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NEW VERSION OF q-DEFORMED SUPERSYMMETRIC QUANTUM MECHANICS

A new version of the q-deformed supersymmetric quantum mechanics (q-SQM), which is inspired by the Tamm–Dankoff-type (TD-type) deformation of quantum harmonic oscillator, is constructed. The obtained algebra of q-SQM is similar to that in Spiridonov's approach. However, within our version of q-SQM, the ground state found explicitly in the special case of superpotential yielding q-superoscillator turns out to be non-Gaussian and takes the form of special (TD-type) q-deformed Gaussian.

Keywords: supersymmetric quantum mechanics, q-deformation, scaling operator, q-superoscillator, ground state, q-Gaussian.

Combining the basic ideas of supersymmetry as incorporated in supersymmetric quantum mechanics or SQM (here, in one dimension), on the one hand, and a q-deformation that has become very popular after the discovery of quantum algebras and especially their Schwinger-type realization through the q-deformed oscillator algebra of Biedenharn and Macfarlane [1, 2], on the other hand, is important and potentially of much interest. Along this root, Spiridonov in Ref. [3] has proposed some rather general deformation of the supersymmetric (SUSY) quantum mechanics [4, 5] on the Hilbert space $\mathcal{H}$ of square integrable functions. As a result of the explicit definition of factorization operators realized in $\mathcal{H}$, (at least) two new features appeared. First, the familiar SUSY algebra became a q-SUSY algebra, i.e., a q-deformed extension of the SUSY algebra. Second, due to a q-deformation of the SUSY algebra, the conventional degeneracy of the familiar SQM gets lifted. Namely, the whole spectrum of H_+, the second of superpartner Hamiltonians, results from that of the first superpartner Hamiltonian H_- (save its lowest state) merely by a definite scaling applied to its spectrum.

Here, we present a new version of the q-deformed supersymmetric quantum mechanics (q-SQM). Its construction is inspired by the TD-type q-deformed oscillator, which was introduced in [6, 7] and whose unusual properties were studied in [8]. The scaling operator T_q is an important ingredient of our model.

It is worth to note that T_q appeared in Spiridonov's version of q-SQM in such special way that it drops from the bilinears $AA^\dagger$ and $A^\dagger A$ of raising/lowering operators. Unlike, in our version of q-SQM, the scaling operator T_q is present, besides the q-supercharges, also both in $AA^\dagger$, $A^\dagger A$ and in the q-SUSY Hamiltonian. An important property of our approach is that this formulation naturally leads to a non-Gaussian ground state when the superpotential is chosen as that corresponding to the q-superoscillator.

Similarly to [3], we define the q-SUSY algebra and provide its explicit realization on the Hilbert space of square integrable functions. Is should be noted that, when a q-deformation is implanted in the SUSY quantum mechanics, there is no degeneracy (natural in standard SUSY models) anymore, both in our version and in previous Spiridonov's one. What concerns the latter one, however, we should stress that while one sequence of eigenvalues (corresponding to H_-) is in fact undeformed and coincides with the case of undeformed superoscillator, the second one (corresponding to H_+) deforms in such way that all its eigenvalues result from respective non-deformed eigenvalues of H_- by a uniform q^2-scaling.

The raising and lowering operators in [3] entering the definition of supercharges as mentioned therein, to generate the q-oscillator algebra of Biedenharn and Macfarlane. The corresponding operators in our version of q-SQM obey a much more involved deformed oscillator algebra than that of Biedenharn and Macfarlane (and hardly known explicitly before). Moreover, we think the model given in [3] implies a more complicated (than BM case) q-oscillator algebra as well, while of course different from ours.

1. SUSYQM (SQM) and Spiridonov's version of q-deformed SQM

1.1. $N = 2$ supersymmetric quantum mechanics

$N = 2$ SQM is defined by the superalgebra

$$\{Q, Q^\dagger\} = H, \quad Q^2 = (Q^\dagger)^2 = 0,$$
$$[H, Q] = [Q^\dagger, H] = 0 \tag{1}$$

with the energy of (nondegenerate in case of exact SUSY) ground state $E_{\text{vac}} \geq 0$ and twofold degenerate spectrum of excited states. The supercharges are conserved, as implied by their commuting with the Hamiltonian. Throughout the paper, it is understood that $\hat{p} \equiv P = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}$.

