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HARMONIC OSCILLATOR CHAIN IN NONCOMMUTATIVE PHASE SPACE WITH ROTATIONAL SYMMETRY

We consider a quantum space with a rotationally invariant noncommutative algebra of coordinates and momenta. The algebra contains the constructed tensors of noncommutativity involving additional coordinates and momenta. In the rotationally invariant noncommutative phase space, the harmonic oscillator chain is studied. We obtain that the noncommutativity affects the frequencies of the system. In the case of a chain of particles with harmonic oscillator interaction, we conclude that, due to the noncommutativity of momenta, the spectrum of the center-of-mass of the system is discrete and corresponds to the spectrum of a harmonic oscillator.

Keywords: harmonic oscillator, composite system, tensors of noncommutativity.

1. Introduction

Owing to the development of String Theory and Quantum Gravity [1, 2], studies of the idea that space coordinates may be noncommutative has attracted much attention. The noncommutativity of coordinates leads to the existence of the minimal length and minimal area [3, 4] and to the space quantization. The canonical version of a noncommutative phase space is characterized by the algebra

$$[X_i, X_j] = i\hbar\theta_{ij}, \tag{1}$$

$$[P_i, P_j] = i\hbar\eta_{ij}, \tag{2}$$

$$[X_i, P_j] = i\hbar(\delta_{ij} + \gamma_{ij}), \tag{3}$$

where θ_{ij}, η_{ij}, γ_{ij} are elements of the constant matrices. The parameters γ_{ij} are considered to be defined as $\gamma_{ij} = \sum_k \theta_{ik}\eta_{jk}/4$ [5].

The noncommutative algebra (1)–(3) with θ_{ij}, η_{ij}, and γ_{ij} being constants is not rotationally invariant [6,7]. Different generalizations of the commutation relations (1)–(3) were considered to solve the problem of rotational symmetry breaking in the noncommutative space [8–11]. Many papers are devoted to studies of the position-dependent noncommutativity [12–18] and spin noncommutativity [19, 20]. The algebras of these types of noncommutativity are rotationally invariant and are not equivalent to noncommutative algebras of the canonical type.

In work [21], a rotationally invariant noncommutative algebra of the canonical type was constructed on the basis of the idea of a generalization of parameters of noncommutativity to tensors. Introducing additional coordinates ($\tilde{a}_i$, $\tilde{b}_i$) and additional momenta ($\tilde{p}^a_i$, $\tilde{p}^b_i$), we proposed to define these tensors in the form

$$\theta_{ij} = \frac{c_\theta l_P^2}{\hbar}\sum_k \varepsilon_{ijk}\tilde{a}_k, \tag{4}$$

$$\eta_{ij} = \frac{c_\eta \hbar}{l_P^2}\sum_k \varepsilon_{ijk}\tilde{p}^b_k, \tag{5}$$

where the constants c_θ and c_η are dimensionless, and l_P is the Planck's length. To preserve the rotational symmetry, the coordinates and momenta ($\tilde{a}_i$, $\tilde{b}_i$ and $\tilde{p}^a_i$, $\tilde{p}^b_i$) are supposed to be governed by rotationally invariant systems. The systems are considered to be harmonic oscillators

$$H^a_{\rm osc} = \hbar\omega_{\rm osc}\left(\frac{(\tilde{p}^a)^2}{2} + \frac{\tilde{a}^2}{2}\right), \tag{6}$$

$$H^b_{\rm osc} = \hbar\omega_{\rm osc}\left(\frac{(\tilde{p}^b)^2}{2} + \frac{\tilde{b}^2}{2}\right), \tag{7}$$

with $\sqrt{\hbar}/\sqrt{m_{\rm osc}\omega_{\rm osc}} = l_P$ and large frequency $\omega_{\rm osc}$ (the distance between energy levels is very large, and oscillators are considered to be in the ground states). The algebra for the additional coordinates and additional momenta reads

$$[\tilde{a}_i, \tilde{a}_j] = [\tilde{b}_i, \tilde{b}_j] = [\tilde{a}_i, \tilde{b}_j] = 0, \tag{8}$$

