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PACS 47.85.Dh, 62.20.+w, 65.20.Fe

NUMERICAL SOLUTION FOR THE EFFECT OF VARIABLE FLUID PROPERTIES ON THE FLOW AND HEAT TRANSFER IN A NON-NEWTONIAN MAXWELL FLUID OVER AN UNSTEADY STRETCHING SHEET WITH INTERNAL HEAT GENERATION

This article looks at the flow and heat transfer in the unsteady two-dimensional boundary layer of a non-Newtonian Maxwell fluid over a stretching sheet in the presence of variable fluid properties and internal heat generation. The governing differential equations are transformed into a set of coupled non-linear ordinary differential equations and then solved numerically, by using the appropriate boundary conditions for various physical parameters. The numerical solution for the governing non-linear boundary-value problem is based on applying the Chebyshev spectral method over the entire range of physical parameters. The effects of various parameters like the viscosity parameter, thermal conductivity parameter, unsteadiness parameter, heat generation parameter, Maxwell parameter, and Prandtl number on the flow and temperature profiles, as well as on the local skin-friction coefficient and the local Nusselt number, are presented and discussed. Comparison of numerical results is made with the earlier published results in limiting cases. A special attention is given to the effect of the viscosity parameter, thermal conductivity parameter, and heat generation parameter on the velocity and temperature fields above the sheet.

Keywords: Maxwell fluid, unsteady stretching sheet, variable fluid properties, internal heat generation, Chebyshev spectral method.

1. Introduction

Recently, the studies of boundary layer flows of non-Newtonian fluids over a stretching surface have attracted a high interest of researchers due to their engineering applications in a number of processes. The familiar examples are the extrusion of polymer fluids, glass blowing, fibers spinning, suspension solutions (see Fig. 1), chemical engineering, solidification of liquid crystals, hot rolling, manufacture of plastic and rubber sheets, crystal growing, continuous cooling, exotic lubricants, *etc.* Crane [1] was the first who studied the motion set up in the ambient fluid due to a linearly stretching surface. Gupta and Gupta [2] have subsequently explored various aspects of the accompanying heat transfer occurring in the infinite fluid medium surrounding the stretching sheet. The flow field of a stretching wall with a power-law velocity variation was discussed by Banks [3]. Grubka and Bobba [4] have analyzed the stretching problem for a surface moving with a linear velocity and with a variable surface temperature. Chen and Char [5] investigated the heat transfer characteristics over a continuous stretching sheet with variable surface temperature. The self-similar boundary flow with identically vanishing skin friction induced by a continuous plane surface with stretching is considered by Magyari and Keller [6]. Mahapatra and Gupta [7] analyzed a stagnation-point flow toward a stretching surface in the presence of a free stream.

In all these mentioned studies [1–7], the flow and temperature fields are considered to be at a steady state. The case where the stretching force and the surface temperature are varying with the time was considered in relatively few papers. The fluid velocity and the skin friction coefficient for an unsteady flow past a wall, which starts to move impulsively from the rest, were calculated by Pop and Na [8], using both the series and numerical solution methods. They found that the unsteady flow would approach the steady flow situation after a long passage of time. The effect of the unsteadiness parameter on heat transfer and the flow field over a stretching sur-

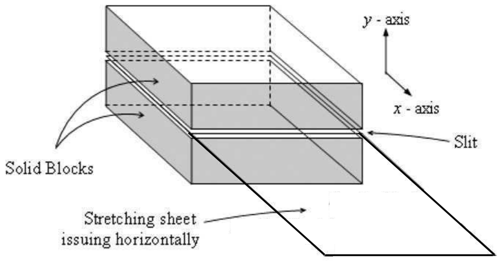

Fig. 1. Schematic of extrusion processes [28]

face with and without heat generation was considered by Elbashbeshy and Bazid [9, 10], respectively. Ali and Magyari [11] studied the problem with unsteady stretching surface condition, by using the similarity method to transform the governing time-dependent boundary layer equations to a set of ordinary differential equations. El-Aziz [12] studied the thermal radiation effects over an unsteady stretching sheet. Ishak *et al.* [13] investigated the boundary layer flow over a continuous stretching permeable surface. They found that the heat transfer rate at the surface increase with the unsteadiness parameter. Effect of Hall's current on the flow and heat transfer over an unsteady stretching surface in the presence of a strong magnetic field was analyzed by El-Aziz [14].

