S_k^y - S_k^z}{\lambda_i - \varepsilon_k} = g. \tag{7}$$

The conjugated variables μ_i are defined by the algebraic curve equation:

$$\mu_i^2 = \mu^2(\lambda_i) = g^2 + \sum_{k=1}^N \frac{S_k^2}{(\lambda_i - \varepsilon_k)^2} + \sum_{k=1}^N \frac{\mathcal{H}_k}{(\lambda_i - \varepsilon_k)}. \tag{8}$$

It can be shown that the Poisson bracket is quasi-canonical

$$\{\lambda_j, \mu_k\} = \imath \delta_{jk}. \tag{9}$$

Equation (7) can be regarded as a linear system of equations which connects λ_i with a combination of the initial variables. Solving it, we obtain

$$(S_i^z - \imath S_i^y) = -g \frac{\prod_k (\varepsilon_i - \lambda_k)}{\prod_{k\neq i}(\varepsilon_i - \varepsilon_k)}. \tag{10}$$

Using the equation $\mu_i = a(\lambda_i)$, we get expression for the variable S_i^x:

$$S_i^x = -\frac{\prod_k (\varepsilon_i - \lambda_k)}{\prod_{k\neq i}(\varepsilon_i - \varepsilon_k)} \sum_j \frac{\prod_{k\neq i}(\lambda_j - \varepsilon_k)}{\prod_{k\neq j}(\lambda_j - \lambda_k)} \mu_j. \tag{11}$$

Since there is the constraint $S_i^2 = \text{const}$, only two of three operators (for every spin) are independent.

The Hamiltonians can be expressed in terms of the separated variables solving (8) with respect to $\mathcal{H}_k$:

$$\mathcal{H}_i = -\frac{\prod_k (\varepsilon_i - \lambda_k)}{\prod_{k\neq i}(\varepsilon_i - \varepsilon_k)} \sum_j \frac{\prod_{k\neq i}(\lambda_j - \varepsilon_k)}{\prod_{k\neq j}(\lambda_j - \lambda_k)} M_j, \tag{12}$$

where $M_j = \mu_j^2 - g^2 - \sum_{k=1}^N \frac{S_k^2}{(\lambda_j - \varepsilon_k)^2}$.

Since we have expressions for all dynamical variables and Hamiltonians in terms of the separated variables, we can quantize the Gaudin model.

3. Quantum Gaudin Model

3.1. Canonical quantization

Inasmuch as we have the set of canonically conjugated variables (9), it is convenient to employ the canonical quantization scheme to quantize the Gaudin model.

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It gives the following recipe of transition from the classical picture to the quantum one:

$$\{\lambda_i, -\imath\mu_j\} = \delta_{ij} \quad \rightarrow \quad \left[\hat{\lambda}_i, -\imath\hat{\mu}_j\right] = \imath\hbar\delta_{ij}\hat{1}.$$

Hereinafter, we set $\hbar = 1$.

One of the possible ways to realize this condition is to chose one of the operators as a multiplication operator. Consequently, the representation space becomes the space of functions depending on the corresponding variable. Then the "conjugated" operator has to be a derivative with respect to this variable (up to a constant factor):

$$\hat{\lambda}_i\Psi(\lambda_1, \lambda_2, ..., \lambda_N) = \lambda_i\Psi(\lambda_1, \lambda_2, ..., \lambda_N),$$

$$\hat{\mu}_i\Psi(\lambda_1, \lambda_2, ..., \lambda_N) = \frac{\partial}{\partial\lambda_i}\Psi(\lambda_1, \lambda_2, ..., \lambda_N).$$

This choice is referred to as the Schrödinger picture.

Such a quantization corresponds to a representation of the algebra of the phase space symmetry group on a Lagrangian submanifold. It is an N-dimensional differentiable submanifold of the phase space such that the exterior form specifying the symplectic structure vanishes identically on it [17]. In terms of the discussed set of canonical coordinates, the submanifold parametrized by a half of them (namely, $\{\lambda_i\}$) is a Lagrangian submanifold.

