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QUANTUM MECHANICS OF A SPIN 1 PARTICLE IN THE MAGNETIC MONOPOLE POTENTIAL, IN SPACES OF EUCLID AND LOBACHEVSKY: NON-RELATIVISTIC APPROXIMATION

A spin-1 particle is treated in the presence of a Dirac magnetic monopole in the non-relativistic approximation. After the separation of variables, the problem is reduced to the system of three interrelated equations, which can be disconnected with the use of a special linear transformation making the mixing matrix diagonal. As a result, there arise three separate differential equations which contain the roots of a cubic algebraic equation as parameters. The algorithm permits the extension to the case where external spherically symmetric fields are present. The cases of the Coulomb and oscillator potentials are treated in detail. The approach is generalized to the case of the Lobachevsky hyperbolic space. The exact solutions of the radial equation are constructed in terms of hypergeometric functions and Heun functions.

K e y w o r d s: magnetic monopole, the Duffin–Kemmer–Petiau equation, non-relativistic approximation, space of constant curvature, Coulomb field, oscillator potential.

1. Introduction

Spin significantly influences the behavior of quantum-mechanical particles in the field of a Dirac monopole (in particular, see [1] and references therein). In the literature, the cases of scalar and spin-1/2 particles are mainly treated. We turn to a spin-1 particle described by the usual 10-component Duffin–Kemmer–Petiau (DKP) equation in the presence of the Dirac monopole potential. Then we add the Coulomb and oscillator potentials, as well as a curved background of the hyperbolic Lobachevsky geometry. Since the radial systems turn out to be very complex, we use the non-relativistic approximation for them, so that the problems can be solved exactly. The extension to a non-Euclidean geometry is performed, and the case of Lobachevsky space is specified in detail.

2. Separation of the Variables

A spin-1 particle in the Dirac monopole potential is treated on the base of the tetrad formalism [2]

$$\left[i\beta^0 \partial_t + i \left(\beta^3 \partial_r + \frac{1}{r} \left(\beta^1 j^{31} + \beta^2 j^{32} \right) \right) + \right.$$

$$\left. + \frac{1}{r} \Sigma^k_{\theta,\phi} - M \right] \Phi(x) = 0, \tag{1}$$

where the angular operator $\Sigma^k_{\theta,\phi}$ is given by [2]

$$\Sigma^k_{\theta,\phi} = i\,\beta^1\,\partial_\theta + \beta^2\,\frac{i\,\partial_\phi + \left(i\,j^{12} - k\right)\cos\theta}{\sin\theta}, \qquad k = eg/\hbar c, \tag{2}$$

and the explicit expressions for three projections of the total angular momentum are

$$J^k_1 = l_1 + \frac{\cos\phi}{\sin\theta}(iJ^{12} - \kappa),$$

$$J_2^k = \; l_2 + \frac{\sin\phi}{\sin\theta}(iJ^{12} - \kappa), \quad J_3^k = l_3.$$

They are of Schrödinger's type [2]. Therefore, the 10-component wave function with quantum numbers $(\epsilon,\ j,\ m)$ is constructed as

$$\Phi_{\epsilon jm}(x) = e^{-i\epsilon t}\,[\, f_1(r)\, D_k, \;\; f_2(r)\, D_{k-1}, \;\; f_3(r)\, D_\kappa,$$
$$f_4(r)\, D_{k+1}, \;\; f_5(r)\, D_{k-1}, \;\; f_6(r)\, D_k, \;\; f_7(r)\, D_{k+1},$$
$$f_8(r)\, D_{k-1}, \;\; f_9(r)\, D_k, \;\; f_{10}(r)\, D_{k+1}]; \qquad (3)$$

D_σ stands for Wigner functions $D^j_{-m,\sigma}(\phi,\theta,0)$. The parameter $k = eg/\hbar c$ is quantized according to Dirac [3]:

$$\mid k \mid = 0,\ 1/2,\ 1,\ 3/2,\ 2,\ 5/2,\ ... \qquad (4)$$

With the use of the recurrent relations [4]

$$\begin{aligned}
&\partial_\theta\, D_{k-1} = a\, D_{\kappa-2} - c\, D_k, \\
&\frac{-m-(k-1)\cos\theta}{\sin\theta}\, D_{k-1} = -a\, D_{k-2} - c\, D_k, \\
&\partial_\theta\, D_k = (c\, D_{\kappa-1} - d\, D_{k+1}), \\
&\frac{-m-k\cos\theta}{\sin\theta}\, D_k = -c\, D_{\kappa-1} - d\, D_{k+1}, \\
&\partial_\theta\, D_{k+1} = (d\, D_k - b\, D_{k+2}), \\
&\frac{-m-(k+1)\cos\theta}{\sin\theta}\, D_{k+1} = -d\, D_k - b\, D_{k+2},
\end{aligned} \qquad (5)$$

where

$$\begin{aligned}
a &= \frac{1}{2}\sqrt{(j+k-1)(j-k+2)}, \\
b &= \frac{1}{2}\sqrt{(j-k-1)(j+k+2)}, \\
c &= \frac{1}{2}\sqrt{(j+k)(j-k+1)}, \\
d &= \frac{1}{2}\sqrt{(j-k)(j+k+1)},
\end{aligned} \qquad (6)$$

we obtain ten radial equations

$$\begin{aligned}
&-i\left(\frac{d}{dr}+\frac{1}{r}\right) f_2 - i\frac{\sqrt{2}c}{r} f_3 - M f_8 = 0, \\
&i\left(\frac{d}{dr}+\frac{1}{r}\right) f_4 + i\frac{\sqrt{2}d}{r} f_3 - M f_{10} = 0, \\
&i\epsilon f_5 + i\left(\frac{d}{dr}+\frac{1}{r}\right) f_8 + i\frac{\sqrt{2}c}{r} f_9 - M f_2 = 0, \\
&i\epsilon f_7 - i\left(\frac{d}{dr}+\frac{1}{r}\right) f_{10} - i\frac{\sqrt{2}d}{r} f_9 - M f_4 = 0, \\
&-i\epsilon f_2 + \frac{\sqrt{2}c}{r} f_1 - M f_5 = 0, \\
&-i\epsilon f_4 + \frac{\sqrt{2}d}{r} f_1 - M f_7 = 0, \\
&-\left(\frac{d}{dr}+\frac{2}{r}\right) f_6 - \frac{\sqrt{2}}{r}(c\, f_5 + d f_7) - M f_1 = 0, \\
&i\epsilon f_6 + \frac{\sqrt{2}i}{r}(-c f_8 + d f_{10}) - M f_3 = 0, \\
&i\frac{\sqrt{2}}{r}(c f_2 - d f_4) - M f_9 = 0, \\
&-i\epsilon f_3 - \frac{d}{dr} f_1 - M f_6 = 0.
\end{aligned} \qquad (7)$$

For the quantum number j, only the following values are allowed:

$$\begin{aligned}
&k = \pm 1/2, \quad j = \mid k \mid, \mid k \mid +1, ...; \\
&k = \pm 1, \pm 3/2, ..., \quad j = \mid k \mid -1, \mid k \mid, ...
\end{aligned} \qquad (8)$$

The states with $j = \mid k \mid -1$ must be treated separately.