Recall that the standard representation of SQM is

$$Q = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$
$$A = \frac{\hat{p} - iW(x)}{\sqrt{2}}, \quad [\hat{x}, \hat{p}] = i,$$
$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} =$$
$$= \frac{1}{2}\left(\hat{p}^2 + W^2(x) + W'(x)\sigma_3\right),$$
$$W'(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}W(x), \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The superpartner Hamiltonians $H_\pm$ are isospectral, which follows from the intertwining relations

$$A^\dagger H_+ = H_- A^\dagger, \quad H_+ A = A H_-. \tag{2}$$

The choice $W(x) = x$ corresponds to the harmonic oscillator problem with standard bosonic algebra

$$[a, a^\dagger] = 1, \quad [\hat{N}, a^\dagger] = a^\dagger, \quad [\hat{N}, a] = -a. \tag{3}$$

1.2. Properties of q-scaling operator

Here and in our main exposition below, we will use like in [3] an important tool of deformed SQM, namely the q-scaling operator T_q. Though its presence in our resulting formulas will be somewhat unconventional, this will cause no problems since its action on functions is well defined.

So, the q-scaling operator T_q is defined on smooth functions as

$$T_q f(x) = f(qx), \tag{4}$$

where $q \in \mathbb{R}$, and $q \geq 1$ or $0 < q \leq 1$.

The list of its main properties reads

$$T_q F(x) \cdot = [T_q F(x)]\, T_q \cdot, \quad T_q \frac{\mathrm{d}}{\mathrm{d}x} = q^{-1}\frac{\mathrm{d}}{\mathrm{d}x} T_q,$$
$$T_q T_p = T_{qp}, \quad T_q^{-1} = T_{q^{-1}}, \quad T_1 = 1, \tag{5}$$
$$T_q^\dagger = q^{-1} T_q^{-1}, \quad (T_q^\dagger)^\dagger = T_q.$$

Note that the operator $\sqrt{q}\,T_q$ is unitary, and the operator $\sqrt{q}\,T_q + \frac{1}{\sqrt{q}} T_{q^{-1}}$ is Hermitian.

The explicit realization of T_q as a pseudo-differential operator is

$$T_q = \left(e^{\ln q}\right)^{x\frac{\mathrm{d}}{\mathrm{d}x}} = q^{x\frac{\mathrm{d}}{\mathrm{d}x}}. \tag{6}$$

Obviously, $T_q x = qx$ and $T_q x^m = q^m x^m$ for any integer m.

1.3. Spiridonov's q-deformation of SQM: defining relations

This deformation is realized by inserting T_q after the factorization operator A

$$A \mapsto A_q = \frac{1}{\sqrt{2}}\left(\hat{p} - iW(x)\right) T_q,$$
$$A^\dagger \mapsto A_q^\dagger = \frac{q^{-1}}{\sqrt{2}} T_q^{-1}\left(\hat{p} + iW(x)\right). \tag{7}$$

From their products (index q is dropped here and below)

$$AA^\dagger = \frac{q^{-1}}{2}\left(\hat{p}^2 + W^2(x) + W'(x)\right), \tag{8}$$

$$A^\dagger A = \frac{q}{2}\left(\hat{p}^2 + q^{-2}W^2(q^{-1}x)\right) - q^{-1}W'(q^{-1}x)), \tag{9}$$

one can get the q-deformed Hamiltonian and supercharges as

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \equiv \begin{pmatrix} qAA^\dagger & 0 \\ 0 & q^{-1}A^\dagger A \end{pmatrix},$$
$$Q = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}. \tag{10}$$

1.4. Spiridonov's q-deformation of SQM: algebra of q-supersymmetry

The above operators satisfy the q-deformed $N = 2$ SUSY algebra

$$\{Q^\dagger, Q\}_q = H, \quad \{Q, Q\}_q = \{Q^\dagger, Q^\dagger\}_q = 0,$$

$$[H, Q]_q = [Q^\dagger, H]_q = 0, \quad (11)$$

where the commutators and anticommutators are now replaced by the corresponding q-brackets:

$$[X, Y]_q \equiv qXY - q^{-1}YX, \quad [Y, X]_q = -[X, Y]_{q^{-1}},$$

$$\{X, Y\}_q \equiv qXY + q^{-1}YX, \quad \{Y, X\}_q = \{X, Y\}_{q^{-1}}.$$

As we see, the supercharges are not conserved.