$$[\tilde{p}^a_i, \tilde{p}^a_j] = [\tilde{p}^b_i, \tilde{p}^b_j] = [\tilde{p}^a_i, \tilde{p}^b_j] = 0, \tag{9}$$

$$[\tilde{a}_i, \tilde{p}^b_j] = [\tilde{b}_i, \tilde{p}^a_j] = 0, \tag{10}$$

$$[\tilde{a}_i, X_j] = [\tilde{a}_i, P_j] = [\tilde{p}^b_i, X_j] = [\tilde{p}^b_i, P_j] = 0, \tag{11}$$

$$[\tilde{a}_i, \tilde{p}^a_j] = [\tilde{b}_i, \tilde{p}^b_j] = i\delta_{ij}. \tag{12}$$

Therefore, we have $[\theta_{ij}, X_k] = [\theta_{ij}, P_k] = [\eta_{ij}, X_k] = [\eta_{ij}, P_k] = [\gamma_{ij}, X_k] = [\gamma_{ij}, P_k] = 0$, as in the case of canonical noncommutativity (1)–(3) with θ_{ij}, η_{ij}, γ_{ij} being constants.

In the present paper, we will study the influence of the noncommutativity of coordinates and noncommutativity of momenta on the spectrum of a harmonic oscillator chain. Studies of a system of harmonic oscillators are important in various fields of physics including molecular spectroscopy and quantum chemistry [22–25], quantum optics [26–28], nuclear physics [29–31], and quantum information processing [28, 32, 33].

Harmonic oscillators were intensively studied in the frame of noncommutative algebras [34–48]. Recently, the experiments with micro- and nanooscillators were implemented for probing the minimal length [49]. In a noncommutative space of the canonical type, two coupled harmonic oscillators were studied in [50–52]. In [53], the spectrum of a system of N oscillators interacting with each other (symmetric network of coupled harmonic oscillators) has been examined in a rotationally invariant noncommutative phase space. In [54], the classical N interacting harmonic oscillators were examined in a noncommutative space-time. In [55, 56], the influence of the noncommutativity of coordinates and the noncommutativity of momenta on the properties of a system of free particles was examined.

The paper is organized as follows. In Section 2, we study the energy levels of a harmonic oscillator chain in a rotationally invariant noncommutative phase space. A particular case of a chain of particles with harmonic oscillator interaction is examined. Conclusions are presented in Section 3.

2. Spectrum of a Harmonic Oscillator Chain in the Rotationally Invariant Noncommutative Phase Space

Let us consider a chain of N interacting harmonic oscillators with masses m and frequencies ω in a space with (1)–(3) and (4), (5) in the case of the closed configuration of the system. So, let us study the Hamiltonian

$$H_s = \sum_{n=1}^{N} \frac{(\mathbf{P}^{(n)})^2}{2m} + \sum_{n=1}^{N} \frac{m\omega^2 (\mathbf{X}^{(n)})^2}{2} + k\sum_{n=1}^{N} (\mathbf{X}^{(n+1)} - \mathbf{X}^{(n)})^2 \tag{13}$$

with the periodic boundary conditions $\mathbf{X}^{(N+1)} = \mathbf{X}^{(1)}$, k is a constant.

In general case, the coordinates and momenta which correspond to different particles satisfy a noncommutative algebra with different tensors of noncommutativity. We have

$$[X_i^{(n)}, X_j^{(m)}] = i\hbar\delta_{mn}\theta_{ij}^{(n)}, \tag{14}$$

$$[X_i^{(n)}, P_j^{(m)}] = i\hbar\delta_{mn}\left(\delta_{ij} + \sum_k \frac{\theta_{ik}^{(n)}\eta_{jk}^{(m)}}{4}\right), \tag{15}$$

$$[P_i^{(n)}, P_j^{(m)}] = i\hbar\delta_{mn}\eta_{ij}^{(n)}, \tag{16}$$

$$\theta_{ij}^{(n)} = \frac{c_\theta^{(n)} l_P^2}{\hbar}\sum_k \varepsilon_{ijk}\tilde{a}_k, \tag{17}$$

$$\eta_{ij}^{(n)} = \frac{c_\eta^{(n)}\hbar}{l_P^2}\sum_k \varepsilon_{ijk}\tilde{p}^b_k, \tag{18}$$

where indices $m, n = (1, ..., N)$ label the particles [57].

