https://doi.org/10.15407/ujpe71.9.727

SARRA GARAH,[1] ABDELMALEK BOUMALI,[1] SLIMANE ZAIEM [2]

[1] Laboratory of Theoretical and Applied Physics, Echahid Cheikh Larbi Tebessi University
(Algeria; e-mails: sarra.garah@univ-tebessa.dz, boumali.abdelmalek@gmail.com)

[2] Department of Physics, Faculty of Sciences of Matter, University of Batna 1
(Batna, Algeria; e-mail: zaim69slimane@yahoo.com)

DYNAMICS OF THE KLEIN–GORDON OSCILLATOR IN ANTI-DE SITTER SPACE

This study investigates the quantum dynamics of bosonic particles within Anti-de Sitter (AdS) space, structured into two complementary analyses. Initially, exact solutions are derived for a bosonic particle subjected to a stationary electric field, elucidating the field's impact on the thermodynamic characteristics of quantum oscillators. Subsequently, the analysis extends to incorporate a linear interaction term into the Klein–Gordon equation, forming the Klein–Gordon oscillator (KGO). Exact analytical solutions for this modified system are obtained and comprehensively interpreted. The theoretical framework employs quantum operator symmetry algebra, offering profound insights into the intricate behavior of bosonic particles in curved spacetime environments. Employing quantum operator symmetry algebra as a foundational approach, this research provides comprehensive insights into both the microscopic quantum states and macroscopic thermodynamic behavior of bosonic systems within curved AdS geometries.

K e y w o r d s: Klein–Gordon equation, curved spacetime, AdS spacetime, relativistic harmonic oscillator, thermodynamic properties, Heun function.

1. Introduction

The theoretical framework of relativistic quantum mechanics in curved spacetime is an advanced concept that describes the dynamics of particles and fields within the geometry of curved spacetime [1–3]. This approach combines the principles of general relativity with the foundational ideas of quantum mechanics. In particular, it employs quantum mechanics to describe the behavior of microscopic particles while also accounting for incorporating the effects of spacetime curvature described by general relativity.

This unified approach aims to reconcile two fundamental pillars of physics – quantum mechanics and general relativity – by providing a cohesive perspective on the interplay between quantum phenomena and the gravitational effects inherent to curved spacetime. It provides a framework for investigating the interplay between quantum phenomena and gravitational effects and thereby contributes to a better understanding of quantum systems in curved spacetime [3–5].

One prominent example of curved spacetime in theoretical physics is Anti-de Sitter (AdS) spacetime, characterized by constant negative curvature and maximal symmetry. AdS spacetime has significant applications in theoretical physics, particularly in the context of the AdS/CFT correspondence, also known as gauge/gravity duality. This duality establishes a mathematical equivalence between two types

C i t a t i o n: Garah S., Boumali A., Zaiem S. Dynamics of the Klein–Gordon oscillator in Anti-de Sitter space. *Ukr. J. Phys.* **71**, No. 9, 727 (2026). https://doi.org/10.15407/ujpe71.9.727.

of theories: a gravitational theory in an AdS space of dimension d and a nongravitational quantum field theory (CFT) in dimension $d-1$ residing on the boundary of the AdS space [6–9].

At the core of this correspondence is the principle that these two distinct theories provide equivalent descriptions of a single underlying physical system. Specifically, it posits that the gravitational theory within AdS space and the quantum field theory on its boundary are mutually equivalent. This equivalence implies that all information and properties describing gravitational dynamics in AdS space can be described in terms of the corresponding quantum field theory on its boundary – and vice versa. This profound duality highlights a remarkable symmetry between gravitational physics and quantum field theory, suggesting that these seemingly unrelated frameworks are intricately connected and capable of capturing the same physical reality.

The AdS/CFT correspondence not only provides a theoretical framework but also serves as a powerful tool for investigating fundamental physical phenomena. It offers profound insights into the nature of space, time, and the interconnected fabric of our universe while bridging gaps between disparate realms of physics. Ultimately, this paradigm-shifting concept contributes to a deeper understanding of the fundamental principles governing our reality [8–10].

An important form of AdS spacetime is static anti-de Sitter spacetime (CAdS), which serves as a solution to Einstein's field equations and represents a ground state for asymptotically CAdS spacetimes–analogous to the role of Minkowski space as a ground state for asymptotically flat spacetimes [11, 12]. This connection forms part of the broader CAdS correspondence, which relates conformal field theories (CFTs) at the boundary of CAdS spacetimes to gravity theories within CAdS spacetimes.