As a result, we can build a representation of spin operators in terms of the separated variables. The rules of ordering are established on the basis of the correspondence principle between the Poisson bracket (3) and commutators of elements of local $sl(2)$ algebras. Due to the above-mentioned realization of the operators $\hat{\lambda}_i$, $\hat{\mu}_i$ and taking expressions (10)-(11) into account, we obtain that

$$\left(\hat{S}_i^z - \imath\hat{S}_i^y\right) = -g\frac{\prod_k(\varepsilon_i - \lambda_k)}{\prod_{k\neq i}(\varepsilon_i - \varepsilon_k)} \tag{13}$$

is a multiplication operator, and

$$\hat{S}_i^x = -\frac{\prod_k(\varepsilon_i - \lambda_k)}{\prod_{k\neq i}(\varepsilon_i - \varepsilon_k)}\sum_j\frac{\prod_{k\neq i}(\lambda_j - \varepsilon_k)}{\prod_{k\neq j}(\lambda_j - \lambda_k)}\frac{\partial}{\partial\lambda_j} \tag{14}$$

is a first-order differential operator. Since the expression for the operator $\left(\hat{S}_i^z + \imath\hat{S}_i^y\right)$ can be obtained using the quadratic constraint $\hat{S}_i^2 = j_i(j_i+1)\hat{1}$, it turns out to be a second-order differential operator. The constructed representation has two peculiar properties: all variables enter the operators symmetrically, and every operator depends on the full set of the separated variables.

We can rewrite these operators in a more usual form. If we introduce the multiplication operator

$$\hat{X}_i\Psi(\lambda_1, ..., \lambda_N) = -g\frac{\prod_k(\varepsilon_i - \lambda_k)}{\prod_{k\neq i}(\varepsilon_i - \varepsilon_k)}\Psi(\lambda_1, ..., \lambda_N) \equiv$$
$$\equiv X_i\Psi(\lambda_1, ..., \lambda_N), \tag{15}$$

the spin operators take the form

$$\left(\hat{S}_i^z - \imath\hat{S}_i^y\right) = X_i, \tag{16}$$

$$\left(\hat{S}_i^z + \imath\hat{S}_i^y\right) = \frac{1}{X_i}\left[X_i^2\frac{\partial^2}{\partial X_i^2} - j_i(j_i+1)\right], \tag{17}$$

$$\hat{S}_i^x = X_i\frac{\partial}{\partial X_i}. \tag{18}$$

The operators given by (16)–(18) form a representation of the $sl(2)$-triple for the i-th spin.

The Gaudin Hamiltonians in terms of the separated variables are as follows:

$$\hat{\mathcal{H}}_i = -\frac{\prod_k(\varepsilon_i - \lambda_k)}{\prod_{k\neq i}(\varepsilon_i - \varepsilon_k)}\sum_j\frac{\prod_{k\neq i}(\lambda_j - \varepsilon_k)}{\prod_{k\neq j}(\lambda_j - \lambda_k)}\hat{M}_j,$$

where $\hat{M}_j = \left[\frac{\partial^2}{\partial\lambda_j^2} - g^2 - \sum_{k=1}^N\frac{\hat{S}_k^2}{(\lambda_j-\varepsilon_k)^2}\right]$ or, equivalently,

$$\hat{\mathcal{H}}_i = \frac{g}{X_i}\left[-X_i^2 + X_i^2\frac{\partial^2}{\partial X_i^2} - j_i(j_i+1)\right] +$$
$$+\sum_k\frac{X_iX_k}{\varepsilon_i - \varepsilon_k}\left[\frac{\partial^2}{\partial X_i^2} + 2\frac{\partial^2}{\partial X_i\partial X_k} + \frac{\partial^2}{\partial X_k^2} -\right.$$
$$\left. - \frac{j_i(j_i+1)}{X_i^2} - \frac{j_k(j_k+1)}{X_k^2}\right].$$