Because of the presence of additional coordinates and momenta in (17), (18), we have to study the Hamiltonian, which includes the Hamiltonians of harmonic oscillators

$$H = H_s + H^a_{\rm osc} + H^b_{\rm osc}. \tag{19}$$

The noncommutative coordinates and noncommutative momenta can be represented as

$$X_i^{(n)} = x_i^{(n)} + \frac{1}{2}[\boldsymbol{\theta}^{(n)} \times \mathbf{p}^{(n)}]_i, \tag{20}$$

$$P_i^{(n)} = p_i^{(n)} - \frac{1}{2}[\mathbf{x}^{(n)} \times \boldsymbol{\eta}^{(n)}]_i, \tag{21}$$

where coordinates and momenta $x_i^{(n)}$, $p_i^{(n)}$ satisfy the ordinary commutation relations

$$[x_i^{(n)}, x_j^{(m)}] = [p_i^{(n)}, p_j^{(m)}] = 0, \tag{22}$$

$$[x_i^{(n)}, p_j^{(m)}] = i\hbar\delta_{mn}, \tag{23}$$

and the vectors $\boldsymbol{\theta}^{(n)}$, $\boldsymbol{\eta}^{(n)}$ have the components $\theta_i^{(n)} = \sum_{jk} \varepsilon_{ijk}\theta^n_{jk}/2$, $\eta_i^{(n)} = \sum_{jk}\varepsilon_{ijk}\eta_{jk}^{(n)}/2$. In our paper [57], we proposed the constants $c_\theta^{(n)}$, $c_\eta^{(n)}$ in the tensors of noncommutativity to be determined by the mass as $c_\theta^{(n)} m_n = \tilde{\gamma} = \text{const}$, $c_\eta^{(n)}/m_n = \tilde{\alpha} = \text{const}$ with $\tilde{\gamma}$, $\tilde{\alpha}$, being the same for different particles. Therefore, one has

$$\theta_{ij}^{(n)} = \frac{\tilde{\gamma} l_P^2}{m_n \hbar}\sum_k \varepsilon_{ijk}\tilde{a}_k, \tag{24}$$

$$\eta_{ij}^{(n)} = \frac{\tilde{\alpha}\hbar m_n}{l_P^2}\sum_k \varepsilon_{ijk}\tilde{p}_k^b. \tag{25}$$

The determination of the tensors of noncommutativity in the forms (24) and (25) gives a possibility to consider the noncommutative coordinates as kinematic variables [57] and to recover the weak equivalence principle [58]. Taking (24) and (25) into account in the case where the system consists of oscillators with the same masses, one has $\theta_{ij}^{(n)} = \theta_{ij}$, $\eta_{ij}^{(n)} = \eta_{ij}$. Using (20)–(21), the Hamiltonian H_s reads

$$\begin{aligned} H_s = \sum_{n=1}^{N} \Bigg(& \frac{(\mathbf{p}^{(n)})^2}{2m} + \frac{m\omega^2 (\mathbf{x}^{(n)})^2}{2} + \\ & + k(\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)})^2 - \frac{(\boldsymbol{\eta}\,[\mathbf{x}^{(n)} \times \mathbf{p}^{(n)}])}{2m} - \\ & - \frac{m\omega^2(\boldsymbol{\theta}\,[\mathbf{x}^{(n)}\times\mathbf{p}^{(n)}])}{2} - \\ & - k(\boldsymbol{\theta}\,[(\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}) \times (\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)})]) + \\ & + \frac{[\boldsymbol{\eta}\times\mathbf{x}^{(n)}]^2}{8m} + \frac{m\omega^2}{8}[\boldsymbol{\theta}\times\mathbf{p}^{(n)}]^2 + \\ & + \frac{k}{4}[\boldsymbol{\theta}\times(\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)})]^2\Bigg). \end{aligned} \tag{26}$$