Recently, there has been renewed interest in relativistic harmonic oscillators, with investigations spanning various approaches. Initial studies by Yukawa [13], followed by contributions from Feynman *et al.* [14], laid the foundation for subsequent work in this area. Ito *et al.* [15], for instance, developed a Dirac equation linear in both coordinates and momenta. Building on this foundation, Aldaya *et al.* [16] propose an extension based on CAdS spacetimes to extend the previous studies reported in Refs [16–18] on relativistic harmonic oscillators. This proposal employs Poincaré algebra – a natural generalization of the symmetry algebra of quantum operators for relativistic free systems–and demonstrates that a static AdS metric naturally defines a relativistic harmonic oscillator in Minkowski space when interpreted geometrically.

The quantum theory associated with this proposal can be solved exactly, yielding wavefunctions with distinct behavior compared to those in the non-relativistic case. Apart from differences in the ground-state energy, the energy spectrum coincides with that of the nonrelativistic oscillator. A notable aspect discussed in Refs [16–19] is that directly substituting classical coordinates and momenta with their quantum mechanical counterparts in classical Hamiltonians often leads to ambiguities due to operator ordering issues. To address these ambiguities, two new parameters (α and β) are introduced and carefully chosen to ensure consistency with both nonrelativistic limits and physical energy spectra. These "special quantum relativistic oscillators" represent free relativistic particles in CAdS spacetime.

In this work, we build on previous studies demonstrating that static AdS metrics give rise to relativistic harmonic oscillators in Minkowski spacetime. In Section 2, we introduce an external field V and perform calculations for two scenarios: $V=$ constant and $V=\kappa x$ (an external electric field). Subsequently, we investigate the thermodynamic properties [20, 21] of bosonic systems under these conditions and discuss the results.

In Section 3, we introduce a Klein–Gordon oscillator with a linear interaction using an approach similar to that employed for Dirac oscillators. Specifically, this involves replacing p^2 with $(p+im\omega_1 x)(p-im\omega_1 x)$ [20–22]. Finally, we conclude with remarks summarizing our findings.

2. The $(1+1)$-Dimensional Special Relativistic Oscillator in the Presence of an External Electric Field

In this section, we will examine the one-dimensional relativistic harmonic oscillator in the universal covering spacetime of the static anti-de Sitter spacetime (CAdS), under the influence of an external field.

Aldaya *et al.* [16,17] address the challenge of identifying a relativistic algebra of quantum operators that generalizes the nonrelativistic algebra generated by the energy, position, and momentum operators, i.e.,

$$\begin{aligned}&\left[\hat{x}^i,\hat{p}^j\right]=i\hbar\delta^{ij},\\&\left[\hat{E},\hat{p}^j\right]=im\hbar\omega^2\hat{x}^j,\\&\left[\hat{E},\hat{x}^i\right]=-i\frac{\hbar}{m}\hat{p}^i.\end{aligned}\tag{1}$$

This algebra must reproduce the algebra of the free relativistic quantum particle as $\omega\to 0$, as well as the algebra (1) in the $c\to\infty$ limit. In this limit, the wave functions should reduce to those of the nonrelativistic harmonic oscillator (NRHO), such that the ground state must approaches a Gaussian in x, and the excited states should correspond to a Hermite polynomial multiplied by the Gaussian. The same authors also suggested a different method described in Refs [16–19]. This method uses the SO(1,2) group algebra for the quantum operators associated with the RHO

$$\begin{aligned}&\left[\hat{x}^i,\hat{p}^j\right]=i\hbar\left(1+\frac{1}{mc^2}\hat{E}\right)\delta^{ij},\\&\left[\hat{E},\hat{p}^j\right]=im\hbar\omega^2\hat{x}^j,\\&\left[\hat{E},\hat{x}^i\right]=-i\frac{\hbar}{m}\hat{p}^i.\end{aligned}\tag{2}$$

In the limit as $c\to\infty$, the relativistic harmonic oscillator reduces to the nonrelativistic harmonic oscillator as given by (1). In the nonrelativistic limit $c\to\infty$ or for the static form of the metric, the line element in anti-de Sitter (AdS) spacetime is given as follows:

$$ds^2=\left(1+\frac{\omega^2}{c^2}x^2\right)c^2dt^2-\frac{1}{1+\frac{\omega^2}{c^2}x^2}dx^2.\tag{3}$$

This form of the metric corresponds to a static AdS spacetime and simplifies further in the nonrelativistic limit ($c\to\infty$), where relativistic corrections vanish. In this limit, time and space decouple, yielding a Newtonian-like framework in which spatial and temporal components are treated independently.