For instance, let $k = +1$ and $j = 0$. Then the initial substitution is

$$\Phi^{(0)}(t,r) = e^{-i\epsilon t}(0,\ f_2,\ 0,\ 0,\ f_5,\ 0,\ 0,\ f_8,\ 0,\ 0); \qquad (9)$$

and we obtain three radial equations

$$\begin{aligned}
&f_5 = -i\frac{\epsilon}{M} f_2, \quad f_8 = -\frac{i}{M}\left(\frac{d}{dr}+\frac{1}{r}\right) f_2, \\
&f_2(r) = r^{-1} F_2(r), \quad \left(\frac{d^2}{dr^2} + \epsilon^2 - M^2\right) F_2 = 0.
\end{aligned} \qquad (10)$$

Such a state is the same as that in the spin-1/2 case.

The case $j = 0$, $k = -1$ is treated similarly:

$$\Phi^{(0)}(t,r) = e^{-i\epsilon t}(0,\ 0,\ 0,\ f_4,\ 0,\ 0,\ f_7,\ 0,\ 0,\ f_{10}),$$

and so on.

Let us consider the case $j = \mid k \mid -1$ with half-integer k: $k = \pm 3/2, \pm 2, \dots$. First, let k be positive. Then we must start with the substitution

$$k \geq 3/2,$$

$$\Phi^{(0)} = e^{-i\epsilon t}[0, f_2 D_{k-1}, 0, 0, f_5 D_{k-1}, 0, 0 f_8 D_{k-1}, 0, 0]. \qquad (11)$$

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With the use of the recurrent relations [4]

$$\partial_\theta \, D_{\kappa-1} = \sqrt{\frac{k-1}{2}} \, D_{k-2},$$

$$\frac{-m-(k-1)\cos\theta}{\sin\theta} \, D_{k-1} = -\sqrt{\frac{k-1}{2}} \, D_{k-2},$$

we obtain (the factor $e^{-i\epsilon t}$ is omitted)

$$i\beta^1 \, \Phi^0 = i\sqrt{\frac{k-1}{2}} \begin{vmatrix} -if_5 \, D_{k-2} \\ 0 \\ +f_8 \, D_{k-2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -f_2 \, D_{k-2} \\ 0 \end{vmatrix},$$

$$\beta^2 \, \frac{i\,\partial_\phi + (ij^{12}-k)\cos\theta}{\sin\theta} \, \Phi^0 = \sqrt{\frac{k-1}{2}} \begin{vmatrix} -f_5 \, D_{k-2} \\ 0 \\ -if_8 \, D_{\kappa-2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ +if_2 \, D_{\kappa-2} \\ 0 \end{vmatrix},$$

and then we get $\Sigma_{\theta,\phi}\Phi^{(0)} = 0$. Therefore, the radial system for $f_2, \; f_5, \; f_8$ will coincide with (10). The case $j = |k| - 1$ for negative k can be treated similarly:

$$k \leq -3/2,$$

$$\Phi^{(0)} = e^{-i\epsilon t}(0,\, 0,\, 0,\, f_4 \, D_{k+1},\, 0,\, 0,$$

$$f_7 \, D_{k+1},\, 0,\, 0,\, f_{10} \, D_{k+1}). \tag{12}$$

Again, we have the identity $\Sigma_{\theta,\phi}\Phi^{(0)} \equiv 0$.

3. Non-Relativistic Approximation

To proceed with the radial system, let us pass to the non-relativistic approximation (we will use the well-elaborated technique exposed in [5]). First, we exclude the non-dynamical components

$$-\left(\frac{d}{dr} + \frac{2}{r}\right) f_6 - \frac{\sqrt{2}}{r}(cf_5 + df_7) = Mf_1,$$

$$-i\left(\frac{d}{dr} + \frac{1}{r}\right) f_2 - i\frac{\sqrt{2}c}{r} f_3 = Mf_8,$$

$$i\frac{\sqrt{2}}{r}(cf_2 - df_4) = Mf_9,$$

$$i\left(\frac{d}{dr} + \frac{1}{r}\right) f_4 + \frac{i\sqrt{2}d}{r} f_3 = Mf_{10} \tag{13}$$

and then translate the equations to the more symmetric notation

$$(f_2,\, f_3,\, f_4) \longrightarrow (\Phi_1,\, \Phi_2,\, \Phi_3),$$

$$(f_5,\, f_6,\, f_7) \longrightarrow (E_1,\, E_2,\, E_3).$$

Thus, we arrive at

$$i\left(\frac{d}{dr} + \frac{1}{\text{sh}\, r}\right)\left[-i\left(\frac{d}{dr} + \frac{1}{\text{sh}\, r}\right)\Phi_1 - i\frac{\sqrt{2}c}{\text{sh}\, r}\Phi_2\right] +$$
$$+ i\frac{\sqrt{2}c}{\text{sh}\, r}\left[i\frac{\sqrt{2}}{r}(c\Phi_1 - d\Phi_3)\right] + i\epsilon M E_1 - M^2\Phi_1 = 0,$$
$$\frac{\sqrt{2}i}{\text{sh}\, r}\left[-c\left(-i\left(\frac{d}{dr} + \frac{1}{\text{sh}\, r}\right)\Phi_1 - i\frac{\sqrt{2}c}{\text{sh}\, r}\Phi_2\right) + \right.$$
$$\left. + d\left(i\left(\frac{d}{dr} + \frac{1}{\text{sh}\, r}\right)\Phi_3 + \frac{i\sqrt{2}d}{\text{sh}\, r}\Phi_2\right)\right] +$$
$$+ i\epsilon M E_2 - M^2\Phi_2 = 0,$$
$$-i\left(\frac{d}{dr} + \frac{1}{\text{sh}\, r}\right)\left[i\left(\frac{d}{dr} + \frac{1}{\text{sh}\, r}\right)\Phi_3 + \frac{i\sqrt{2}d}{\text{sh}\, r}\Phi_2\right] -$$
$$- i\frac{\sqrt{2}d}{\text{sh}\, r}\left[i\frac{\sqrt{2}}{\text{sh}\, r}(c\Phi_1 - d\Phi_3)\right] + i\epsilon M E_3 - M^2\Phi_3 = 0,$$
$$\frac{\sqrt{2}c}{\text{sh}\, r}\left[-\left(\frac{d}{dr} + \frac{2}{\text{sh}\, r}\right)E_2 - \frac{\sqrt{2}}{\text{sh}\, r}(cE_1 + dE_3)\right] -$$
$$- i\epsilon M\Phi_1 - M^2 E_1 = 0,$$
$$-\frac{d}{dr}\left[-\left(\frac{d}{dr} + \frac{2}{\text{sh}\, r}\right)E_2 - \frac{\sqrt{2}}{\text{sh}\, r}(cE_1 + dE_3)\right] -$$
$$- i\epsilon M\Phi_2 - M^2 E_2 = 0,$$
$$\frac{\sqrt{2}d}{\text{sh}\, r}\left[-\left(\frac{d}{dr} + \frac{2}{\text{sh}\, r}\right)E_2 - \frac{\sqrt{2}}{\text{sh}\, r}(cE_1 + dE_3)\right] -$$
$$- i\epsilon M\Phi_3 - M^2 E_3 = 0. \tag{14}$$