The intertwining relations for the Hamiltonians $H_\pm$ encoded in (11) obviously change:

$$A^\dagger H_+ = q^2 H_- A^\dagger, \quad H_+ A = q^2 A H_-. \quad (12)$$

This implies that H_- and H_+ (without the lowest state of H_-) are not isospectral to each other, but rather q-isospectral: the spectrum of H_+ results from the spectrum of H_- (without its lowest state) by applying the uniform q^2-scaling, that is,

$$H_+ \psi^{(+)} = E^{(+)}\psi^{(+)}, \quad H_- \psi^{(-)} = E^{(-)}\psi^{(-)},$$

$$E^{(+)} = q^2 E^{(-)}, \quad \psi^{(+)} \propto A\psi^{(-)}, \quad \psi^{(-)} \propto A^\dagger\psi^{(+)}.$$

Special case of q-supersymmetric oscillator

Consider the simplest physical example of a q-superoscillator, for which the superpotential is $W(x) = -x$ (or $W(x) = x$). In that case, we have

$$AA^\dagger = \frac{q^{-1}}{2}(\hat{p}^2 + x^2 - 1), \quad A^\dagger A = \frac{q}{2}(\hat{p}^2 + q^{-4}x^2 + q^{-2}),$$

that yields the anticommutator and the commutator

$$\{A, A^\dagger\} = \frac{q^{-1}}{2}\Big((1 + q^2)\hat{p}^2 + (1 + q^{-2})x^2\Big),$$

$$[A, A^\dagger] = \frac{q^{-1}}{2}\Big((1 - q^2)\hat{p}^2 + (1 - q^{-2})x^2 - 2\Big),$$

along with such versions of q-commutators:

$$AA^\dagger - qA^\dagger A = \frac{1}{2}\Big((q^{-1} - q^2)\hat{p}^2 + (q^{-1} - q^{-2})x^2 - q^{-1} - 1\Big),$$

$$qAA^\dagger - A^\dagger A = \frac{1}{2}\Big((1 - q)\hat{p}^2 + (1 - q^{-3})x^2 - q^{-1} - 1\Big),$$

$$qAA^\dagger - q^{-1}A^\dagger A = \frac{1 + q^{-2}}{2}\Big((1 - q^{-2})x^2 - 1\Big),$$

$$q^{-1}AA^\dagger - qA^\dagger A = \frac{1}{2}\Big((q^{-2} - q^2)\hat{p}^2 - q^{-2} - 1\Big).$$

The Hamiltonian of this q-superoscillator takes the form (with I_2 being a 2×2 unit matrix)

$$4H = \left[2\hat{p}^2 + \left(1 + q^{-4}\right)x^2 + 1 - q^{-2}\right] I_2 +$$

$$+ \left[\left(1 - q^{-4}\right)x^2 + 1 + q^{-2}\right]\sigma_3. \quad (13)$$

It describes a spin-1/2 particle in the harmonic potential and with transverse magnetic field.

Remark. It is important to note that the spectrum (it is obviously equidistant) of the superpartner Hamiltonian H^- coincides, up to an overall multiplier q^{-1}, with that of the corresponding superpartner Hamiltonian in the usual (non-deformed) SUSY quantum mechanics (see, e.g., [5]).

Let us comment on the physical meaning of the deformation parameter q: it plays a role of some additional interaction constant. Note also that this model possesses the exact q-deformed SUSY. As mentioned in [3], for the value of q^2 being a simple rational number, the spectrum of the q-superoscillator shows an accidental degeneracy.

2. New version of q-SQM inspired by the Tamm–Dankoff deformation

2.1. Tamm–Dankoff (TD) deformed oscillator

The "Tamm–Dankoff cutoff" deformed oscillator is given in terms of a q-bracket of the TD type:

$$\hat{N} \mapsto \{\hat{N}\}_q \equiv \hat{N}q^{\hat{N}-1}, \quad \{\hat{N}\}_q \xrightarrow{q\to 1} N,$$

$$a^\dagger a = \{\hat{N}\}_q, \quad aa^\dagger = \{\hat{N} + 1\}_q,$$

with the algebra (denote $a \equiv a^-$ and $a^\dagger \equiv a^+$)

$$aa^\dagger - qa^\dagger a = q^{\hat{N}}, \quad \left[\hat{N}, a^\pm\right] = \pm a^\pm, \quad (14)$$

$\hat{N}$ being the number operator, and the Hamiltonian

$$H_{\text{oscil.}} = \frac{1}{2}\left(\{N\}_q + \{N + 1\}_q\right).$$

The energy spectrum of this TD-type q-deformed oscillator,

$$E_n = \frac{1}{2}\left(nq^{n-1} + (n + 1)q^n\right),$$

is very special. As noticed in [8], the TD-type q-deformed oscillator shows various patterns of the accidental pairwise energy level degeneracy, always within a definite single pair of levels.