The Lagrangian in the presence of an external potential is given as follows [23–25]:

$$\mathcal{L}=-mc^2\sqrt{g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}-V.\tag{4}$$

In our case, (4) takes the form

$$\mathcal{L}=-mc^2\sqrt{1-\frac{1}{1+\frac{\omega^2}{c^2}x^2}\frac{v^2}{c^2}+\frac{\omega^2}{c^2}x^2}-V\tag{5}$$

from (5), the Hamiltonian can be readily calculated (see Appendix A):

$$\begin{aligned}&H^2=m^2c^4+p^2c^2+m^2\omega^2c^2x^2+2\omega^2x^2p^2+\\&+\frac{\omega^4}{c^2}x^4p^2-V^2+2VH.\end{aligned}\tag{6}$$

This Hamiltonian describes the dynamics of a special-relativistic harmonic oscillator system in the center-of-momentum frame in the presence of an interaction V [16–19].

The inclusion of AdS symmetry accounts for the nonlinear terms in Eq. (6). It should be noted that Eq. (6) fundamentally differs from the naive definition:

$$\left(H-\frac{1}{2}m\omega^2x^2\right)^2=m^2c^4+p^2c^2.\tag{7}$$

Unlike Eq. (7), Eq. (6) leads to square-integrable quantum wavefunctions.

By applying the standard operator substitutions,

$$H\to i\hbar\frac{\partial}{\partial t},\quad p\to -i\hbar\frac{\partial}{\partial x},\quad x\to x,\tag{8}$$

To resolve the normal ordering ambiguities present in Eq. (6), we express each term explicitly as a linear combination of all possible Hermitian forms. For this purpose, the parameters α and β are introduced into Eq. (6) as follows:

$$x^4p^2\to -\hbar^2\left(x^4\frac{d^2}{dx^2}+4x^3\frac{d}{dx}+\alpha x^2\right),\tag{9}$$

$$x^2p^2\to -\hbar^2\left(x^2\frac{d^2}{dx^2}+2x\frac{d}{dx}+\beta\right).\tag{10}$$

It should be noted that an ambiguity arises from the ordering of operators when passing to the quantum mechanical formulation of Eq. (6). Due to the different possible orderings of terms such as x^4p^2 and x^2p^2 in various configurations, it is possible to construct multiple distinct Hermitian quantum mechanical counterparts of Eq. (6). Applying symmetrization or normal ordering techniques ensures that the solutions are physically consistent, preventing unexpected modifications to the wave function and preserving measurable physical quantities, including energy eigenvalues, eigenstates, and expectation values. In Minkowski space, the Hamiltonian is expressed as $H^2 = m^2c^4 + p^2c^2$. The additional terms arise from Anti-de Sitter (AdS) spacetime – a curved spacetime that, in this context, gives rise to a harmonic oscillator due to the chosen metric – and the external potential V.

To derive the quantum counterpart of Eq. (6), we substitute Eqs (8), (9), and (10), which yields

$$\left[-\frac{1}{\hbar^2c^2}\left(i\frac{\partial}{\partial t}-V\right)^2-\left(1+\frac{\omega^2}{c^2}x^2\right)^2\frac{\partial^2}{\partial x^2}-\right.$$
$$-4\frac{\omega^2}{c^2}x\left(1+\frac{\omega^2}{c^2}x^2\right)\frac{\partial}{\partial x}+\left[\left(\frac{mc}{\hbar}\right)^2-2\beta\frac{\omega^2}{c^2}\right]\times$$
$$\left.\times\left(1+\frac{\omega^2}{c^2}x^2\right)-(\alpha-2\beta)\frac{\omega^4}{c^4}x^2\right]\psi(x,t)=0. \quad (11)$$

The resulting equation is a relativistic wave equation in Anti-de Sitter (AdS) spacetime, incorporating additional terms arising from the external potential and the introduced parameters. Furthermore, Eq. (11) must consistently reduce to the standard Klein–Gordon equation in AdS spacetime

$$\left(\Box+\frac{mc^2}{\hbar^2}+\xi R\right)=0. \quad (12)$$

Here, ξ is a real, dimensionless coupling constant that governs the nonminimal interaction between the scalar field and spacetime curvature. It plays a pivotal role in mediating the interaction between the scalar field and spacetime curvature within general relativity. When $\xi = 0$, the coupling is minimal, meaning that the field interacts solely through the geometry of the metric. The specific case $\xi = 1/6$ corresponds to conformal coupling in four-dimensional spacetime, ensuring the equation remains invariant under conformal transformations. In our spacetime, where $R = \frac{\omega^2}{c^2}$, this coupling introduces an effective mass term given by $m^2_{\text{eff}} = m^2 - \xi\frac{\omega^2}{c^2}$, while $\Box$ denotes the d'Alembertian operator.