Big Ψ_j and small ψ_j components are introduced by the linear combinations

$$\Psi_j = \Phi_j + iE_j, \quad \psi_j = \Phi_j - iE_j. \tag{15}$$

Then we regroup Eqs. (14) in pairs and separate the rest energy by the formal change $\epsilon = (M + E)$. So, we get

$$+ i\left(\frac{d}{dr} + \frac{1}{\operatorname{sh} r}\right)\left[-i\left(\frac{d}{dr} + \frac{1}{\operatorname{sh} r}\right)\Phi_1 - i\frac{\sqrt{2}c}{\operatorname{sh} r}\Phi_2\right] +$$
$$+ i\frac{\sqrt{2}c}{\operatorname{sh} r}\left[i\frac{\sqrt{2}}{\operatorname{sh} r}(c\Phi_1 - d\,\Phi_3)\right] +$$
$$+ i(M+E)ME_1 - M^2\,\Phi_1 = 0,$$
$$\frac{\sqrt{2}c}{\operatorname{sh} r}\left[-\left(\frac{d}{dr} + \frac{2}{\operatorname{sh} r}\right)E_2 - \frac{\sqrt{2}}{\operatorname{sh} r}(cE_1 + dE_3)\right] -$$
$$- i(M+E)M\Phi_1 - M^2E_1 = 0,$$
$$\frac{\sqrt{2}i}{\operatorname{sh} r}\left[-c\left(-i\left(\frac{d}{dr} + \frac{1}{\operatorname{sh} r}\right)\Phi_1 - i\frac{\sqrt{2}c}{\operatorname{sh} r}\Phi_2\right) +\right.$$
$$\left. + d\left(i\left(\frac{d}{dr} + \frac{1}{\operatorname{sh} r}\right)\Phi_3 + \frac{i\sqrt{2}d}{\operatorname{sh} r}\Phi_2\right)\right] +$$
$$+ i(M+E)ME_2 - M^2\Phi_2 = 0,$$
$$-\frac{d}{dr}\left[-\left(\frac{d}{dr} + \frac{2}{\operatorname{sh} r}\right)E_2 - \frac{\sqrt{2}}{\operatorname{sh} r}(cE_1 + dE_3)\right] -$$
$$- i(M+E)M\Phi_2 - M^2E_2 = 0,$$
$$- i\left(\frac{d}{dr} + \frac{1}{\operatorname{sh} r}\right)\left[i\left(\frac{d}{dr} + \frac{1}{\operatorname{sh} r}\right)\Phi_3 + \frac{i\sqrt{2}d}{\operatorname{sh} r}\Phi_2\right] -$$
$$- i\frac{\sqrt{2}d}{\operatorname{sh} r}\left[i\frac{\sqrt{2}}{\operatorname{sh} r}(c\Phi_1 - d\Phi_3)\right] +$$
$$+ (M+E)ME_3 - M^2\Phi_3 = 0,$$
$$\frac{\sqrt{2}d}{\operatorname{sh} r}\left[-\left(\frac{d}{dr} + \frac{2}{\operatorname{sh} r}\right)E_2 - \frac{\sqrt{2}}{\operatorname{sh} r}(cE_1 + dE_3)\right] -$$
$$- i(M+E)M\Phi_3 - M^2E_3 = 0.$$

From whence, we derive the system of radial equations in the Pauli approximation:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2EM\right)\Psi_1 - \frac{2\sqrt{2}c}{r^2}\Psi_2 - \frac{4c^2}{r^2}\Psi_1 = 0,$$
$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2EM\right)\Psi_2 - \frac{2\left(c^2 + d^2 + 1\right)}{r^2}\Psi_2 -$$
$$- \frac{2\sqrt{2}c}{r^2}\Psi_1 - \frac{2\sqrt{2}d}{r^2}\Psi_3 = 0, \qquad (16)$$
$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2EM\right)\Psi_3 - \frac{2\sqrt{2}d}{r^2}\Psi_2 - \frac{4d^2}{r^2}\Psi_3 = 0.$$

4. Solving the Radial Equations

With the notation

$$\frac{1}{2}r^2\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2EM\right) = \bar{\Delta},$$

the problem takes the form

$$\bar{\Delta}\Psi(r) = A\Psi(r), \quad \Psi = \begin{vmatrix} \Psi_1(r) \\ \Psi_2(r) \\ \Psi_3(r) \end{vmatrix},$$
$$A = \begin{vmatrix} 2c^2 & \sqrt{2}c & 0 \\ \sqrt{2}c & (c^2 + d^2 + 1) & \sqrt{2}d \\ 0 & \sqrt{2}d & 2d^2 \end{vmatrix}. \qquad (17)$$

The non-relativistic wave function is given by

$$\Phi_{\epsilon jm}(x) = e^{-i\epsilon t}\begin{vmatrix} \Psi_1(r)D_{k-1} \\ \Psi_2(r)D_{\kappa} \\ \Psi_3(r)D_{k+1} \end{vmatrix}.$$

By a special similarity transformation, we can reduce the problem to the diagonal form

$$S\Psi' = \begin{vmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{vmatrix},$$
$$\bar{\Delta}\begin{vmatrix} \Psi_1' \\ \Psi_2' \\ \Psi_3' \end{vmatrix} = \begin{vmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{vmatrix}\begin{vmatrix} \Psi_1' \\ \Psi_2' \\ \Psi_3' \end{vmatrix}, \qquad (18)$$

and can construct three independent solutions explicitly:

$$\begin{vmatrix} \Psi_1' \\ 0 \\ 0 \end{vmatrix}, \quad \begin{vmatrix} 0 \\ \Psi_2' \\ 0 \end{vmatrix}, \quad \begin{vmatrix} 0 \\ 0 \\ \Psi_3' \end{vmatrix}; \qquad (19)$$

and after that turn back to initial basis

$$\Psi_1 = S\begin{vmatrix} \Psi_1' \\ 0 \\ 0 \end{vmatrix}, \quad \Psi_2 = S\begin{vmatrix} 0 \\ \Psi_2' \\ 0 \end{vmatrix}, \quad \Psi_3 = S\begin{vmatrix} 0 \\ 0 \\ \Psi_3' \end{vmatrix}.$$

The diagonal elements A_1, A_2, and A_3 are solutions of the cubic equation

$$\left(c^2 + d^2 - 1\right)4c^2d^2 + A\left[-4c^2d^2 - 2\left(c^2 + d^2\right)^2\right] +$$
$$+ A^2\left[\left(c^2 + d^2 + 1\right) + 2\left(c^2 + d^2\right)\right] - A^3 = 0. \qquad (20)$$