Below, we will study the q-deformation of SUSY QM in the spirit of the TD-type q-deformation,

$$\hat{N} \mapsto \hat{N} q^{\hat{N}-1} \quad \Longleftrightarrow \quad \hat{p} \mapsto q^{-1} \hat{p}\, T_q, \tag{15}$$

that is, we will merely adopt the replacement of the momentum operator just as it is indicated here.

2.2. Tamm–Dankoff type q-deformed SQM

We start by introducing a q-deformation in the momentum part of undeformed factorization operators A and $A^\dagger$:

$$\begin{aligned} &A \mapsto B = \frac{1}{\sqrt{2}} (T_q \hat{p} - iW(x)), \\ &A^\dagger \mapsto B^\dagger = \frac{1}{\sqrt{2}} \left(q^{-1} \hat{p} T_q^{-1} + iW(x)\right). \end{aligned} \tag{16}$$

Then,

$$BB^\dagger = \frac{1}{2q}(q^{-2}\hat{p}^2 + qW^2(x) + W'(qx)T_q +$$

$$+ iW(qx)\hat{p}T_q - iW(x)\hat{p}T_q^{-1}), \tag{17}$$

$$B^\dagger B = \frac{1}{2q}(\hat{p}^2 + qW^2(x) - W'(q^{-1}x)T_q^{-1} +$$

$$+ iW(x)\hat{p}T_q - iW(q^{-1}x)\hat{p}T_q^{-1}). \tag{18}$$

Setting $B^\dagger B \equiv H_-$ and $BB^\dagger \equiv H_+$ we come to the q-deformed algebra for $H, Q, Q^\dagger$ (see also Eq. (11)):

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \equiv \begin{pmatrix} qBB^\dagger & 0 \\ 0 & q^{-1}B^\dagger B \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 \\ B^\dagger & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix};$$

$$\{Q^\dagger, Q\}_q = H, \quad \{Q, Q\}_q = \{Q^\dagger, Q^\dagger\}_q = 0,$$

$$[H, Q]_q = [Q^\dagger, H]_q = 0.$$

As seen, the supercharges in our model are not conserved as well.

2.3. Tamm–Dankoff type deformation: q-deformed supersymmetric oscillator

We consider a q-supersymmetric oscillator with the superpotential $W(x) = -x$. In this case,

$$B = \frac{1}{\sqrt{2}} (T_q \hat{p} - iX)), \quad B^\dagger = \frac{1}{\sqrt{2}} \left(q^{-1}\hat{p}T_q^{-1} + iX\right), \tag{19}$$

where $X \equiv x\cdot$. Now (16)–(17) turn into

$$BB^\dagger = \frac{1}{2q}\Big(q^{-2}\hat{p}^2 + qX^2 + qT_q \ + iX\hat{p}(qT_q - T_q^{-1})\Big),$$

$$B^\dagger B = \frac{1}{2q}\Big(\hat{p}^2 + qX^2 - q^{-1}T_q^{-1} + iX\hat{p}(T_q - q^{-1}T_q^{-1})\Big).$$

From these, different versions of the permutation relation involving different (q-)commutators result:

$$BB^\dagger - B^\dagger B = \frac{1}{2q}\Big((q^{-2} - 1)\hat{p}^2 +$$

$$+ iX\hat{p}(1 - q^{-1})(qT_q - T_q^{-1}) + qT_q + q^{-1}T_q^{-1}\Big),$$

$$qBB^\dagger - q^{-1}B^\dagger B = \frac{1}{2q}\Big((q^2 - 1)X^2 +$$

$$+ iX\hat{p}(q - q^{-2})(qT_q - T_q^{-1}) + q^2T_q + q^{-2}T_q^{-1}\Big),$$

$$BB^\dagger - qB^\dagger B = \frac{1}{2q}\Big((q - q^2)X^2 +$$

$$+ (q^{-2} - q)\hat{p}^2 + +qT_q + T_q^{-1}\Big),$$

$$qBB^\dagger - B^\dagger B = \frac{1}{2q}\Big((q^{-1} - 1)\hat{p}^2 + (q^2 - q)X^2 +$$

$$+ \mathrm{i}X\hat{p}(q - q^{-1})(qT_q - T_q^{-1}) + (q^2T_q + q^{-1}T_q^{-1})\Big).$$

Here we observe the following:

– if we use a usual commutator, the dependence on X^2 drops on the r.h.s.;

– if we use the q-commutator $qAB - q^{-1}BA$, the dependence on $\hat{p}^2$ drops on the r.h.s.;

– if we use the q-commutator $AB - qBA$, the terms with $\mathrm{i}X\hat{p}$ cancel out.