All the above studies [1–14] deal with the flow and heat transfer for a Newtonian fluid. A vast majority of reactions involved specifically in the polymer processing, food processing, biochemical industries, *etc.*, are also typical examples of a non-Newtonian behavior. The modeling studies that deal with non-Newtonian fluids offer interesting challenges to the mathematicians, physicists, engineers, and computer scientists. Rajagopal [15] studied exact solutions for a class of unsteady unidirectional flows of a second-grade fluid under four different flow situations. The fractional calculus approach in the constitutive relationship model of a generalized Maxwell fluid was introduced by Tan *et al.* [16]. The problem of the flow and heat transfer in a thin film of a power-law fluid on an unsteady stretching surface was investigated by Chen [17, 18]. He also studied the effect of viscous dissipation on heat transfer in a non-Newtonian thin liquid film over an unsteady stretching sheet. The fractional calculus were found to be quite flexible for describing the rheological and viscoelastic properties of fluids [19]. Abel *et al.* [20] investigated the effect of a non-uniform heat source on MHD heat transfer in a liquid film over an unsteady stretching sheet. Mahmoud and Megahed [21] investigated the effects of variable viscosity and thermal conductivity on the flow and heat transfer of an electrically conducting power-law fluid within a thin liquid film over an unsteady stretching sheet in the presence of a transverse magnetic field.

The purpose of the present work is to study the effects of variable fluid properties and internal heat generation on the flow and heat transfer of a non-Newtonian Maxwell fluid over an unsteady stretching sheet. To achieve this purpose, we use the well-known numerical technique, namely the Chebyshev spectral method.

Chebyshev polynomials are examples of eigenfunctions of singular Sturm–Liouville problems. Chebyshev polynomials have been used widely in the numerical solutions of boundary-value problems [22] and in the computational fluid dynamics and many applications [23–25]. The existence of a fast Fourier transformation for Chebyshev polynomials to efficiently compute matrix-vector products has meant that they are more widely used than other sets of orthogonal polynomials. Moreover, they have good properties in the approximation of functions.

The well-known family of orthogonal polynomials on $[-1,1]$ are Chebyshev polynomials, which can be determined with the aid of the following recurrence formula:

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x), \qquad n = 1, 2, \dots .$$

The first three Chebyshev polynomials are

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1.$$

These polynomials have been implemented to solve the linear and non-linear differential equations and the integral and integro-differential equations [23]. This method is also adopted for solving the fractional diffusion equation [24] and fractional order integro-differential equations [25].

The organization of this paper is as follows: In the next section, the formulation of the problem is given. Section 3 summarizes the solution procedure using the Chebyshev spectral method. In Section 4, the results and their discussion are presented. The conclusion is given in Section 5.

2. Formulation of the Problem

Consider an unsteady two-dimensional laminar boundary layer flow of a non-Newtonian Maxwell fluid over a stretching sheet immersed in the incompressible fluid. The x axis is chosen along the plane of the sheet and the y axis is taken to be normal to the plane. We assume that the surface starts stretching from rest with velocity $U(x,t)$, (see Fig. 2).

The viscosity μ and the thermal conductivity κ of the fluid are assumed to vary with the temperature as follows [21]:

$$\mu = \mu_\infty e^{-\alpha\theta}, \quad (1)$$

$$\kappa = \kappa_\infty(1+\varepsilon\theta), \quad (2)$$

where μ_∞ and κ_∞ are the coefficients of viscosity and thermal conductivity at the ambient, α is the viscosity parameter, and ε is the thermal conductivity parameter.

After using the usual boundary layer approximations, the basic equations for the mass, momentum, and energy in the boundary layer take the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (3)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) -$$

$$-\lambda_1\left[u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial x\partial y}\right], \quad (4)$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\rho c_p}\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right) + \frac{q'''}{\rho c_p}, \quad (5)$$

where u and v are the velocity components along the x and y directions, respectively, ρ is the fluid density, T is the fluid temperature, t is the time, λ_1 is the relaxation time, c_p is the specific heat at a constant pressure, and q''' is the rate of internal heat generation. The appropriate boundary conditions for the present problem are as follows:

$$u = U, \quad v = 0, \quad T = T_w \quad \text{at} \quad y = 0, \quad (6)$$

$$u \to 0, \quad T \to T_\infty \quad \text{as} \quad y \to \infty, \quad (7)$$

where U is the surface velocity of the stretching sheet, T_w is the surface temperature, and T_∞ is the free stream temperature. The flow is caused by stretching

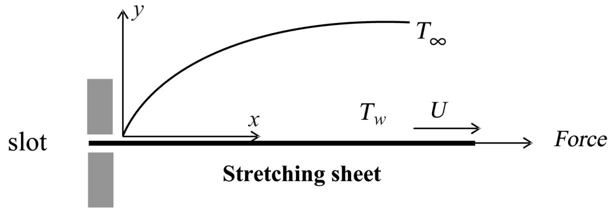

Fig. 2. Flow geometry and coordinate system

the elastic surface at $y = 0$ such that the continuous sheet moves in the x direction with the velocity

$$U = \frac{bx}{1-at}, \quad (8)$$

where a and b are positive constants with dimension (time^{-1}). We note that our problem is valid only for $at < 1$.