With the notation

$$c^2+d^2=\frac{j(j+1)-k^2}{2}=M>0,$$

$$4c^2d^2=\frac{j^2-k^2}{2}\frac{(j+1)^2-k^2}{2}=N\geq 0$$

Eq. (20) reads

$$A^3+rA^2+sA+t=0,\quad r=-(3M+1),\\ s=(N+2M^2),\quad t=-(M-1)N. \tag{21}$$

The identity

$$\lambda^3+r\lambda^2+s\lambda+t=(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)=0$$

yields

$$\begin{aligned} &r=-(A_1+A_2+A_3)\Longrightarrow\\ &\Longrightarrow 3M+1=A_1+A_2+A_3>0,\\ &s=A_1A_2+A_1A_3+A_2A_3\Longrightarrow\\ &\Longrightarrow N+2M^2=A_1A_2+A_1A_3+A_2A_3>0,\\ &t=-A_1A_2A_3\Longrightarrow (M-1)N=A_1A_2A_3\geq 0. \end{aligned} \tag{22}$$

Let us specify some first numerical solutions A_1, A_2, and A_3:

$k=\pm 1/2$	$j=3/2$	$j=5/2$	$j=7/2$	$j=9/2$
	0.31	1.79	4.28	7.77
	1.73	4.24	7.75	12.25
	4.21	7.72	12.23	17.73
$k=\pm 1$	$j=2$	$j=3$	$j=4$	$j=5$
	0.68	2.62	5.59	9.57
	2.45	5.48	9.49	14.49
	5.36	9.40	14.42	20.43
$k=\pm 3/2$	$j=5/2$	$j=7/2$	$j=9/2$	$j=11/2$
	1.09	3.49	6.94	11.40
	3.17	6.71	11.23	16.73
	6.49	11.05	16.59	23.12
$k=\pm 2$	$j=3$	$j=4$	$j=5$	$j=6$
	1.53	4.38	8.30	13.25
	3.89	7.94	12.96	18.97
	7.57	12.68	18.74	25.78

The roots are real and positive. These roots can be described analytically as well. To this end, let us change the variable

$$B=A+\frac{r}{3}. \tag{23}$$

So, we get the reduced cubic equation

$$B^3+pB+q=0,\quad p=\frac{3s-r^2}{3},\quad q=\frac{2r^3}{27}-\frac{rs}{3}+t. \tag{24}$$

Further, we obtain

$$p=-\left(j(j+1)-\frac{3}{4}k^2+\frac{1}{3}\right)<0,\\ q=-\left(\frac{1}{3}j(j+1)+\frac{2}{27}\right)<0. \tag{25}$$

As is known, the discriminant

$$D=\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2$$

determines the nature of three roots of the cubic equation. When $D<0$, all roots are real. The quantity D is given by

$$D=-\left(\frac{j(j+1)-k^2}{3}+\frac{k^2}{12}+\frac{1}{9}\right)^3+\\ +\left(\frac{j(j+1)}{6}+\frac{1}{27}\right)^2. \tag{26}$$

The sign of D can be established explicitly, if one uses the following substitutions:

$$j=k+n,\quad (k>0)\quad n=1,2,3,...,$$

$$\begin{aligned} D=&-\frac{1}{72}k^5n-\frac{17}{144}k^4n^2-\frac{17}{144}k^4n-\frac{11}{27}k^3n^3-\\ &-\frac{11}{18}k^3n^2-\frac{17}{36}k^2n^4-\frac{17}{18}k^2n^3-\frac{2}{9}kn^5-\frac{5}{9}kn^4-\\ &-\frac{1}{54}kn-\frac{1}{108}n^2-\frac{11}{54}k^3n-\frac{7}{12}k^2n^2-\frac{1}{9}k^2n-\\ &-\frac{13}{27}kn^3-\frac{1}{6}kn^2-\frac{1}{432}k^4-\frac{13}{108}n^4-\\ &-\frac{1}{18}n^3-\frac{1}{1728}k^6-\frac{1}{144}k^5-\frac{1}{27}n^6-\frac{1}{9}n^5<0; \end{aligned}$$

$$j=-k+n,\quad (k<0)\quad n=1,2,3,...,$$

$$\begin{aligned} D=&\frac{1}{72}k^5n-\frac{17}{144}k^4n^2-\frac{17}{144}k^4n+\frac{11}{27}k^3n^3+\\ &+\frac{11}{18}k^3n^2-\frac{17}{36}k^2n^4-\frac{17}{18}k^2n^3+\frac{2}{9}kn^5+\\ &+\frac{5}{9}kn^4+\frac{1}{54}kn-\frac{1}{108}n^2+\frac{11}{54}k^3n-\frac{7}{12}k^2n^2-\\ &-\frac{1}{9}k^2n+\frac{13}{27}kn^3+\frac{1}{6}kn^2-\frac{1}{432}k^4-\frac{13}{108}n^4-\\ &-\frac{1}{18}n^3-\frac{1}{1728}k^6+\frac{1}{144}k^5-\frac{1}{27}n^6-\frac{1}{9}n^5<0. \end{aligned}$$

According to the conventional method, we can construct the expressions for roots as follows:

$$\rho = \sqrt{-\frac{p^3}{27}}, \quad \cos\phi = -\frac{q}{2\rho},$$
$$B_1 = 2\rho^{1/3}\cos\frac{\phi}{3} = 2\sqrt{-\frac{p}{3}}\cos\frac{\phi}{3},$$
$$B_2 = 2\rho^{1/3}\cos\left[\frac{\phi}{3}+\frac{2\pi}{3}\right] = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\phi}{3}+\frac{2\pi}{3}\right),$$
$$B_3 = 2\rho^{1/3}\cos\left[\frac{\phi}{3}+\frac{4\pi}{3}\right] = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\phi}{3}-\frac{2\pi}{3}\right),$$

where

$$2\sqrt{-\frac{p}{3}} = 2\sqrt{\frac{j(j+1)-k^2}{3}+\frac{k^2}{12}+\frac{1}{9}},$$
$$\rho = \sqrt{-\frac{p^3}{27}} = \left(\frac{j(j+1)-k^2}{3}+\frac{k^2}{12}k^2+\frac{1}{9}\right)^{3/2},$$
$$\cos\phi = -\frac{q/2}{\rho} = \frac{\left(\frac{1}{6}j(j+1)+\frac{1}{27}\right)}{\left(\frac{j(j+1)-k^2}{3}+\frac{k^2}{12}+\frac{1}{9}\right)^{3/2}}.$$

5. Particle in the Coulomb Field in the Presence of a Magnetic Charge

Let us consider the Coulomb attractive force

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+2M\left(E+\frac{\alpha}{r}\right)-\frac{L(L+1)}{r^2}\right)f(r)=0,$$
$$L(L+1) = 2A = \{2A_1,\ 2A_2,\ 2A_3\}, \tag{27}$$
$$L = -\frac{1}{2}\pm\sqrt{\frac{1}{4}+2A}$$

or

$$x = \sqrt{-2ME}\,r, \quad -\frac{\alpha\sqrt{-2ME}}{E} = B > 0,$$
$$\left(\frac{d^2}{dx^2}+\frac{2}{x}\frac{d}{dx}-\frac{L(L+1)}{x^2}-1+\frac{B}{x}\right)f=0. \tag{28}$$