3.2. Functional Bethe ansatz

Now, we can formulate a spectral problem that involves the solution of the stationary Schrödinger equation

$$\hat{\mathcal{H}}_i\Psi_{\{h_1,h_2,...,h_N\}}(\{\lambda_k\}) = h_i\Psi_{\{h_1,h_2,...,h_N\}}(\{\lambda_k\}),$$

i.e., finding the wave-function and the allowed values of energy h_i. Evidently, this problem is equivalent to the eigenvalue problem for the generating function $\hat{\mu}$, which means the simultaneous finding of eigenvalues and eigenfunctions of all Hamiltonians:

$$\hat{\mu}_i^2\Psi\left(\{\lambda_k\}\right) = $$
$$= \left[g^2 + \sum_{k=1}^{N}\frac{S_k^2}{(\lambda_i-\varepsilon_k)^2} + \sum_{k=1}^{N}\frac{\hat{\mathcal{H}}_k}{(\lambda_i-\varepsilon_k)}\right]\Psi(\{\lambda_k\}). \tag{19}$$

As follows from the last formula, the variables can be really separated, and we can represent the wave-function of the system as a product of "one-particle" wave-functions:

$$\Psi_{\{h_1,h_2,\dots,h_N\}}(\{\lambda_k\}) = \prod_{k=1}^{N}\varphi(\lambda_k).$$

Then the "one-particle" equation (index is omitted)

$$\left[\frac{\partial^2}{\partial\lambda^2} - g^2 - \sum_k\frac{j_k(j_k+1)}{(\lambda-\varepsilon_k)^2}\right]\varphi(\lambda) =$$
$$= \sum_k\frac{h_k(\lambda-\varepsilon_k)}{\varphi}(\lambda) \tag{20}$$

is an equation of the Fuchsian type. The analysis of a behavior near the singular points of the equation enables us to present a solution in the form

$$\varphi(\lambda) = C\prod_l(\lambda-\varepsilon_l)^{-j_l}e^{-g\lambda}F(\lambda), \tag{21}$$

where $F(\lambda)$ is a holomorphic function, and C is a constant. The substitution of ansatz (21) into (20) gives the equation for this unknown function:

$$F'' - \left[2g + \sum_k\frac{2j_k}{(\lambda-\varepsilon_k)}\right]F' - \sum_k\frac{a_k}{(\lambda-\varepsilon_k)}F = 0,$$

where $a_k = h_k - 2gj_k - 2\sum_{k\neq m} j_kj_m/(\varepsilon_k-\varepsilon_m)$. It is easy to check that this equation has a polynomial solution

$$F(\lambda) = \prod_{\alpha=1}^{M}(\lambda-\eta_\alpha)$$

under condition that its roots satisfy the system of algebraic equations

$$\sum_{k=1}^{N}\frac{j_k}{(\eta_\alpha-\varepsilon_k)} - \sum_{\beta=1}^{M}\frac{1}{\eta_\alpha-\eta_\beta} + g = 0. \tag{22}$$

They are referred to as algebraic Bethe equations. Even in the case $M = 1$ when system (22) degenerates into one algebraic equation of the N-th degree, it is still hardly to be solved. Thus, the analytic expressions for Bethe roots are found only in a few very special cases [14], while the problem is treated numerically in the majority of cases [15].

If the solutions of (22) called Bethe roots are found, the spectrum is expressed as

$$h_k = 2gj_k + 2j_k\sum_{m=1,m\neq k}^{N}\left[\frac{j_m}{\varepsilon_k-\varepsilon_m} - \sum_{\alpha=1}^{M}\frac{1}{\eta_\alpha-\varepsilon_k}\right],$$

where M is interpreted as the number of excitations. If $M = 0$, the spin system is in the ground ("vacuum") state with maximal value of energy, while the case of $M \geq 1$ corresponds to spin configurations with lower energy.

This approach for the Gaudin model referred to as the functional Bethe ansatz was suggested by E. Sklyanin in [3].