As a check of consistency, we verify: in the limit $q \to 1$, each of these relations turns into the standard commutation relation $[B, B^\dagger] = 1$ for the boson operators B and $B^\dagger$.

2.4. TD-type deformation: relation with deformed Heisenberg algebra

We wish to find explicitly the relation of our deformed oscillator algebra with some version of the deformed Heisenberg algebra, say along the lines described in [12]. For this, we solve Eq. (19) for X and P in terms of the operators B, $B^\dagger$ (and also T_q, $T_{q^{-1}}$):

$$P = q\sqrt{2}\left(B\frac{T_q}{1+T_{q^2}} + B^\dagger\frac{T_q^{-1}}{1+T_{q^{-2}}}\right) =$$

$$= q\sqrt{2}(B+B^\dagger)\frac{1}{T_q+T_{q^{-1}}}, \qquad (20)$$

$$X = \mathrm{i}\sqrt{2}\left(B\frac{1}{1+T_{q^2}} - B^\dagger\frac{1}{1+T_{q^{-2}}}\right) =$$

$$= \mathrm{i}\sqrt{2}(BT_{q^{-1}} - B^\dagger T_q)\,\frac{1}{T_q+T_{q^{-1}}}. \qquad (21)$$

To check once more the Hermiticity of X, it is better to use another formula for X stemming from (19):

$$X = \frac{\sqrt{2}}{2\mathrm{i}}(B^\dagger - B) + \frac{q^{-1}}{2\mathrm{i}}P(T_q - T_{q^{-1}}). \qquad (22)$$

Now the Hermiticity follows from the skew-Hermiticity of $B^\dagger - B$ and skew-Hermiticity of the product operator $P(T_q - T_{q^{-1}})$. Likewise, the Hermiticity of P stems, see (20), from that of $B + B^\dagger$ and the fact that the product $P(T_q + T_{q^{-1}})$ is Hermitian.

The operators P and X can be expressed through q- or q^{-1}-commutators of the operators B and T_q^{-1} as follows:

$$P = \frac{\sqrt{2}}{q^{-1}-q}[B, T_q^{-1}]_q, \quad X = \frac{\mathrm{i}\sqrt{2}}{q-q^{-1}}T_q[B, T_q^{-1}]_{q^{-1}},$$

where $[A, B]_q \equiv AB - qBA$. With the equality $[A, B]_q A^{-1} = A[B, A^{-1}]_q$ taken into account, we have yet another formulas for X:

$$X = \frac{\mathrm{i}\sqrt{2}}{q-q^{-1}}[T_q, B]_{q^{-1}}T_q^{-1} = \frac{\mathrm{i}\sqrt{2}}{1-q^2}T_q[T_q^{-1}, B]_q =$$

$$= \frac{\mathrm{i}\sqrt{2}}{1-q^2}[B, T_q]_q T_q^{-1}.$$

From these expressions after some algebra, we obtain (note that $[X, P] = \mathrm{i}$) is intact:

$$\mathrm{i} = XP - PX = \frac{2\mathrm{i}}{(q-q^{-1})^2}\big(T_q^{-1}[T_q, B]_q[B, T_q^{-1}]_{q^{-1}} -$$

$$- T_q[B, T_q^{-1}]_{q^{-1}}[B, T_q^{-1}]_q\big).$$

That implies the validity of two identities:

$$T_q^{-1}[T_q, B]_q[B, T_q^{-1}]_{q^{-1}} - T_q[B, T_q^{-1}]_{q^{-1}}[B, T_q^{-1}]_q =$$

$$= (q - q^{-1})^2/2,$$

$$T_q[T_q^{-1}, B]_q[B, T_q^{-1}]_q - T_q^{-1}[T_q, B]_q[T_q^{-1}, B]_q =$$

$$= q(q - q^{-1})^2/2.$$

On the other hand, we can deduce a q-deformed extension of Heisenberg algebra for the pair of operators X and $\tilde{P} = T_q P = q^{-1}PT_q$, which is

$$[X, \tilde{P}] = q^{-1}\big(\mathrm{i}T_q + (1-q)\tilde{P}X\big). \qquad (23)$$

If we compare (23) with the known deformations of the Heisenberg algebra, e.g., those from [9], we notice the presence of a q-scaling operator T_q times $\mathrm{i}q^{-1}$ and of the bilinear $\tilde{P}X$ multiplied by $1 - q$.