Using the substitution $f = x^L e^{-x}F(x)$, (at positive L) for the variable $z = 2x$, we get the hypergeometric equation

$$z\frac{d^2F}{dz^2}+(2L+2-z)\,\frac{dF}{dz}+\left(\frac{B}{2}-L-1\right)\,F=0,$$

which gives the energy spectrum

$$\alpha = -n, \quad n = 0,1,2,..., \quad E = -\frac{1}{2}\,\frac{\alpha^2 M}{(n+L+1)^2}.$$

Ultimately, we have three series of energy levels:

$$E_i = -\frac{1}{2}\,\frac{\alpha^2 M}{(n+L_i(j,k)+1)^2}. \tag{29}$$

6. Particle in the Oscillator Potential in the Presence of a Magnetic Charge

Let us consider the oscillator potential

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+2M\left(E-\frac{kr^2}{2}\right)-\frac{L(L+1)}{r^2}\right)f=0. \tag{30}$$

In the variable $x = \sqrt{Mk}\,r^2$:

$$\left(\frac{d^2}{dx^2}+\frac{3}{2x}\frac{d}{dx}-\frac{1}{4}+\frac{E\sqrt{M}}{2\sqrt{k}x}-\frac{L(L+1)}{4x^2}\right)f(x)=0,$$
$$L(L+1) = 2A = \{2A_1, 2A_2, 2A_3\},$$
$$L = -\frac{1}{2}+\sqrt{\frac{1}{4}+2A} > 0,$$

with the substitution $f(x) = x^a e^{-x/2}\,F(x)$; at $a = +L/2$, we get the confluent hypergeometric equation

$$x\frac{d^2F}{dx^2}+(L+3/2-x)\frac{dF}{dx}-\left(\frac{3}{4}+\frac{L}{2}-\frac{E\sqrt{M}}{2\sqrt{k}}\right)F=0.$$

The quantization rule gives the energy spectrum

$$A = -n, \quad E = \frac{1}{2}\sqrt{\frac{k}{M}}\left(\frac{3}{2}+L+2n\right).$$

We have three series of energy levels

$$E_i = \frac{1}{2}\sqrt{\frac{k}{M}}\left(\frac{3}{2}+L_i(j,k)+2n\right),$$
$$L_i(j,k) = -\frac{1}{2}\pm\sqrt{\frac{1}{4}+2A_i(j,k)}. \tag{31}$$

7. Separation of Variables in the Lobachevsky Space in the Presence of a Magnetic Monopole

In spherical coordinates and tetrads

$$dS^2 = c^2dt^2 - dr^2 - \mathrm{sh}^2 r\,\left(d\theta^2+\mathrm{sh}^2\theta d\phi^2\right),$$
$$e^{\alpha}_{(0)} = (1,0,0,0), \quad e^{\alpha}_{(3)} = (0,1,0,0),$$
$$e^{\alpha}_{(1)} = \left(0,0,\frac{1}{\mathrm{sh}\,r},0\right),$$
$$e^{\alpha}_{(2)} = \left(1,0,0,\frac{1}{\mathrm{sh}\,r\sin\theta}\right), \tag{32}$$

the DKP equation looks as

$$\left[i\beta^0\partial_t + i\left(\beta^3\partial_r+\frac{1}{\mathrm{sh}\,r}\left(\beta^1 j^{31}+\beta^2 j^{32}\right)\right)+\right.$$
$$\left.+\frac{1}{\mathrm{sh}\,r}\Sigma^k_{\theta,\phi}-M\right]\Phi=0,$$
$$\Sigma^k_{\theta,\phi} = i\beta^1\partial_\theta+\beta^2\,\frac{i\,\partial_\phi+\left(ij^{12}-k\right)\cos\theta}{\sin\theta},$$
$$k = eg/\hbar c. \tag{33}$$

Then the radial equations are

$$-\left(\frac{d}{dr}+\frac{2}{\operatorname{sh} r}\right) f_6-\frac{\sqrt{2}}{\operatorname{sh} r}(cf_5+df_7)-Mf_1=0,$$
$$i\epsilon f_5+i\left(\frac{d}{dr}+\frac{1}{\operatorname{sh} r}\right) f_8+i\frac{\sqrt{2}c}{\operatorname{sh} r}f_9-Mf_2=0,$$
$$i\epsilon f_6+\frac{\sqrt{2}i}{\operatorname{sh} r}(-cf_8+df_{10})-Mf_3=0,$$
$$i\epsilon f_7-i\left(\frac{d}{dr}+\frac{1}{\operatorname{sh} r}\right) f_{10}-i\frac{\sqrt{2}d}{\operatorname{sh} r}f_9-Mf_4=0, \tag{34}$$

$$-i\epsilon f_2+\frac{\sqrt{2}c}{\operatorname{sh} r}f_1-Mf_5=0,$$
$$-i\epsilon f_3-\frac{d}{dr}f_1-Mf_6=0,$$
$$-i\epsilon f_4+\frac{\sqrt{2}d}{\operatorname{sh} r}f_1-Mf_7=0,$$
$$-i\left(\frac{d}{dr}+\frac{1}{\operatorname{sh} r}\right) f_2-i\frac{\sqrt{2}c}{\operatorname{sh} r}f_3-Mf_8=0,$$
$$i\frac{\sqrt{2}}{\operatorname{sh} r}(cf_2-df_4)-Mf_9=0,$$
$$i\left(\frac{d}{dr}+\frac{1}{\operatorname{sh} r}\right) f_4+\frac{i\sqrt{2}d}{\operatorname{sh} r}f_3-Mf_{10}=0. \tag{35}$$

The quantum number j takes the values

$k=\pm 1/2, \quad j=|k|, \quad |k|+1, ...;$

$k=\pm 1, \quad \pm 3/2, ..., \quad j=|k|-1, \quad |k|, ...$

Let $k=+1$ and $j=0$. Then

$$\Phi^{(0)}(t,r)=e^{-i\epsilon t}\,(0,\, f_2,\, 0,\, 0,\, f_5,\, 0,\, 0,\, f_8,\, 0,\, 0),$$

and we get three radial equations

$$\left(\frac{d^2}{dr^2}+\frac{2}{\operatorname{sh} r}\frac{d}{dr}+\frac{1-\operatorname{ch} r}{\operatorname{sh}^2 r}+\epsilon^2-M^2\right) f_2=0,$$
$$f_8=-\frac{i}{M}\left(\frac{d}{dr}+\frac{1}{\operatorname{sh} r}\right) f_2, \quad f_5=-i\frac{\epsilon}{M}f_2. \tag{36}$$

By a special substitution, we simplify the problem to

$$f_2=\frac{1+\operatorname{ch} r}{2\operatorname{sh} r}F_2, \quad \left(\frac{d^2}{dr^2}+\epsilon^2-M^2\right) F_2=0, \tag{37}$$

which coincides with the equation arising in the flat space. The case $j=0$, $k=-1$ looks much the same.