4. Basis in the Space of Spin States in Terms of Laguerre Polynomials

4.1. States for $N = 1$ in terms of Laguerre polynomials

Let us start from the degenerate case of one spin. There is no interaction and the Hamiltonian of the system is proportional to $\hat{S}_1^z$

$$\hat{\mathcal{H}}_1 = 2g\hat{S}_1^z.$$

Thus, the eigenproblem for $N = 1$ reduces to solving the equation

$$\hat{\mathcal{H}}_1\Psi_m^{[j_1]} = 2g(j_1-m)\Psi_m^{[j_1]}. \tag{23}$$

As is easy to deduce from (16) and (17), $\hat{S}_i^z$ is a differential operator of the second order

$$\hat{S}_i^z = \frac{1}{2}\left[X_i\frac{\partial^2}{\partial X_i^2} - X_i - \frac{j_i(j_i+1)}{X_i}\right].$$

In the case $N = 1$, there is one separated variable, so $X_1 = g(\lambda_1-\varepsilon_1)$. It follows that Eq. (23) is a confluent hypergeometric equation which can be solved in terms of Whittaker functions [16].

According to (23), the vacuum state, which corresponds to $m = 0$, is given by

$$\Psi_0^{[j_1]}(X_1) = e^{-X_1}(2X_1)^{-j_1}.$$

This expression is in agreement with (21) as $M = 0$.

Taking into account that j_1 is half-integer (or integer), the arbitrary state can be presented as

$$\Psi_m^{[j_1]} = (-1)^{j_1-m}\mathcal{L}_{j_1-m}^{(\alpha)}(2X_1)\Psi_0^{[j_1]}(X_1),$$

where $\mathcal{L}_n^{(\alpha)}(2X_1)$ is a generalized Laguerre polynomial [18], $\alpha = -(2j_1+1)$. We will use this result as a hypothesis for the structure of a state for an N-spin system.

4.2. Description of states in terms of Laguerre polynomials for arbitrary N

The vacuum state (eigenstate of the Gaudin Hamiltonian with maximal energy) for any number of spins is a product of "one-particle" vacuum states:

$$\Psi_0^{[j]} = C_1 \exp\left[-g\sum_{i=1}^{N}(\lambda_i - \varepsilon_i)\right]\prod_{i,k}(\lambda_i - \varepsilon_k)^{-j_k} = $$

$$= C_1\prod_{i=1}^{N} e^{-X_i}X_i^{-j_i}. \qquad (24)$$

Obviously, this state is an eigenvector of the operator $\hat{S}^z = \sum_i \hat{S}_i^z$ and is the highest weight vector in this space:

$$\hat{S}^z\Psi_0^{[j]} = \left(\sum_k j_k\right)\Psi_0^{[j]}, \quad \hat{S}^+\Psi_0^{[j]} = 0. \qquad (25)$$

Our goal is to construct the vectors that satisfy the condition

$$\hat{S}^z\Phi_m^{[j]} = (j-m)\Phi_m^{[j]}. \qquad (26)$$

Obviously, $\Phi_0^{[j]} = \Psi_0^{[j]}$. For the further analysis, it is important that these states can be obtained also as an m-tuple action of the "lowering" operator

$$\Phi_m^{[j]} = (\hat{S}^-)^m\Psi_0^{[j]},$$

where $S^- = \sum_i S_i^-$. Let us rewrite $\hat{S}^-$ as

$$\hat{S}^- = -2X - 2\hat{S}_3 + \hat{S}^+,$$

where $X = \sum_i X_i$. Now, the $\hat{S}_z$ eigenstate takes the form

$$\Phi_m^{[j]} = (-2X - 2\hat{S}_3 + \hat{S}^+)^m\Psi_0^{[j]}.$$

Taking (25) into account and setting $m = 1$, we see that the state $\Phi_1^{[j]}$ is proportional to the generalized Laguerre polynomial $\mathcal{L}_1^{-(2j+1)}(2X)$.