The mathematical analysis of the problem is simplified by introducing the following dimensionless characteristics:

$$\eta = \left(\frac{b}{(\mu_\infty/\rho)}\right)^{1/2}(1-at)^{-1/2}y, \quad (9)$$

$$\psi = \left(\frac{\mu_\infty b}{\rho}\right)^{1/2}(1-at)^{-1/2}xf(\eta), \quad (10)$$

$$T = T_\infty + T_0\left(\frac{dx^2}{2(\mu_\infty/\rho)}\right)(1-at)^{-3/2}\theta(\eta), \quad (11)$$

$$T_w = T_\infty + T_0\left(\frac{dx^2}{2(\mu_\infty/\rho)}\right)(1-at)^{-3/2}, \quad (12)$$

where f is the dimensionless stream function, θ is the dimensionless temperature of the fluid, and d is a constant.

The internal heat generation or absorption q''' is modeled according to the equation [26]

$$q''' = \left(\frac{\kappa_\infty \mathrm{Re}_x}{x^2}\right)[a^*(T_w - T_\infty)e^{-\eta} + b^*(T - T_\infty)], \quad (13)$$

where $\mathrm{Re}_x = \frac{\rho U x}{\mu_\infty}$ is the local Reynolds number, a^* is the temperature-dependent heat source/sink parameter, and b^* is the space-dependent heat source/sink parameter.

In Eq. (13), the first term represents the dependence of the internal heat generation or absorption on the space coordinates, whereas the last term represents its dependence on the temperature. Note that when $a^* > 0$ and $b^* > 0$, we have the case with internal heat generation. But it will be the case with internal heat absorption for $a^* < 0$ and $b^* < 0$.

Using Eqs.(9)–(12), the mathematical problem defined in Eqs.(3)–(5) is then transformed in a set of ordinary differential equations and in their associated boundary conditions:

$$e^{-\alpha\theta}(f''' - \alpha\theta' f'') + ff'' - f'^2 - S\left(f' + \frac{\eta}{2}f''\right) - - De(f^2 f''' - 2ff'f'') = 0, \quad (14)$$

$$(1+\varepsilon\theta)\theta'' + \varepsilon\theta'^2 + Pr\left(f\theta' - 2f'\theta - \frac{S}{2}(3\theta + \eta\theta')\right) + + a^* e^{-\eta} + b^*\theta = 0, \quad (15)$$

$$f(0) = 0, \quad f'(0) = 1, \quad \theta(0) = 1, \quad (16)$$

$$f' \to 0, \quad \theta \to 0, \quad \text{as} \quad \eta \to \infty, \quad (17)$$

where a prime denotes the differentiation with respect to η, $S = \frac{a}{b}$ is the unsteadiness parameter, $Pr = \frac{\mu_\infty c_p}{\kappa_\infty}$ is the Prandtl number, and $De = \frac{\lambda_1(t)b}{(1-at)}$ is the Maxwell parameter. For a similarity solution, $\lambda_1(t)$ can be taken in the form $\lambda_1(t) = \lambda_0(1 - at)$ for some constant λ_0, then $De = b\lambda_0$.

The physical quantities of interest are the local skin-friction coefficient Cf_x and the local Nusselt number Nu_x that are defined as follows:

$$Cf_x = -\frac{2\mu_\infty(\frac{\partial u}{\partial y})_{y=0}}{\rho U^2} = -2\mathrm{Re}_x^{-1/2} f''(0), \quad (18)$$

$$Nu_x = -\frac{x(\frac{\partial T}{\partial y})_{y=0}}{(T_w - T_\infty)} = -\mathrm{Re}_x^{1/2}\theta'(0). \quad (19)$$

3. Solution Procedure within the Chebyshev Spectral Method

We solve the resulting system of non-linear ODEs of the form (14)–(15) with boundary conditions by using the Chebyshev spectral method. Since the Gauss–Lobatto nodes lie in the computational interval $[-1, 1]$, the transformation $\eta = \frac{\eta_\infty}{2}(x+1)$ is used in the first step of this method to change Eqs.(14)–(15) to the following form:

$$e^{-\alpha\theta}\left(\left(\frac{2}{\eta_\infty}\right)^3 f''' - \alpha\left(\frac{2}{\eta_\infty}\right)^3 \theta' f''\right) + \left(\frac{2}{\eta_\infty}\right)^2 ff'' - - \left(\frac{2}{\eta_\infty}\right)^2 f'^2 - S\left(\left(\frac{2}{\eta_\infty}\right) f' + \frac{\eta_\infty}{4}(x+1)\left(\frac{2}{\eta_\infty}\right)^2 f''\right) - - De\left(\left(\frac{2}{\eta_\infty}\right)^3 f^2 f''' - 2\left(\frac{2}{\eta_\infty}\right)^3 ff'f''\right) = 0, \quad (20)$$

$$(1+\varepsilon\theta)\left(\frac{2}{\eta_\infty}\right)^2 \theta'' + \varepsilon\left(\frac{2}{\eta_\infty}\right)^2 \theta'^2 + Pr\left(\left(\frac{2}{\eta_\infty}\right) f\theta' - - 2\left(\frac{2}{\eta_\infty}\right) f'\theta - \frac{S}{2}\left(3\theta + \frac{\eta_\infty}{2}(x+1)\left(\frac{2}{\eta_\infty}\right)\theta'\right)\right) + + a^* e^{-\frac{\eta_\infty}{2}(x+1)} + b^*\theta = 0, \quad (21)$$