Written through the q-commutator, it takes another simpler though equivalent form

$$X\tilde{P} - \frac{1}{q}\tilde{P}X = \frac{\mathrm{i}}{q}T_q. \qquad (24)$$

Clearly, in the limit $q \to 1$, the r.h.s. of both (23) and (24) turns into familiar "i".

2.5. Ground state of TD-type q-superoscillator

Let us find the ground state (zero mode) for the ladder operators B and $B^\dagger$, namely,

$$\begin{cases} Bf(x) = 0 \\ B^\dagger\tilde{f}(x) = 0 \end{cases} \Rightarrow \begin{cases} T_q f'(x) + xf(x) = 0, \\ q^{-2}T_{q^{-1}}\tilde{f}'(x) - x\tilde{f}(x) = 0; \end{cases} \qquad (25)$$

$$\begin{cases} f(x) = \sum\limits_{k=0}^{\infty} C_k x^k, \\ \tilde{f}(x) = \sum\limits_{k=0}^{\infty} \tilde{C}_k x^k. \end{cases} \qquad (26)$$

These turn into recurrence relations for the expansion coefficients

$$\begin{cases} f(x): \begin{cases} C_k + (k+2)q^{k+1}C_{k+2} = 0, \\ C_1 = 0, \end{cases} \\ \tilde{f}(x): \begin{cases} -\tilde{C}_k + (k+2)q^{-k-3}\tilde{C}_{k+2} = 0, \\ \tilde{C}_1 = 0. \end{cases} \end{cases} \qquad (27)$$

These can be solved (e.g., by using Mathematica), that gives

$$\begin{cases} f(x) = C_0 \sum\limits_{k=0}^{\infty} \frac{q^{-k^2}}{k!}\left(-\frac{x^2}{2}\right)^k \\ \tilde{f}(x) = \tilde{C}_0 \sum\limits_{k=0}^{\infty} \frac{q^{k(k+2)}}{k!}\left(\frac{x^2}{2}\right)^k \end{cases} \xrightarrow{q\to 1} \begin{cases} f(x) = e^{-\frac{x^2}{2}}, \\ \tilde{f}(x) = e^{\frac{x^2}{2}}. \end{cases} \quad (28)$$

Here, C_0 and $\tilde{C}_0$ are arbitrary constants depending on the deformation parameter q, with the obvious property $\{C_0, \tilde{C}_0\} \xrightarrow{q\to 1} 1$ in order to recover the undeformed case. From these two functions, only $f(x)$ at $q \to 1$ recovers the ground state of standard supersymmetric oscillator. Moreover, the function $\tilde{f}(x)$ in (28) is not square integrable. Thus, we find

$$\psi_0 = C_0 \sum_{k=0}^{\infty} \frac{q^{-k^2}}{k!}\left(-\frac{x^2}{2}\right)^k. \quad (29)$$

We have obtained the unique, i.e. non-degenerate, ground state as the (non-Gaussian) eigenfunction of B^- and hence of H^-. Below, it will be shown that this non-Gaussian wave function naturally takes the form of a specially introduced TD-type q-deformed Gaussian exponent.

2.6. Elements of TD-analysis

We will need some more elements of TD-analysis. First, let us introduce the q-number of the Tamm–Dankoff type

$$(n]_q \equiv nq^{n-1}. \quad (30)$$

This form can be easily obtained from the (p,q)-number $[n]_{p,q} = \frac{p^n - q^n}{p-q}$ by taking the limit $p \to q$. The TD-factorial is given as

$$\begin{aligned} &(n]_q! = (1]_q(2]_q \ldots (n]_q = q^{n(n-1)/2} n!, \\ &(0]_q! = 0! = 1, \quad (1]_q! = 1! = 1. \end{aligned} \quad (31)$$

Remembering that $q^{z\frac{d}{dz}} f(z) = f(qz)$, one can introduce the TD-derivative

$$z\mathcal{D}_z^{(\mathrm{TD})} \equiv z\frac{d}{dz} q^{z\frac{d}{dz}-1}, \quad (32)$$

which acts on monomials as

$$\mathcal{D}_z^{(\mathrm{TD})} z^n = (n]_q z^{n-1}. \quad (33)$$

From these definitions, it is natural to introduce the TD-exponent

$$\exp_q^{(\mathrm{TD})}(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{(n]!} = \sum_{n=0}^{\infty} \frac{q^{-n(n-1)/2}}{n!} z^n, \quad (34)$$

with the property

$$\mathcal{D}_z^{(\mathrm{TD})} \exp_q^{(\mathrm{TD})}(\alpha z) = \alpha \exp_q^{(\mathrm{TD})}(\alpha z). \quad (35)$$