We are now in the position to prove that an arbitrary eigenstate of $\hat{S}_z$ has the form

$$\Phi_m^{[j]} = \mathcal{L}_m^{(\alpha)}(2X)\Psi_0^{[j]}(X) \qquad (27)$$

with $X = g\sum_i(\lambda_i - \varepsilon_i)$, $\alpha = -(2j+1)$ and $\Psi_0^{[j]}$ is defined in (24). Therefore, such states form the basis in the Verma module with the highest weight j.

This assertion can be proven by induction. For the proof, it is necessary to compute such commutators as $[S^\pm, X] = \pm 2(S^\pm \mp S_3)$, $[S_3, X] = (S^+ + S^-)$, as well as their analogs with higher powers of X. We have used the recurrence formula

$$\mathcal{L}_{n+1}^{(\alpha)}(x) = \mathcal{L}_n^{(\alpha)}(x)\mathcal{L}_1^{(\alpha)}(x) + 2n\mathcal{L}_n^{(\alpha)}(x) -$$

$$-n(\alpha+n)\mathcal{L}_{n-1}^{(\alpha)}(x) \qquad (28)$$

for Laguerre polynomials. (Hereafter, the Laguerre polynomials are normalized by the fixation of the leading coefficient as $\mathcal{L}_m^{(\alpha)}(x) = (-1)^m x^m + ...$). Using these relations and computing the commutator $\left[\hat{S}^-, \mathcal{L}_m^{(\alpha)}(2X)\right]$, we can show that

$$S^-\mathcal{L}_m^{(\alpha)}(2X)\Psi_0^{[j]} = \mathcal{L}_{m+1}^{(\alpha)}(2X)\Psi_0^{[j]}.$$

This proves that any eigenstate for the operator $\hat{S}^z$ is represented by (27).

The normalization of these vectors can be done, by using the Shapovalov form

$$\langle\Phi_n^{[j]}, \Phi_m^{[j]}\rangle = \delta_{nm}\frac{(2j)!n!}{(2j-n)!}.$$

Apart from the fact that we presented the basis of spin states in the convenient form, we also built the extended state space, from which vectors of the Gaudin model can be constructed. The obtained Verma module is not bounded from below and is extended in the sense that each state is described by N separated variables.

4.3. Tensor structure of the state space in terms of Laguerre polynomials

The state space of an N-spin system is the space of an $sl(2)$ N-fold tensor product representation. It can be decomposed into a direct sum of irreducible components.

To recover the tensor structure of the state space, we need to introduce the notation of singular vectors that are the highest weight vectors in every irreducible component:

$$S^+\Phi_0^{[j-l]} = 0,$$

where $l = 1, ..., j$.

For example, for $N = 2$, such a vector has the form

$$\Phi_{j-M}^{[j-M]} = \sum_{m_k, m_p} C_{m_k;m_p}(S_k^-)^{j_k - m_k}(S_p^-)^{j_p - m_p}\Psi_0^{[j]},$$

where $C(m_k; m_p) = C^{\,j,j}_{j_k, m_k; j_p, m_p}$ are Clebsch–Gordan coefficients.

Analogously to the previous subsection, we can show that

$$(S_k^-)^n\Psi_0^{[j]} = \mathcal{L}_n^{-(2j_k+1)}(2X_k)\Psi_0^{[j]}, \tag{29}$$

using also the fact that the vacuum state is a product of one-spin vacuum states (24).

Therefore, the singular vector for $N = 2$ has the form

$$\Psi_{j-M}^{[j-M]} = \sum_{m_k, m_p} C_{m_k;m_p}\mathcal{L}_{j_k - m_k}^{-(2j_k+1)}(2X_k) \times$$
$$\times\, \mathcal{L}_{j_p - m_p}^{-(2j_p+1)}(2X_p)\Psi_0^{[j]}. \tag{30}$$

The construction of singular vectors for arbitrary N is a group-theoretic problem, which can be effectively solved (see the formula, e.g., in [7]). The corresponding expressions are composed of vectors like $(S_k^-)^l\Psi_0^{[j]}$ which we evaluated explicitly in terms of the separated variables (29).