with the transformed boundary conditions

$$f(-1) = 0, \quad f'(-1) = \frac{\eta_\infty}{2}, \quad f'(1) = 0, \quad \theta(-1) = 1, \quad \theta(1) = 0, \quad (22)$$

where $f(x)$ and $\theta(x)$ are the unknown functions from $C^m[-1, 1]$. The differentiation in Eqs.(20) and (21) will be with respect to the new variable x. Our technique is accomplished by starting with the Chebyshev approximation for the highest order derivatives, $f^{(3)}$ and $\theta^{(2)}$, and generating approximations to the lower order derivatives $f^{(i)}$, $i = 0, 1, 2$ and $\theta^{(i)}$, $i = 0, 1$ as follows. We set $f^{(3)}(x) = \phi(x)$ and $\theta^{(2)}(x) = \xi(x)$. Then, by integration, we obtain $f^{(2)}(x)$, $f^{(1)}(x)$, $f(x)$, $\theta^{(1)}(x)$ and $\theta(x)$ as follows:

$$f^{(2)}(x) = \int_{-1}^{x} \phi(x)dx + c_0,$$

$$f^{(1)}(x) = \int_{-1}^{x}\int_{-1}^{x} \phi(x)dxdx + (x+1)c_0 + c_1,$$

$$f(x) = \int_{-1}^{x}\int_{-1}^{x}\int_{-1}^{x} \phi(x)dxdxdx + + \frac{(x+1)^2}{2!}c_0 + \frac{(x+1)}{1!}c_1 + c_2. \quad (23)$$

$$\theta^{(1)}(x) = \int_{-1}^{x} \xi(x) + d_0,$$

$$\theta(x) = \int_{-1}^{x}\int_{-1}^{x} \xi(x)dxdx + (x+1)d_0 + d_1. \quad (24)$$

From the boundary conditions (22), we can obtain the constants of integration $c_k,\ d_k,\ \ k=0,1,2$, where

$$c_0=-\frac{\eta_\infty}{4}-\frac{1}{2}\int_{-1}^{1}\int_{-1}^{x}\phi(x)dxdx,\quad c_1=\frac{\eta_\infty}{2},\quad c_2=0,$$

$$d_0=-\frac{1}{2}-\frac{1}{2}\int_{-1}^{1}\int_{-1}^{x}\xi(x)dxdx,\quad d_1=1.$$

Therefore, we can give approximations to Eqs.(20) and (21) as follows:

$$f_i=\sum_{j=0}^{n}\ell_{ij}^{f}\phi_j+c_i^f,\quad f_i^{(1)}=\sum_{j=0}^{n}\ell_{ij}^{f1}\phi_j+c_i^{f1},$$

$$f_i^{(2)}=\sum_{j=0}^{n}\ell_{ij}^{f2}\phi_j+c_i^{f2},\quad \theta_i=\sum_{j=0}^{n}\ell_{ij}^{\theta}\xi_j+d_i^{\theta},$$

$$\theta_i^{(1)}=\sum_{j=0}^{n}\ell_{ij}^{\theta1}\xi_j+d_i^{\theta1}, \tag{25}$$

for all $i=0,1,2,...,n$, where

$$\ell_{ij}^{f}=b_{ij}^3-\frac{1}{4}(x_i+1)^2b_{nj}^2,\quad \ell_{ij}^{f1}=b_{ij}^2-\frac{1}{2}(x_i+1)b_{nj}^2,$$

$$\ell_{ij}^{f2}=b_{ij}-\frac{1}{2}b_{nj}^2,\quad \ell_{ij}^{\theta}=b_{ij}^2-\frac{1}{2}(x_i+1)b_{nj}^2,$$

$$\ell_{ij}^{\theta1}=b_{ij}-\frac{1}{2}b_{nj}^2,\quad c_i^f=\frac{\eta_\infty}{2}(x_i+1)-\frac{\eta_\infty}{8}(x_i+1)^2,$$

$$c_i^{f1}=\frac{\eta_\infty}{2}-\frac{\eta_\infty}{4}(x_i+1),\quad c_i^{f2}=-\frac{\eta_\infty}{4},$$

$$d_i^{\theta}=-\frac{1}{2}(x_i+1)+1,\quad d_i^{\theta1}=-\frac{1}{2}.$$

Here, $b_{ij}^2=(x_i-x_j)b_{ij}, b_{ij}^3=\frac{(x_i-x_j)^2}{2!}b_{ij}$, and b_{ij} are the elements of the matrix B given in [23]. By using Eq. (25), one can transform Eqs. (20)–(21) to the following system of non-linear equations in the highest derivative:

$$e^{-\alpha\theta_i}\left(\phi_i-\alpha\theta_i^{(1)}f_i^{(2)}\right)+\left(\frac{\eta_\infty}{2}\right)\left(f_if_i^{(2)}-\left(f_i^{(1)}\right)^2\right)-$$