8. Non-Relativistic Approximation in the Lobachevsky Space

The general procedure is much the same as in the flat space. The radial equations in the Pauli limit look as follows:

$$\left(\frac{d^2}{dr^2}+2EM-\frac{4c^2}{\operatorname{sh}^2 r}\right) F_1=\frac{1+\operatorname{ch} r}{\operatorname{sh}^2 r}\sqrt{2}cF_2,$$
$$\left(\frac{d^2}{dr^2}+2EM-\frac{2(c^2+d^2)}{\operatorname{sh}^2 r}\right) F_2=$$
$$=\frac{1+\operatorname{ch} r}{\operatorname{sh}^2 r}\left(\sqrt{2}cF_1+F_2+\sqrt{2}dF_3\right),$$
$$\left(\frac{d^2}{dr^2}+2EM-\frac{4d^2}{\operatorname{sh}^2 r}\right) F_3=\frac{1+\operatorname{ch} r}{\operatorname{sh}^2 r}d\sqrt{2}dF_2. \tag{38}$$

Unfortunately, this system turns out to be very difficult for solving, since the method used in the flat space cannot be applied here. However, we can solve exactly the case of the minimal value of $j=|\,k\,|-1$ in the presence of the additional Coulomb or oscillator potentials.

9. Minimal Value of *j*, Coulomb and Oscillator Potentials

The case of the minimal j in the presence of the monopole and Coulomb fields is described by the equation

$$\left(\frac{d^2}{dr^2}+(\epsilon+\frac{\alpha}{\operatorname{th} r})^2-M^2\right) F_2=0. \tag{39}$$

In the variable $x=1-e^{-2r}$, with substitution $F_1= = x^a(1-x)^b f(x)$, at positive

$$a=\frac{1+\sqrt{1-4\alpha^2}}{2}, \quad b=+\frac{1}{2}\sqrt{-(\epsilon+\alpha)^2+M^2},$$

we get the hypergeometric equation

$$x(1-x)f''+[2a-(2a+2b+1)x]\;f'-$$
$$-\left[(a+b)^2+\left(\frac{\epsilon}{2}-\frac{\alpha}{2}\right)^2-\frac{M^2}{4}\right] f=0\,.$$

The quantization rule $\alpha=-n$ provides us with the energy spectrum

$$\epsilon=\frac{M}{\sqrt{1+\alpha^2/\nu^2}}\,\sqrt{1-\frac{\alpha^2+\nu^2}{M^2}},$$
$$\nu=n+\frac{1+\sqrt{1-4\alpha^2}}{2}. \tag{40}$$

In usual units, it reads

$$E = \frac{mc^2}{\sqrt{1+\alpha^2/\nu^2}} \sqrt{1 - \frac{\hbar^2}{m^2c^2R^2}(\alpha^2+\nu^2)}. \qquad (41)$$

The solutions constructed are good only when the restriction given below holds

$$\nu^2 \le \frac{m^2c^2R^2}{\hbar^2} - \alpha^2. \qquad (42)$$

So, the number of bound states is finite.

Now, for the states with minimal j, let us take the oscillator potential into account:

$$\left(\frac{d^2}{dr^2} + 2M\left(E - \frac{K\,\text{th}^2 r}{2}\right)\right) F_2 = 0. \qquad (43)$$

The solutions are found in terms of the hypergeometric function

$$E = N\sqrt{\frac{K}{M} + \left(\frac{1}{2M}\right)^2} - \frac{1}{2M}\left(N^2 + \frac{1}{4}\right), \qquad N = 2n + \frac{3}{2}. \qquad (44)$$

In usual units, we have

$$\epsilon = \hbar\left(N\sqrt{\frac{k}{m} + \frac{\hbar^2}{4m^2R^4}} - \frac{\hbar}{2mR^2}\left(N^2 + \frac{1}{4}\right)\right).$$

To obtain the solutions tending to zero at infinity, we must impose the following restriction:

$$2n + \frac{3}{2} < \frac{1}{2}\sqrt{1 + \frac{4km}{\hbar^2}R^4}. \qquad (45)$$

The number of discrete energy levels is finite and governed by the curvature radius.

10. Spin-1 Particle in the Absence of a Monopole

In the absence of a monopole, the identity $d = c = \frac{1}{2}\sqrt{j(j+1)}$ holds, and the task becomes simpler:

$$\left(\frac{d^2}{dr^2} + 2EM\right)F_1 - \sqrt{2}c\frac{1+\text{ch}\,r}{\text{sh}^2 r}F_2 - \frac{4c^2}{\text{sh}^2\,r}F_1 = 0,$$
$$\left(\frac{d^2}{dr^2} + 2EM\right)F_3 - \sqrt{2}c\frac{1+\text{ch}\,r}{\text{sh}^2 r}F_2 - \frac{4c^2}{\text{sh}^2\,r}F_3 = 0;$$
$$\left(\frac{d^2}{dr^2} + 2EM\right)F_2 - \frac{4c^2}{\text{sh}^2 r}F_2 - \frac{(1+\text{ch}\,r)}{\text{sh}^2 r}F_2 - \frac{\sqrt{2}c(1+\text{ch}\,r)}{\text{sh}^2 r}F_1 - \frac{\sqrt{2}c(1+\text{ch}\,r)}{\text{sh}^2 r}F_3 = 0. \qquad (46)$$

One can diagonalize the space reflection operator

$$P = (-1)^{j+1}, \quad F_2 = 0, \quad F_3 = -F_1; \qquad P = (-1)^j, \quad F_3 = +F_1, \qquad (47)$$

so that the system is divided into two ones (1+2):

$$\left(\frac{d^2}{dr^2} + 2EM - \frac{4c^2}{\text{sh}^2\,r}\right)F_1 = 0; \qquad (48)$$

and

$$\bar{\Delta}\begin{vmatrix} F_1 \\ F_2 \end{vmatrix} = \begin{vmatrix} 0 & \nu \\ 2\nu & 1 \end{vmatrix}\begin{vmatrix} F_1 \\ F_2 \end{vmatrix}, \quad \nu = c\sqrt{2}, \qquad (49)$$

where

$$\bar{\Delta} = \frac{\text{sh}^2 r}{1+\text{ch}\,r}\left(\frac{d^2}{dr^2} - \frac{j(j+1)}{\text{sh}^2\,r} + 2EM\right),$$

In the case of the subsystem of two equations, let us diagonalize the mixing matrix

$$F = SF', \quad \bar{\Delta}\,F' = S^{-1}A\,SF', \qquad S = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}, \quad \bar{\Delta}F' = \begin{vmatrix} j+1 & 0 \\ 0 & -j \end{vmatrix}F'. \qquad (50)$$