In [57], we showed that, up to the second order in ΔH defined as

$$\Delta H = H_s - \langle H_s\rangle_{ab}, \tag{27}$$

the Hamiltonian

$$H_0 = \langle H_s\rangle_{ab} + H^a_{\rm osc} + H^b_{\rm osc} \tag{28}$$

can be studied, because the corrections to the spectrum of H_0 caused by terms $\Delta H = H - H_0 = H_s - \langle H_s\rangle_{ab}$ vanish up to the second order in perturbation theory. Here, the notation $\langle ...\rangle_{ab}$ is used for the averaging over the well-known eigenstates of $H^a_{\rm osc}$ and $H^b_{\rm osc}$ $\langle ...\rangle_{ab} = \langle\psi^a_{0,0,0}\psi^b_{0,0,0}|...|\psi^a_{0,0,0}\psi^b_{0,0,0}\rangle$. For the harmonic oscillator chain, we have

$$\begin{aligned} \Delta H = \sum_{n=1}^{N}\Bigg(& \frac{[\boldsymbol{\eta}\times\mathbf{x}^{(n)}]^2}{8m} + \frac{m\omega^2}{8}[\boldsymbol{\theta}\times\mathbf{p}^{(n)}]^2 - \\ & - \frac{m\omega^2(\boldsymbol{\theta}\,[\mathbf{x}^{(n)}\times\mathbf{p}^{(n)}])}{2} - \frac{(\boldsymbol{\eta}\,[\mathbf{x}^{(n)}\times\mathbf{p}^{(n)}])}{2m} - \\ & - k\boldsymbol{\theta}\,[(\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}) \times (\mathbf{p}^{(n+1)} - \mathbf{p}^{(n+1)})] + \\ & + \frac{k}{4}[\boldsymbol{\theta}\times(\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)})]^2 - \frac{\langle\eta^2\rangle(\mathbf{x}^{(n)})^2}{12m} - \\ & - \frac{\langle\theta^2\rangle m\omega^2(\mathbf{p}^{(n)})^2}{12} - \frac{k}{6}\langle\theta^2\rangle(\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)})^2\Bigg). \end{aligned} \tag{29}$$

Here, we take into account that $\langle\psi^a_{0,0,0}|\tilde{a}_i|\psi^a_{0,0,0}\rangle = \langle\psi^b_{0,0,0}|\tilde{p}_i|\psi^b_{0,0,0}\rangle = 0$ and use the notations

$$\langle\theta_i\theta_j\rangle = \frac{c_\theta^2 l_P^4}{\hbar^2}\langle\psi^a_{0,0,0}|\tilde{a}_i\tilde{a}_j|\psi^a_{0,0,0}\rangle = \frac{c_\theta^2 l_P^4}{2\hbar^2}\delta_{ij} = \frac{\langle\theta^2\rangle\delta_{ij}}{3}, \tag{30}$$

$$\langle\eta_i\eta_j\rangle = \frac{\hbar^2 c_\eta^2}{l_P^4}\langle\psi^b_{0,0,0}|\tilde{p}^b_i\tilde{p}^b_j|\psi^b_{0,0,0}\rangle = \frac{\hbar^2 c_\eta^2}{2l_P^4}\delta_{ij} = \frac{\langle\eta^2\rangle\delta_{ij}}{3}. \tag{31}$$

So, analyzing the form of ΔH (29), we see that, up to the second order in the parameters of noncommutativity, one can study the Hamiltonian H_0. This Hamiltonian can be rewritten for convenience as

$$\begin{aligned} H_0 = \sum_{n=1}^{N}\Bigg(& \frac{(\mathbf{p}^{(n)})^2}{2m_{\rm eff}} + \frac{m_{\rm eff}\omega_{\rm eff}^2(\mathbf{x}^{(n)})^2}{2} + \\ & + k(\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)})^2 + \\ & + \frac{k}{6}\langle\theta^2\rangle(\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)})^2 + H^a_{\rm osc} + H^b_{\rm osc}\Bigg) \end{aligned} \tag{32}$$

with

$$m_{\rm eff} = m\left(1 + \frac{m^2\omega^2\langle\theta^2\rangle}{6}\right)^{-1}, \tag{33}$$

$$\omega_{\text{eff}} = \left(\omega^2 + \frac{\langle\eta^2\rangle}{6m^2}\right)^{1/2}\left(1 + \frac{m^2\omega^2\langle\theta^2\rangle}{6}\right)^{1/2}. \tag{34}$$