$$-S\left(\frac{\eta_\infty}{2}\right)^2\left(f_i^{(1)}+\frac{1}{2}(x_i+1)f_i^{(2)}\right)-$$

$$-De\left((f_i)^2\phi_i-2f_if_i^{(1)}f_i^{(2)}\right)=0, \tag{26}$$

$$1+\varepsilon\theta_i)\xi_i+\varepsilon\left(\theta_i^{(1)}\right)^2+Pr\left(\frac{\eta_\infty}{2}\right)\times$$

$$\times\left(f_i\theta_i^{(1)}-2f_i^{(1)}\theta_i-\left(\frac{\eta_\infty}{2}\right)\left(\frac{S}{2}\right)\left(3\theta_i+(x_i+1)\theta_i^{(1)}\right)\right)+$$

$$+\left(\frac{\eta_\infty}{2}\right)^2\left(a^*e^{-\frac{\eta_\infty}{2}(x_i+1)}+b^*\theta_i\right)=0. \tag{27}$$

This scheme is a non-linear system of $2n+2$ algebraic equations for $2n+2$ unknowns ϕ_i and ξ_i $(i=0,1,...,n)$, which are solved, by using Newton's iteration method. After solving this system and substituting ϕ_i and ξ_i in Eq. (25), we can obtain the numerical solution of Eqs. (14) and (15).

4. Results and Discussion

The boundary layer problem of the flow and heat transfer over an unsteady stretching sheet with variable fluid properties and internal heat generation is numerically solved by applying the Chebyshev spectral method. The correctness of the present numerical method is checked with the results obtained by Abel *et al.* [27] for the values of skin friction coefficient $-f''(0)$ under the limiting condition ($S=\alpha=\varepsilon=0$ and $a^*=b^*=0$). Thus, it is seen from Table 1 that the numerical results are in excellent agreement with those published previously.

In Fig. 3, the velocity profiles are shown for various values of S. From this figure, it is seen that the velocity along the sheet decreases with increase of the unsteadiness parameter S. This implies an accompanying reduction of the thickness of the momentum boundary layer. Figure 4 represents the temperature

Table 1. **Comparison of $-f''(0)$ values when $S=\alpha=\varepsilon=0$ and $a^*=b^*=0$ with those of Abel *et al.* [27] (in the absence of magnetic number)**

De	Abel *et al.* [27]	Present work
0.0	0.999962	0.999978
0.2	1.051948	1.051945
0.4	1.101850	1.101848
0.6	1.150163	1.150160
0.8	1.196692	1.196690
1.2	1.285257	1.285253
1.6	1.368641	1.368641
2.0	1.447617	1.447616