So, we have two similar equations

$$\left(\frac{d^2}{dr^2} + 2EM - \frac{j(j+1)}{\text{sh}^2\,r} - \frac{1+\text{ch}\,r}{\text{sh}^2\,r}(j+1)\right)F'_1 = 0,$$
$$\left(\frac{d^2}{dr^2} + 2EM - \frac{j(j+1)}{\text{sh}^2\,r} + \frac{1+\text{ch}\,r}{\text{sh}^2\,r}j\right)F'_2 = 0. \qquad (51)$$

The above three equations are of the same type. In the variable $y = (\text{ch}\,r + 1)/2$, they read

$$\left(y(y-1)\frac{d^2}{dy^2} + \left(y - \frac{1}{2}\right)\frac{d}{dy} + 2ME - \frac{j(j+1)}{4y(y-1)} + \frac{\mu}{2(y-1)}\right)F = 0, \qquad (52)$$

where $\mu = 0$, $\mu = -j-1$, $\mu = +j$. Their solutions have been constructed in terms of hypergeometric functions

$$F = y^a(1-y)^bF(\alpha,\beta,\gamma;y), \quad \gamma = 2a + \frac{1}{2},$$
$$\alpha = a + b + i\sqrt{2ME}, \quad \beta = a + b - i\sqrt{2ME}, \qquad (53)$$
$$a = \frac{j+1}{2}, \quad a' = -\frac{j}{2};$$
$$\mu = 0, \quad b = \frac{j+1}{2}, \quad b' = -\frac{j}{2};$$

$$\mu = -j-1, \quad b = \frac{j+2}{2}, \quad b' = -\frac{j+1}{2};$$
$$\mu = +j, \quad b = \frac{j}{2}, \quad b' = -\frac{j+1}{2}.$$

The value $y = 1$ corresponds to the point $r = 0$. So, to get the solutions finite at $r = 0$, we must take $b > 0$. The asymptotic behavior at infinity is given by

$$f(y) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)} \left(-\frac{e^r}{4}\right)^{-i\sqrt{2EM}} + \\ + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\alpha)} \left(-\frac{e^r}{4}\right)^{+i\sqrt{2EM}}. \quad (54)$$

Therefore, we have constructed real standing wave solutions regular at $r = 0$.

11. Spin-1 Particle in the Coulomb Potential

In the presence of an external Coulomb field, the equations in the Lobachevsky space take the form

$$\left(\frac{d^2}{dr^2} + 2M\left(E + \frac{\alpha}{\tanh r}\right) - \frac{j(j+1)}{\operatorname{sh}^2 r}\right) F_1 = 0; \quad (55)$$
$$\left(\frac{d^2}{dr^2} + 2M\left(E + \frac{\alpha}{\tanh r}\right) - \frac{j(j+1)}{\operatorname{sh}^2 r} - \\ - \frac{1+\operatorname{ch} r}{\operatorname{sh}^2 r}(j+1)\right) F_1' = 0, \quad (56)$$
$$\left(\frac{d^2}{dr^2} + 2M\left(E + \frac{\alpha}{\tanh r}\right) - \frac{j(j+1)}{\operatorname{sh}^2 r} + \\ + \frac{1+\operatorname{ch} r}{\operatorname{sh}^2 r} j\right) F_2' = 0. \quad (57)$$

The first equation (55) is much simpler than two others. It is solved in terms of hypergeometric functions and gives the energy levels (in usual units)

$$\epsilon = -mc^2 \frac{\alpha^2}{2(j+1+n)^2} - \frac{\hbar^2}{mR^2} \frac{(j+1+n)^2}{2}. \quad (58)$$

Two other equations can be reduced to those for the Heun function (ODE with four singular points). Applying only the first of the two conditions for polynomial solutions (so, we do not construct polynomials), we have arrived at else two series of energy levels. Let us specify some details. In these cases, we can use the variable $z = \operatorname{th} \frac{r}{2}$. Respectively, the equations will read

$$\left[\frac{d^2}{dz^2} - \frac{2z}{1-z^2}\frac{d}{dz} + 8M\left(E + \alpha\frac{1+z^2}{2z}\right)\frac{1}{(1-z^2)^2} - \\ - \frac{j(j+1)}{z^2} - \frac{2(j+1)}{z^2(1-z^2)}\right] F_1' = 0, \quad (59)$$
$$\left[\frac{d^2}{dz^2} - \frac{2z}{1-z^2}\frac{d}{dz} + 8M\left(E + \alpha\frac{1+z^2}{2z}\right)\frac{1}{(1-z^2)^2} - \\ - \frac{j(j+1)}{z^2} + \frac{2j}{z^2(1-z^2)}\right] F_2' = 0; \quad (60)$$

the singular points are $z = 0, \infty, \pm 1$; two of them are physical $r = 0$, $z = 0$; $r = \infty$, $z = +1$.

In Eq. (60), let us use the simplifying substitutions

$$F_1'(z) = \frac{f_1'(z)}{\sqrt{z^2-1}}, \quad f_1' = z^A(1-z)^B(-1-z)^C f_1.$$

Then

$$\frac{d^2 f_1}{dz^2} + \left[\frac{2A}{z} - \frac{2B}{1-z} - \frac{2C}{-1-z}\right]\frac{df_1}{dz} + \\ + \left[\frac{A(A-1)-(j+1)(j+2)}{z^2} + \\ + \frac{(2B-1)^2 + 8M(E+\alpha)}{4(z-1)^2} + \\ + \frac{(2C-1)^2 + 8M(E-\alpha)}{4(z+1)^2} + \frac{-2A(B-C)+4M\alpha}{z} + \\ + \frac{4B(2A+C)+3+4j-8M(E+\alpha)}{4(z-1)} + \\ + \frac{-4C(2A+B)-3-4j+8M(E-\alpha)}{4(z+1)}\right] f_1 = 0. \quad (61)$$

At

$$A = -(j+1),\ j+2, \quad B = \frac{1}{2} \pm \sqrt{-2M(E+\alpha)},$$
$$C = \frac{1}{2} \pm \sqrt{-2M(E-\alpha)},$$

Eq. (61) becomes simpler

$$\frac{d^2 f_1}{dz^2} + \left[\frac{2A}{z} - \frac{2B}{1-z} - \frac{2C}{-1-z}\right]\frac{df_1}{dz} + \\ + \left[\frac{-2A(B-C)+4M\alpha}{z} - \\ - \frac{4B(2A+C)+3+4j-8M(E+\alpha)}{4(1-z)} + \\ + \frac{-4C(2A+B)-3-4j+8M(E-\alpha)}{4(z+1)}\right] f_1 = 0.$$