The terms $H^a_{\text{osc}} + H^b_{\text{osc}}$ commute with H_0. The coordinates and momenta $\mathbf{x}^{(n)}$, $\mathbf{p}^{(n)}$ satisfy (22) and (23). Let us rewrite H_0 as

$$H_0 = \frac{\hbar\omega_{\text{eff}}}{2}\sum_n \left(1+\frac{4km_{\text{eff}}\langle\theta^2\rangle}{3}\sin^2\frac{\pi n}{N}\right)\tilde{\mathbf{p}}^{(n)}(\tilde{\mathbf{p}}^{(n)})^\dagger + \frac{\hbar\omega^2_{\text{eff}}}{2}\sum_n\left(1+\frac{8k}{m_{\text{eff}}\omega^2_{\text{eff}}}\sin^2\frac{\pi n}{N}\right)\tilde{\mathbf{x}}^{(n)}(\tilde{\mathbf{x}}^{(n)})^\dagger, \tag{35}$$

using

$$\mathbf{x}^{(n)} = \sqrt{\frac{\hbar}{Nm_{\text{eff}}\omega_{\text{eff}}}}\sum_{l=1}^N \exp\left(\frac{2\pi inl}{N}\right)\tilde{\mathbf{x}}^{(l)}, \tag{36}$$

$$\mathbf{p}^{(n)} = \sqrt{\frac{\hbar m_{\text{eff}}\omega_{\text{eff}}}{N}}\sum_{l=1}^N \exp\left(-\frac{2\pi inl}{N}\right)\tilde{\mathbf{p}}^{(l)} \tag{37}$$

(see, e.g., [28]). Introducing operators $a^{(n)}_j$ defined as

$$a^{(n)}_j = \frac{1}{\sqrt{2w_n}}\left(w_n\tilde{x}^{(n)}_j + i\tilde{p}^{(n)}_j\right), \tag{38}$$

$$w_n = \left(1+\frac{8k}{m_{\text{eff}}\omega^2_{\text{eff}}}\sin^2\frac{\pi n}{N}\right)^{1/2}\times \left(1+\frac{4km_{\text{eff}}\langle\theta^2\rangle}{3}\sin^2\frac{\pi n}{N}\right)^{-1/2}, \tag{39}$$

we have

$$H_0 = \hbar\omega_{\text{eff}}\sum_{n=1}^N\sum_{j=1}^3\left(1+\frac{4km_{\text{eff}}\langle\theta^2\rangle}{3}\sin^2\frac{\pi n}{N}\right)^{1/2}\times \left(1+\frac{8k}{m_{\text{eff}}\omega^2_{\text{eff}}}\sin^2\frac{\pi n}{N}\right)^{1/2}\left((a^{(n)}_j)^\dagger a^{(n)}_j + \frac{1}{2}\right). \tag{40}$$

The spectrum of H_0 reads

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \hbar\sum_{a=1}^N\left(\omega^2_{\text{eff}} + \frac{8k}{m_{\text{eff}}}\sin^2\frac{\pi a}{N}\right)^{1/2}\times \left(1+\frac{4km_{\text{eff}}\langle\theta^2\rangle}{3}\sin^2\frac{\pi a}{N}\right)^{1/2}\left(n^{(a)}_1 + n^{(a)}_2 + n^{(a)}_3 + \frac{3}{2}\right) = \sum_{a=1}^N\hbar\omega_a\left(n^{(a)}_1 + n^{(a)}_2 + n^{(a)}_3 + \frac{3}{2}\right), \tag{41}$$

where $n^{(a)}_i$ are quantum numbers ($n^{(a)}_i = 0, 1, 2, ...$). In view of (33) and (34), the frequencies read

$$\omega^2_a = \left(\omega^2 + \frac{\langle\eta^2\rangle}{6m^2}\right)\left(1 + \frac{m^2\omega^2\langle\theta^2\rangle}{6} + \frac{4k^2m\langle\theta^2\rangle}{3}\sin^2\frac{\pi a}{N}\right) + \frac{8k}{m}\sin^2\frac{\pi a}{N} + \frac{32k^2\langle\theta^2\rangle}{3}\sin^4\frac{\pi a}{N}. \tag{42}$$