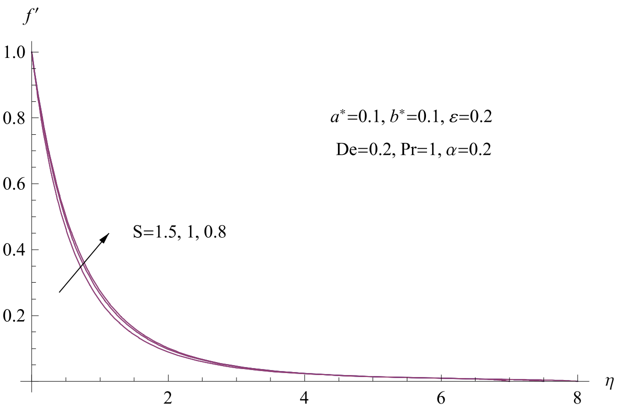

Fig. 3. Behavior of the velocity distribution for various values of S

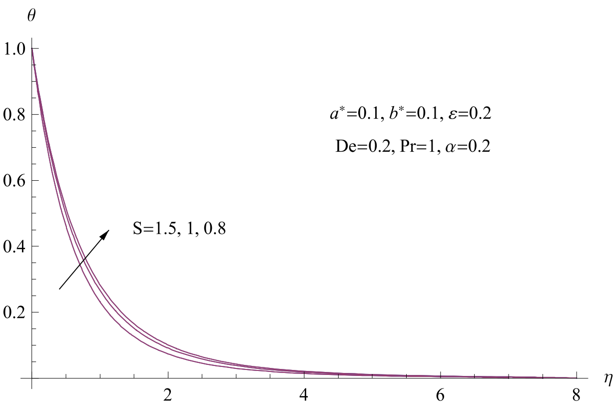

Fig. 4. Behavior of the temperature distribution for various values of S

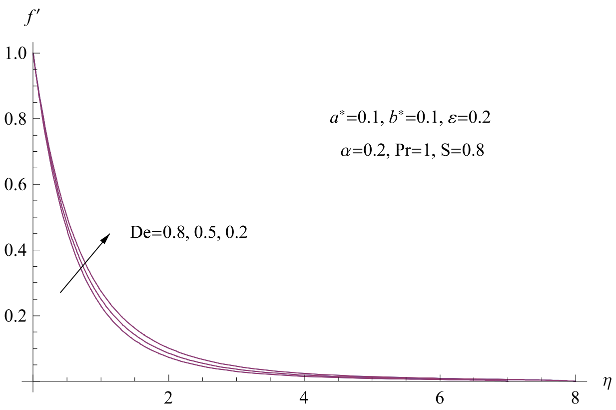

Fig. 5. Behavior of the velocity distribution for various values of De

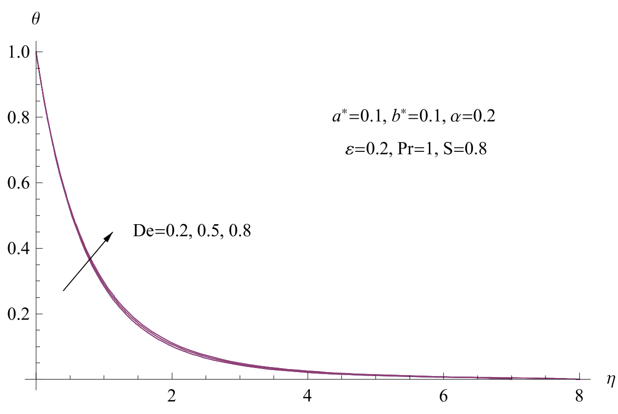

Fig. 6. Behavior of the temperature distribution for various values of De

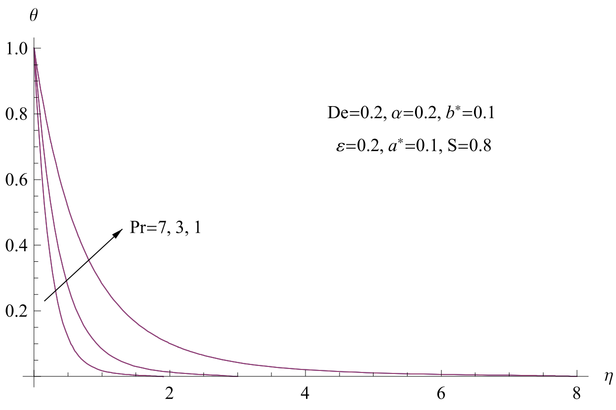

Fig. 7. Behavior of the temperature distribution for various values of Pr

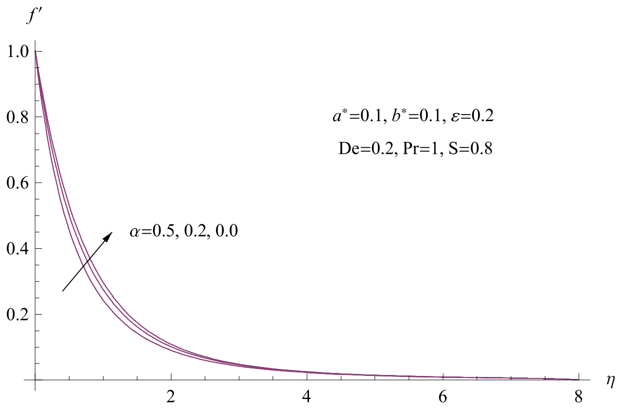

Fig. 8. Behavior of the velocity distribution for various values of α

profiles for the same parameter values as in Fig. 3. For all values of S under consideration, the temperature is found to decrease monotonically with the distance η from the sheet. This shows the important fact that the rate of cooling is much faster for the higher values of unsteadiness parameter, whereas it may take a longer time of cooling for the smaller values of unsteadiness parameter. It is noteworthy that the impact of S on the temperature profiles is more pronounced than that on the velocity profiles in Fig. 3.

The influence of the Maxwell parameter De on the velocity $f'(\eta)$ is shown in Fig. 5. It is found that an increase in the Maxwell parameter De causes a decrease of the velocity at any given point above the sheet. Figure 6 demonstrates that, at any point, the dimensionless temperature $\theta(\eta)$ increases with the Maxwell parameter De. Therefore, the cooling of a heated sheet can be improved by choosing a coolant having a small Maxwell parameter.

Figure 7 shows the effect of the Prandtl number on the temperature profiles above the sheet. It is noticed that an increase in the Prandtl number leads to a decrease of the fluid temperature $\theta(\eta)$ above the sheet. This is because the fluid with a higher value of Pr possesses a large heat capacity. Hence, the heat transfer is intensified. That is not surprising in view of the fact that the larger the Prandtl number, the thinner is the thermal boundary. Therefore, the cooling of a heated sheet can be improved by choosing a coolant having a large Prandtl number.