In the $q\to 1$ (no-deformation) limit: $e_q^{(\mathrm{TD})}(z) \xrightarrow{q\to 1} e^z$, $\mathcal{D}_z^{(\mathrm{TD})} \xrightarrow{q\to 1} \frac{d}{dz}$.

The ground state ψ_0 has a natural record in terms of the TD-exponent

$$\psi_0(z) = C_0 \exp_{q^2}^{(\mathrm{TD})}\left(-\frac{1}{2}q^{-1}z^2\right). \quad (36)$$

In the limit $q\to 1$, we have $\psi_0(z) \to C_0 \exp\left(-\frac{z^2}{2}\right)$.

2.7. Relation to bibasic and twin-basic hypergeometric functions

First, let us recall the (p,q)-exponent

$$\exp_{p,q}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{p,q}!}, \quad (37)$$

where $[n]_{p,q}! = [1]_{p,q}[2]_{p,q}...[n]_{p,q}$, and the (p,q)-number was given after Eq. (30). The TD-exponent can be recovered from (37) in the $p \to q$ limit. The family of the so-called twin-basic hypergeometric functions is given as

$$\begin{aligned} &{}_r\Phi_s(\{a,b\};\{c,d\};(p,q);z) \equiv \\ &\equiv \sum_{n=0}^{\infty} \frac{((a_1,b_1),(a_2,b_2),...,(a_r,b_r);(p,q))_n}{((c_1,d_1),(c_2,d_2),...,(c_s,d_s);(p,q))_n} \times \\ &\times \frac{\left[(-1)^n (q/p)^{n(n-1)/2}\right]^{1+s-r}}{((p,q);(p,q))_n} z^n, \end{aligned} \quad (38)$$

where we introduced the shorthand notations

$$\begin{aligned} &\{a,b\} \doteq ((a_1,b_1),(a_2,b_2),...,(a_r,b_r)) \\ &((a_1,b_1),(a_2,b_2),...,(a_r,b_r);(p,q))_n \doteq \\ &\doteq ((a_1,b_1);(p,q))_n ((a_2,b_2);(p,q))_n \cdots \\ &\cdots ((a_r,b_r);(p,q))_n, \\ &((a,b);(p,q))_n \doteq (a-b)(ap-bq)(ap^2-bq^2)... \\ &...(ap^{n-1}-bq^{n-1}), \quad ((a,b);(p,q))_0 = 1. \end{aligned}$$

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Some special cases are:

$$((0,b);(p,q))_n = (-b)^n q^{n(n-1)/2}, \quad (39)$$
$$((a,0);(p,q))_n = a^n p^{n(n-1)/2}, \quad (40)$$
$$((a,b);(q,q))_n = (a-b)^n q^{n(n-1)/2}, \quad (41)$$
$$((a,a);(p,q))_n = 0, \quad (42)$$
$$((p,q);(p,q))_n = (p-q)^n [n]_{p,q}, \quad (43)$$
$$\left((q^{-1},q);(q^{-1},q)\right)_n = \left(q-q^{-1}\right)^n [n]_q!. \quad (44)$$

The requirement of convergence of (38) implies that $|q/p| < 1$ and also $|z| < 1$. Now, we establish the relation between different deformed exponents and the twin-basic hypergeometric function ${}_1\Phi_1$, namely,

$$\exp_{p,q}(z) = {}_1\Phi_1((1,0);(0,1);(p,q);(p-q)z), \quad (45)$$

$$\exp_q(z) = {}_1\Phi_1((1,0);(0,1);(q^{-1},q);(q^{-1}-q)z), \quad (46)$$

$$\exp_q^{(\mathrm{TD})}(z) = \lim_{p\to q} {}_1\Phi_1((1,0);(0,1);(p,q);(p-q)z). \quad (47)$$

That is, when the limit on the r.h.s. of (47) is performed, we obtain nothing but the TD-type q-exponent from (36).