For a chain of particles with harmonic oscillator interaction described by Hamiltonian (13) with $\omega = 0$, up to the second order in the parameters of noncommutativity, one has

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \sum_{a=1}^N \hbar\omega_a\left(n^{(a)}_1 + n^{(a)}_2 + n^{(a)}_3 + \frac{3}{2}\right) \tag{43}$$

with

$$\omega^2_a = \frac{8k}{m}\sin^2\frac{\pi a}{N} + \frac{\langle\eta^2\rangle}{6m^2} + \frac{32k^2\langle\theta^2\rangle}{3}\sin^4\frac{\pi a}{N}. \tag{44}$$

It is worth noting that, in the case of a space with noncommutative coordinates and commutative momenta (1)–(3) with (4) and $\eta_{ij} = 0$, the spectrum of a chain of particles with harmonic oscillator interaction reads as (43) with

$$\omega^2_a = \frac{8k}{m}\sin^2\frac{\pi a}{N} + \frac{32k^2\langle\theta^2\rangle}{3}\sin^4\frac{\pi a}{N}. \tag{45}$$

Note that ω^2_N equals zero and corresponds to the spectrum of the center-of-mass of the system. The noncommutativity of momenta leads to a discrete spectrum of the center-of-mass of a chain of interacting particles. From (43) and (44), we have that the spectrum of the center-of-mass of the system corresponds to the spectrum of a three-dimensional harmonic oscillator with the frequency determined as

$$\omega^2_N = \frac{\langle\eta^2\rangle}{6m^2}. \tag{46}$$

In the limit $\langle\theta^2\rangle \to 0$, $\langle\eta^2\rangle \to 0$, relation (42) yields the well-known result $\omega^2_a = \omega^2 + \frac{8k}{m}\sin^2\frac{\pi a}{N}$, which, for instance, was presented in [28, 62].

3. Conclusions

We have considered a rotationally invariant algebra with the noncommutativity of coordinates and the noncommutativity of momenta. The algebra is constructed, by involving additional coordinates and additional momenta (1)–(3) with (4), (5). We have studied the influence of the noncommutativity on the spectrum of a harmonic oscillator chain with periodic boundary conditions. For this purpose, the total Hamiltonian has been examined (19), and the energy levels of the harmonic oscillator chain have been obtained up to the second order in the parameters of noncommutativity. We have found that the noncommutativity does not change the form of chain's spectrum (41). The noncommutativity of coordinates and the noncommutativity of momenta affect the frequencies of the system (42).

The case of a chain of particles with harmonic oscillator interaction described by Hamiltonian (13) with $\omega = 0$ has been studied. We have obtained that the spectrum of the center-of-mass of the system is discrete because of noncommutativity of momenta. This spectrum corresponds to the the spectrum of a harmonic oscillator with frequency (46).

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ЛАНЦЮЖОК ГАРМОНІЧНИХ ОСЦИЛЯТОРІВ У НЕКОМУТАТИВНОМУ ФАЗОВОМУ ПРОСТОРІ З СФЕРИЧНОЮ СИМЕТРІЄЮ

Р е з ю м е

Ми розглядаємо квантовий простір з сферично-симетричною некомутативною алгеброю координат та імпульсів. Алгебра містить тензори некомутативності, побудованими з залученням додаткових координат та імпульсів. У сферично-симетричному просторі досліджується ланцюжок гармонічних осциляторів. Ми отримали, що некомутативність впливає на частоти системи. У випадку ланцюжка частинок з осциляторною взаємодією ми прийшли до висновку про те, що спектр центра мас системи є дискретним і відповідає спектру гармонічного осцилятора, що зумовлено некомутативністю імпульсів.