Table 2. **Values for $-f''(0)$ and $-\theta'(0)$ for various values of S, α, ε, De, a^*, b^*, and Pr using the Chebyshev spectral method**

S	α	ε	De	a^*, b^*	Pr	$-f''(0)$	$-\theta'(0)$
0.8	0.2	0.2	0.2	0.1	1	1.4775	1.3860
1.0	0.2	0.2	0.2	0.1	1	1.5436	1.4582
1.5	0.2	0.2	0.2	0.1	1	1.6987	1.6233
0.8	0.0	0.2	0.2	0.1	1	1.2994	1.4040
0.8	0.2	0.2	0.2	0.1	1	1.4775	1.3860
0.8	0.5	0.2	0.2	0.1	1	1.7797	1.3602
0.8	0.2	0.0	0.2	0.1	1	1.4806	1.5708
0.8	0.2	0.2	0.2	0.1	1	1.4775	1.3860
0.8	0.2	0.5	0.2	0.1	1	1.4738	1.1933
0.8	0.2	0.2	0.2	0.1	1	1.4775	1.3860
0.8	0.2	0.2	0.5	0.1	1	1.5398	1.3747
0.8	0.2	0.2	0.8	0.1	1	1.6006	1.3639
0.8	0.2	0.2	0.2	-0.3	1	1.4841	1.6453
0.8	0.2	0.2	0.2	-0.1	1	1.4809	1.5196
0.8	0.2	0.2	0.2	0.1	1	1.4775	1.3960
0.8	0.2	0.2	0.2	0.3	1	1.4737	1.2424
0.8	0.2	0.2	0.2	0.5	1	1.4694	1.0858
0.8	0.2	0.2	0.2	0.1	1	1.4775	1.3860
0.8	0.2	0.2	0.2	0.1	3	1.5032	2.6120
0.8	0.2	0.2	0.2	0.1	7	1.5503	4.1300

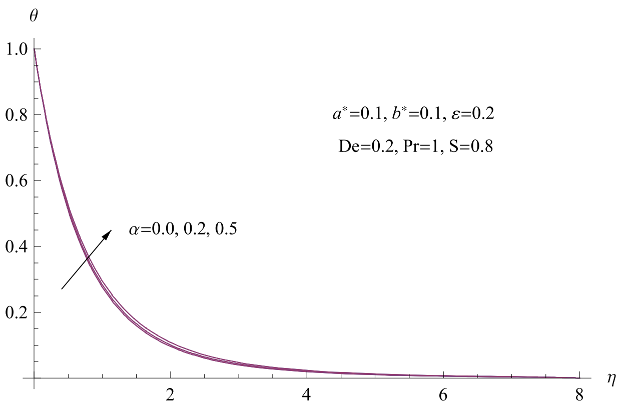

Fig. 9. Behavior of the temperature distribution for various values of α

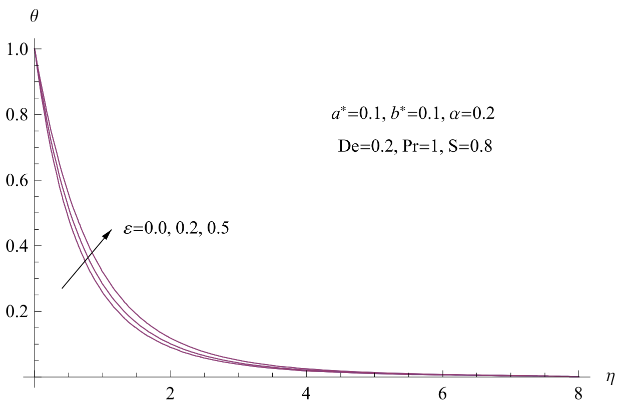

Fig. 10. Behavior of the temperature distribution for various values of ε

Effects of the viscosity parameter α on the velocity and temperature profiles are clearly exhibited in Figs. 8 and 9, respectively. The fluid velocity decreases with increase in the values of viscosity parameter, but the temperature increases in this case. This is due to the fact that increasing the viscosity parameter leads to an increase in the skin friction coefficient, which causes a decrease in the velocity of the fluid. Physically, when friction increases, the area of the stretching surface in contact with the flow increases. Therefore, the generated heat from the friction on the surface is transferred to the flow. This leads to a rise in the surface temperature, and the flow is heated.

Figure 10 displays the effects of the thermal conductivity parameter ε on the temperature profiles. The fluid temperature is found to increase with ε,

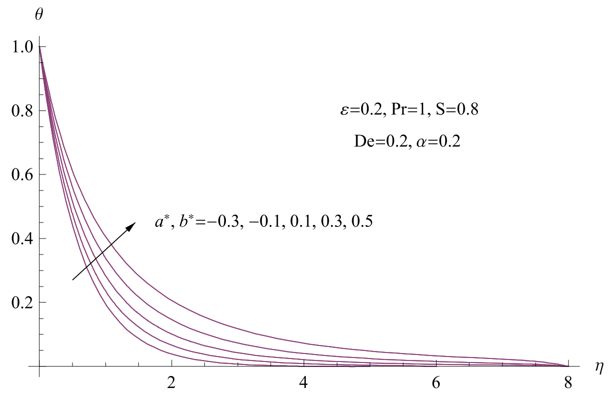

Fig. 11. Behavior of the temperature distribution for various values of a^*, b^*

which leads to a fall in the rate of heat transfer from the source to the flow.