4.4. Bethe states in terms of Laguerre polynomials

Hereinbefore, we constructed a basis of spin states that are the eigenstates of $\hat{S}^z = \sum_i \hat{S}_i^z$. It turns out that they can be expressed in terms of the generalized Laguerre polynomials depending on symmetric combinations of the separated variables. Now we can use this result for the eigenproblem for Gaudin Hamiltonians.

Let us recall the algebraic Bethe ansatz. Due to it, the eigenstate of the Gaudin Hamiltonian can be presented in the form

$$|\Psi(\eta_1, ..., \eta_M)\rangle = \hat{B}(\eta_1) ... \hat{B}(\eta_M)|0\rangle,$$

where $|0\rangle$ is the vacuum state, and the operator $B(\eta_\alpha)$ is constructed from "lowering" spin operators of each spin

$$\hat{B}(\eta_\alpha) = \sum_{k=1}^{N} \frac{\hat{S}_k^-}{\eta_\alpha - \varepsilon_k},$$

and η_α are defined from the Bethe equations (22).

Inasmuch as we succeeded to express explicitly the vacuum state (24) and the result of action of the operator $\hat{S}_k^-$ on it (29), we are able to formulate the Bethe ansatz in terms of Laguerre polynomials. This means that, with given Bethe roots η_α, one can construct explicitly the eigenstate of the Gaudin Hamiltonian:

$$\Psi(\{\eta_\alpha\}) = \prod_{\alpha=1}^{M}\left(\sum_{k=1}^{N} \frac{\mathcal{L}_{j_k - m_k}^{-(2j_k+1)}(2X_k)}{\eta_\alpha - \varepsilon_k}\right)\Psi_0^{[j]}.$$

This representation is convenient for the further study of Bethe vectors and for establishing some peculiarities of their dependence on the parameters η_α.

5. Conclusions and Discussion

In this paper, we consider the rational $sl(2)$ Gaudin model with non-zero magnetic field. Physically, it corresponds, in particular, to the central spin problem, which describes one distinguished spin interacting with other non-interacting spins.

It is well known that the classical version of this model admits separated variables that are associated with hyperelliptic curves. We introduce an extended set of such variables (their number is equal to the number of spins) on the generalized Jacobian of the model. The quantum counterpart of the Gaudin model is obtained by canonical quantization (in the Schrödinger picture) in terms of these separated variables. This quantization is equivalent to the construction of the phase space symmetry algebra representation on a Lagrangian submanifold. As a result, we present the state space of the model by the rational (up to an exponential factor) functions of separated variables.

We have built the representation of an N-fold tensor product of the $sl(2)$ algebra of the model in terms of generalized Laguerre polynomials, which depend on the separated variables. It is shown that this representation gives a convenient form of the functional Bethe ansatz for the rational Gaudin model.

It is known that an adequate understanding of the Gaudin model can be achieved if one extends the $sl(2)$

symmetry to the affine $\widetilde{sl}(2)$ [13]. We hope for that our results will be useful for the explicit construction and the investigation of such an extension.

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РОЗШИРЕНИЙ ПРОСТІР СТАНІВ
РАЦІОНАЛЬНОЇ $sl(2)$ МОДЕЛІ ГОДЕНА
В ТЕРМІНАХ ПОЛІНОМІВ ЛЯГЕРА

Р е з ю м е

Ми досліджуємо раціональну модель Годена з ненульовим магнітним полем, яка відповідає фізичній задачі центрального спіна. Простір станів описано в термінах змінних розділення. Стани спінової системи представлено як раціональні (з точністю до експоненційного множника) функції від цих змінних на лагранжевому многовиді. Ми будуємо представлення алгебри симетрій моделі $sl(2)$ в термінах узагальнених поліномів Лягера і формулюємо функціональний анзац Бете, використовуючи його.