On the other hand, the TD-exponent can be expressed as some special case of bibasic hypergeometric functions (see, e.g., [10, 11]):

$$ {}_{r,r'}\mathcal{F}_{s,s'}(\underline{a},\underline{c};\underline{b},\underline{d};(p,q);z) \equiv $$
$$ \equiv \sum_{n=0}^{\infty} \frac{(a_1,a_2,...,a_r;p)_n (c_1,c_2,...,c_{r'};q)_n}{(b_1,b_2,...,b_s;p)_n (d_1,d_2,...,d_{s'};q)_n\,(q;q)_n} \times $$
$$ \times \left[(-1)^n p^{n(n-1)/2}\right]^{1+s-r} \left[(-1)^n q^{n(n-1)/2}\right]^{s'-r'} z^n. \quad (48)$$

Here $\underline{a} = (a_1,a_2,...,a_r)$ and $(a_1,a_2,...,a_r;p)_n \equiv \equiv (a_1;p)_n(a_2;p)_n \ldots (a_r;p)_n$; and $(a;q)_n \equiv \frac{(a;q)_\infty}{(aq^n;q)_\infty}$ are the q-Pochhammer symbols. Then we have

$$\exp_q^{(\mathrm{TD})}(z) \equiv \lim_{p\to 1} {}_{0,0}\mathcal{F}_{0,1}(-,-;-,0;p,q^{-1};(1-p)z), \quad (49)$$

where it is taken into account that $\frac{(p,p)_n}{(1-p)^n} \xrightarrow{p\to 1} n!$. Now it is clear that, using (49), our ground-state wave function ψ_0 in (36) can be presented as a particular (limit of) bibasic hypergeometric function.

3. Concluding Remarks

In this work, we have presented a new version of the q-deformed supersymmetric quantum mechanics and a new q-deformed superoscillator. Our way of deformation is rooted in the special Tamm–Dankoff-type deformation of quantum harmonic oscillator. Though our main defining relation differs from Spiridonov's variant of q-SQM only slightly at first sight (compare (7) and (16)), the consequences are more principal. The basic distinction lies in the presence, in our case, of the scaling operator T_q (or $T_{q^{-1}}$) in all subsequent formulas: for bilinears $BB^\dagger$, $B^\dagger B$, q-supercharges, and q-super-Hamiltonian, while the analogous operators do not involve T_q in Spiridonov's version (only traces of its action can be seen in these operators). The second important distinction concerns the ground states in the two versions: it is a usual Gaussian for the superpotential $W(x) = -x$ of a superoscillator in Spiridonov's case, and the special TD type of a q-Gaussian in our paper, see (36)–(37) above.

At last, let us make three final remarks:

– In the version of q-SQM presented above, the q-SUSY algebra is exact, as seen from the relations at the end of subsection 2.2, and there is exactly one state with the lowest (zero) energy. However, the twofold degeneracies of exited states of standard SUSY models are lifted. Moreover, the whole set of (q-dependent) eigenvalues of the q-superpartner Hamiltonian H_+ is obtained from the set of q-superpartner Hamiltonian H_- eigenvalues (other than zero and as well q-dependent) merely by the q^2-scaling.

– The particular $W(x) = \pm x$ case of a q-deformed superoscillator was considered, and the ground state is found and expressed through the $p \to q$ limit of the bibasic p,q-hypergeometric function, or through the appropriate limit of the twin-basic hypergeometric function.

– A complete system of eigenfunctions for the excited states of such a q-superoscillator yet remains to be found, and it is an exciting problem!

– A very interesting question is, of course, about a possible relation(s), if any, to the recently developed nonlinear extensions of supersymmetric Quantum Mechanics, see, e.g., [9].

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НОВА ВЕРСІЯ Q-ДЕФОРМОВАНОЇ СУПЕРСИМЕТРИЧНОЇ КВАНТОВОЇ МЕХАНІКИ

Р е з ю м е

Запропонована нова версія q-деформованої суперсиметричної квантової механіки (q-СКМ), інспірована q-деформацією квантового гармонічного осцилятора у формі Тама–Данкова. Отримано алгебру q-СКМ, яка за виглядом подібна до отриманої у підході Спірідонова. Однак, у рамках нашої версії q-СКМ, найнижчий стан для часткового випадку суперпотенціалу, що відповідає q-суперосцилятору, знайдено явно – він відмінний від гаусіана, і має вигляд спеціальної (типу Тама–Данкова) q-деформації гаусової експоненти. Встановлено зв'язок останньої з частковими випадками (границями) бібазисної, а також і твін-базисної узагальнених гіпергеометричних функцій.