Figure 11 illustrates the effect of the heat generation or absorption parameters a^* (space-dependent heat source/sink parameter) and b^* (temperature-dependent heat source/sink parameter) on the temperature profile. It is shown that the effect of the heat absorption parameter $a^* < 0$ and $b^* < 0$ causes a drop in the temperature, as the heat following from the sheet is absorbed. When $a^* > 0$ and $b^* > 0$, it is clear that, as the heat generation or absorption parameter $a^* or b^*$ increases, the temperature of the fluid increases. Physically, the presence of $a^* > 0$ and $b^* > 0$ has the effects of increasing the fluid temperature, by causing the thermal boundary layer thickness to increase.

The numerical values of local skin-friction coefficient and local Nusselt number in terms of $-f''(0)$ and $-\theta'(0)$ for various values of unsteadiness parameter S, viscosity parameter α, thermal conductivity parameter ε, the Maxwell parameter De, heat generation or absorption parameter a^* or b^*, and the Prandtl number Pr are tabulated in Table 2. It can be seen that the local skin-friction coefficient increases by increasing both the viscosity parameter and the Maxwell parameter, whereas the local Nusselt number decreases with the increasing values of the same parameters. However, it is found that both the skin friction coefficient and the local Nusselt number increase with the unsteadiness parameter. Moreover, we note that an increase in the values of heat generation parameter leads to a decrease in the local Nusselt number. This is because the heat generation mechanism will increase the fluid temperature near the surface, and, thus, the temperature gradient at the surface decreases, thereby decreasing the heat transfer at the sheet. It is also observed that the local Nusselt number increases with the heat absorption parameter. This is due to the fact that increasing the heat absorption creates a cold fluid layer near the heated surface. Therefore, the heat transfer rate from the surface increases. However, an increase in the Prandtl number causes an increase in the local Nusselt number. This is because the fluid with a higher values of Prandtl number possesses a large heat capacity, and, hence, the heat transfer is intensified.

5. Conclusions

In this paper, we have presented a numerical solution in order to investigate the unsteady heat and fluid flow of a non-Newtonian Maxwell fluid with variable fluid properties and internal heat generation, which occurs over a stretching sheet of a model system. The resulting non-linear system of ordinary differential equations is solved numerically using the Chebyshev spectral method. We observe that the effect of the Maxwell parameter and the viscosity parameter on the Maxwell fluid over the stretching sheet suppresses the velocity field, which causes, in turn, an enhancement of the temperature field. It is found that increasing the unsteadiness parameter causes a fall in the flow velocity and the temperature. On the other hand, the unsteadiness parameter increases the skin friction coefficient and the local Nusselt number. Likewise, it is observed that an increase of the Prandtl number results in decreasing the thermal boundary layer thickness and a less uniform temperature distribution across the boundary layer. The reason is that the smaller values of Prandtl number are equivalent to increasing the thermal conductivities. Therefore, heat is able to diffuse away from the heated surface more rapidly than for the higher values of Prandtl number. It is seen that the growth in the values of thermal conductivity parameter and heat generation parameter leads to a decrease in the local Nusselt number, but the reverse is true for the heat absorption parameter. From our study, we can conclude that there exist the parameters governing the velocity and temperature distributions (more sensible) and the others governing the rate of heat transfer and the skin friction coefficient (more sensible) (see Table 2).

Thus, we can conclude that the Chebyshev spectral method allowed us to obtain the numerical results, which are in excellent agreement with the known physical data for such problem. The obtained results demonstrate the reliability and the efficiency of the proposed method. All computations in the presented figures and tables use the Mathematica programming version 6.

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Received 25.05.12

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ЧИСЕЛЬНЕ РІШЕННЯ ДЛЯ ВПЛИВУ
ЗМІННИХ ВЛАСТИВОСТЕЙ РІДИНИ НА ПОТІК
І ТЕПЛОПЕРЕДАЧУ В НЕНЬЮТОНІВСЬКІЙ
МАКСВЕЛЛІВСЬКІЙ РІДИНІ
НАД НЕСТАЦІОНАРНОЮ СТИСКУВАНОЮ
ОБОЛОНКОЮ З ВНУТРІШНЬОЮ
ГЕНЕРАЦІЄЮ ТЕПЛА

Р е з ю м е

Розглянуто потік і теплопередачу у нестаціонарному двовимірному граничному шарі неньютонівської максвеллівської рідини над розтяжною оболонкою у разі мінливих властивостей рідини з внутрішньою генерацією тепла. Визначальні диференціальні рівняння перетворено в систему взаємозв'язаних нелінійних звичайних диференціальних рівнянь, які вирішені чисельно за відповідних граничних умов для різних величин фізичних параметрів. Чисельне рішення нелінійної граничної задачі ґрунтується на спектральному методі Чебишева. Розглянуто ефекти таких параметрів, як в'язкість, теплопровідність, параметр нестаціонарності, інтенсивність генерації тепла, параметр Максвелла і число Прандтля на профілі потоку і температури, локальний коефіцієнт скін-тертя і локальне число Нуссельта. Числові результати порівняно з відомими в граничних випадках. Окремо вивчено ефекти в'язкості, теплопровідності та інтенсивності генерації тепла на поля швидкості й температури над оболонкою.