It can be recognized as the general Heun equation for $HeunG(p, q, a, b, c, d, z)$,

$$\frac{d^2Y(z)}{dz^2} + \left[\frac{c}{z} - \frac{d}{1-z} - \frac{a+b-c-d+1}{p-z}\right]\frac{dY(z)}{dz} + $$
$$+\left[-\frac{q}{pz} + \frac{ab-q}{(p-1)(1-z)} + \frac{abp-q}{p(p-1)(z-p)}\right]Y(z) = 0,$$

with the parameters

$$p = -1, \quad q = 4M\alpha - 2A(B-C), \quad c = 2A, \quad d = 2B.$$

$$a = -\frac{1}{2} + A + B + C +$$
$$+\frac{1}{2}\sqrt{(2A-1)^2 + (2B-1)^2 + (2C-1)^2 + 8(2ME - j - 1)},$$
$$b = -\frac{1}{2} + A + B + C -$$
$$-\frac{1}{2}\sqrt{(2A-1)^2 + (2B-1)^2 + (2C-1)^2 + 8(2ME - j - 1)}. \tag{62}$$

Note the identity

$$\frac{1}{2}\sqrt{(2A-1)^2 + (2B-1)^2 + (2C-1)^2 + 8(2ME - j - 1)} =$$
$$= j + \frac{1}{2}.$$

Respectively, the parameters a, b read

$$a = (-j-1) + A + B + C, \quad b = j + A + B + C.$$

We will use the quantization condition in the form (at this, we do not arrive at polynomials) $b = -n$. It turns out that the choice

$$A = j + 2, \quad B = \frac{1}{2} + \sqrt{-2M(E+\alpha)},$$
$$C = \frac{1}{2} - \sqrt{-2M(E-\alpha)}.$$
$$a = 2-$$
$$-(-\sqrt{-2ME - 2M\alpha} + \sqrt{-2ME + 2M\alpha}), \tag{63}$$
$$b = 2j + 3-$$
$$-(-\sqrt{-2ME - 2M\alpha} + \sqrt{-2ME + 2M\alpha})$$

is appropriate. The quantization rule takes the form

$$-\sqrt{-2ME - 2M\alpha} + \sqrt{-2ME + 2M\alpha} =$$
$$= 2j + 3 + n,$$

which yields the formula for energy levels

$$E = -\frac{M\alpha^2}{2(j + 3/2 + n/2)^2} - \frac{(j + 3/2 + n/2)^2}{2M}. \tag{64}$$

In a similar manner, Eq. (60) provides us with the energy levels

$$E = -\frac{M\alpha^2}{2(j + 1/2 + n/2)^2} - \frac{(j + 1/2 + n/2)^2}{2M}. \tag{65}$$

Thus, we have found three series of energy levels: (58), (64), and (65). The presence of n' and $n'/2$ is due to the use of different variables in solving the respective differential equations, $z = \text{th}\,\frac{r}{2}$ and $x = 1 - e^{-2r}$ are connected by the quadratic relations:

$$x = \frac{2\text{th}\,r}{1 + \text{th}\,r}, \quad \text{th}\,r = \frac{2z}{1+z^2},$$
$$x = \frac{4z(1+z^2)}{(1+z^2)^2 + 4z^2} = \frac{4(z + z^{-1})}{4 + (z + z^{-1})}.$$

12. Spin-1 Particle in an Oscillator Field

In the presence of the oscillator potential, we get three radial equations

$$\left(\frac{d^2}{dr^2} + 2M\left(E - \frac{K\text{th}^2\,r}{2}\right) - \frac{j(j+1)}{\text{sh}^2\,r}\right)F_1 = 0,$$
$$\left(\frac{d^2}{dr^2} + 2M\left(E - \frac{K\text{th}^2\,r}{2}\right) - \frac{j(j+1)}{\text{sh}^2\,r} - \right.$$
$$\left. -\frac{1 + \text{ch}\,r}{\text{sh}^2\,r}(j+1)\right)F_1' = 0, \tag{66}$$
$$\left(\frac{d^2}{dr^2} + 2M\left(E - \frac{K\text{th}^2\,r}{2}\right) - \frac{j(j+1)}{\text{sh}^2\,r} + \right.$$
$$\left. + \frac{1 + \text{ch}\,r}{\text{sh}^2\,r}j\right)F_2' = 0.$$

The first one is solved in hypergeometric functions, and two others are solved in Heun functions. In this way, we have found three series of energy levels:

$$\epsilon = \hbar\left(N\sqrt{\frac{k}{m} + \frac{\hbar^2}{4m^2R^4}} - \frac{\hbar}{2mR^2}\left(N^2 + \frac{1}{4}\right)\right),$$
$$N = 2n + j + \frac{3}{2},$$
$$\epsilon = \hbar\left(N\sqrt{\frac{k}{m} + \frac{\hbar^2}{4m^2R^4}} - \frac{\hbar}{2mR^2}\left(N^2 + \frac{1}{4}\right)\right), \tag{67}$$
$$N = 2n' + j + 2,$$
$$\epsilon = \hbar\left(N\sqrt{\frac{k}{m} + \frac{\hbar^2}{4m^2R^4}} - \frac{\hbar}{2mR^2}\left(N^2 + \frac{1}{4}\right)\right),$$
$$N = 2n'' + j + 1.$$

13. Conclusions

Spin-1 particles are treated in the presence of a magnetic monopole in the non-relativistic approximation. After the separation of variables, the problem is reduced to a system of three coupled equations, which can be disconnected with the use of a special linear transformation making the mixing matrix diagonal. As a result, there arise three separated differential equations, which contain the roots of a cubic algebraic equation as parameters. This consideration is extended to the case with the presence of external spherically symmetric fields, in particular, Coulomb and oscillator ones. We have found the energy spectrum and the exact solutions in terms of hypergeometric functions. In the same manner, a spin-1 particle is treated against the Lobachevsky geometry background in the non-relativistic approximation. After the separation of variables, the problem is reduced to a system of second-order differential coupled equations, which cannot be disconnected in the presence of a monopole. However, in the absence of a monopole, the equations have been solved exactly, for instance, in the presence of the Coulomb and oscillator potentials. The energy spectra have been found and the solutions are constructed in terms of the hypergeometric and Heun functions.

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КВАНТОВА ЧАСТИНКА
ЗІ СПІНОМ 1 В ПОЛІ МАГНІТНОГО ЗАРЯДУ
В ПРОСТОРАХ ЕВКЛІДА І ЛОБАЧЕВСЬКОГО:
НЕРЕЛЯТИВІСТСЬКЕ НАБЛИЖЕННЯ

Р е з ю м е

Частинка зі спіном 1 досліджується за наявності магнітного монополя Дірака в нерелятивістському наближенні. Після розділення змінних задача зводиться до системи трьох взаємопов'язаних рівнянь, які можна розщепити, використовуючи спеціальне лінійне перетворення, яке приводить змішуючу матрицю до діагонального вигляду. В результаті виникають три окремі диференціальні рівняння другого порядку, які в ролі параметрів містять корені кубічного алгебраїчного рівняння. Додатково враховано зовнішні сферично-симетричні електричні поля, детально розглянуті випадки кулонівського і осциляторного потенціалів. Задача узагальнена на випадок гіперболічного простору Лобачевського; точні розв'язки радіального рівняння будуються в гіпергеометричних функціях і функціях Гойна.