We observe an equivalence under the transformation:

$$\psi=\frac{1}{\sqrt{1+\frac{\omega^2}{c^2}x^2}}\varphi. \quad (13)$$

This transformation (13) requires the parameters α and β to take specific values that depend on the numerical factor ξ:

$$\beta=\xi+\frac{1}{2},\quad \alpha=2\xi+2. \quad (14)$$

The d'Alembertian operator for the AdS metric is given by:

$$\Box=-\frac{1}{\hbar^2c^2}\frac{1}{1+\frac{\omega^2}{c^2}x^2}\left(i\hbar\frac{\partial}{\partial t}-V\right)^2-$$
$$-\left(1+\frac{\omega^2}{c^2}x^2\right)\frac{\partial^2}{\partial x^2}-2\frac{\omega^2}{c^2}x\frac{\partial}{\partial x}. \quad (15)$$

The first step involves seeking positive frequency states to solve the Klein–Gordon equation. The next step is to determine the spatial dependence of the wave functions, for which the following ansatz is used:

$$\varphi(x,t)=e^{-i\lambda\omega t}\left(1+\frac{\omega^2}{c^2}x^2\right)^{-\frac{\lambda}{2}}\varphi^{\lambda}(x), \quad (16)$$

where λ is a constant. Substituting this ansatz into the equation yields the following differential equation for $\varphi^{\lambda}(x)$:

$$\left\{\left(1+\frac{\omega^2}{c^2}x^2\right)\frac{d^2}{dx^2}-2\frac{\omega^2}{c^2}(\lambda-1)x\frac{d}{dx}+\frac{\omega^2}{c^2}\times\right.$$
$$\left.\times\left[\lambda(\lambda-1)-r^2+2\xi+\frac{V^2-2V\hbar\lambda\omega}{\hbar^2c^2(1+\frac{\omega^2}{c^2}x^2)}\right]\right\}\varphi^{\lambda}(x)=0, \quad (17)$$

where $r = \frac{mc^2}{\hbar\omega} = 1, \frac{3}{2}, 2, ...$ is a dimensionless parameter [16]. Thus, two possible scenarios arise:

(i) When the external field V is constant ($V = \text{const}$).

(ii) When the external field depends linearly on position ($V = \kappa x$). Additionally, in the following calculations, it is assumed that $\xi = 0$ (minimal coupling).

2.1. The first scenario: V = const

In the first scenario, we assume $V = \text{const}$, which yields the following differential equation:

$$\left\{\left(1+\frac{\omega^2}{c^2}x^2\right)\frac{d^2}{dx^2}-2\frac{\omega^2}{c^2}(\lambda-1)x\frac{d}{dx}+\frac{\omega^2}{c^2}\times\right.$$
$$\left.\times\left[\lambda(\lambda-1)-r^2+\frac{V^2-2\hbar V\lambda\omega}{\hbar^2c^2\left(1+\frac{\omega^2}{c^2}x^2\right)}\right]\right\}\varphi(x)=0. \quad (18)$$

An analytical solution to differential equation (18) can be expressed in terms of general Heun functions [26–29] as follows:

$$\varphi(x) = C_1 \left(\omega^2 x^2 + c^2\right)^{\frac{2\hbar\lambda\omega+V}{2\omega\hbar}} \times$$
$$\times \operatorname{HeunG}\left(-\frac{c^2}{\omega^2}, q, \alpha, \beta, \gamma, 0; x^2\right) +$$
$$+ C_2 \left(\omega^2 x^2 + c^2\right)^{\frac{2\hbar\lambda\omega+V}{2\omega\hbar}} x \times$$
$$\times \operatorname{HeunG}\left(-\frac{c^2}{\omega^2}, q_1, \alpha_1, \beta_1, \gamma_1, 0; x^2\right), \tag{19}$$

where C_1 and C_2 are constants. The parameters α, α_1, β, β_1, γ, γ_1, q, and q_1 are defined as follows:

$$q = \frac{-\hbar^2\left(r^2-\lambda^2-\lambda\right)\omega^2 - 2\hbar V\left(\lambda+\frac{1}{2}\right)\omega + V^2}{4\hbar^2\omega^2},$$
$$q_1 = \frac{-\hbar^2\left(r^2-\lambda^2-3\lambda-2\right)\omega^2 - 2\hbar\left(\lambda+\frac{3}{2}\right)V\omega + V^2}{4\hbar^2\omega^2}, \tag{20}$$

$$\alpha = \frac{\sqrt{\left(4r^2+1\right)\hbar^2\omega^2} + (2\lambda+1)\hbar\omega - 2V}{4\omega\hbar},$$
$$\alpha_1 = \frac{\sqrt{\left(4r^2+1\right)\hbar^2\omega^2} + (2\lambda+3)\hbar\omega - 2V}{4\omega\hbar}, \tag{21}$$

$$\beta = \left\{-\sqrt{(4r^2+1)\hbar^2\omega^2}\,\hbar\omega + 4\hbar^2\left(r^2-\lambda^2-\frac{1}{2}\lambda+\frac{1}{4}\right)\times\right.$$
$$\left.\times\,\omega^2 + 8\hbar V\left(\lambda+\frac{1}{4}\right)\omega - 4V^2\right\}\Big/$$
$$\Big/\left\{8\hbar\omega\left(-\hbar\lambda\omega + V - \frac{\sqrt{(4r^2+1)\hbar^2\omega^2}}{2}\right)\right\}, \tag{22}$$

$$\beta_1 = \left\{-3\sqrt{(4r^2+1)\hbar^2\omega^2}\,\hbar\omega + 4\hbar^2\left(r^2-\lambda^2-\frac{3}{2}\lambda+\frac{1}{4}\right)\times\right.$$
$$\left.\omega^2 + 8\hbar V\left(\lambda+\frac{3}{4}\right)\omega + 4V^2\right\}\Big/$$
$$\Big/\left\{8\hbar\omega\left(-\hbar\lambda\omega + V - \frac{\sqrt{(4r^2+1)\hbar^2\omega^2}}{2}\right)\right\}, \tag{23}$$

and

$$\gamma = \frac{1}{2}, \quad \gamma_1 = \frac{3}{2}. \tag{24}$$

The boundary conditions $\varphi(+\infty) = \varphi(-\infty) = 0$ require retaining only the first solution when $\frac{2\hbar\lambda\omega+V}{2\omega\hbar} < 0$. Thus, the solution takes the form:

$$\varphi(x) = C_1 \left(\omega^2 x^2 + c^2\right)^{\frac{2\hbar\lambda\omega+V}{2\omega\hbar}} \times$$
$$\times \operatorname{Heun} G\left(-\frac{c^2}{\omega^2}, \Delta_1, \Delta_2, \Delta_3, \frac{1}{2}, 0, x^2\right), \tag{25}$$

where

$$\Delta_1 = \frac{-\hbar^2(r^2-\lambda^2-\lambda)\omega^2 + 2\hbar V(\lambda+\frac{1}{2})\omega + V^2}{4\hbar^2\omega^2}, \tag{26}$$

$$\Delta_2 = \frac{\sqrt{(4r^2+1)\hbar^2\omega^2} + \hbar(2\lambda+1)\omega + 2V}{4\omega\hbar}, \tag{27}$$

$$\Delta_3 = \left(\sqrt{(4r^2+1)\hbar^2\omega^2}\,\hbar\omega - 4\hbar^2\left(r^2-\lambda^2-\frac{1}{2}\lambda+\frac{1}{4}\right)\times\right.$$
$$\times\,\omega^2 + 8\hbar V\left(\lambda+\frac{1}{4}\right)\omega + 4V^2\Big)\Big/$$
$$\Big/\left(8\hbar\omega\left(\hbar\lambda\omega + V + \frac{\sqrt{(4r^2+1)\hbar^2\omega^2}}{2}\right)\right). \tag{28}$$

The condition $\frac{2\hbar\lambda\omega+V}{2\omega\hbar} < 0$ implies that λ must be negative. By applying the so-called polynomial α-condition associated with the general Heun function, the energy of the system can be determined.

The energy of the system is given by

$$E_n = \left(2n + \frac{1}{2} + \frac{1}{2}\sqrt{\left(4r^2+1\right)\hbar^2\omega^2}\right)\hbar\omega - V. \tag{29}$$

Here again, $r = \frac{mc^2}{\hbar\omega} = 1, \frac{3}{2}, 2, ...$ is a dimensionless parameter associated with an additional quantization condition [16].

In this case, the constant potential V appears as an additional term in the energy, and the corresponding wave function is given by

$$\frac{\varphi(x,t)}{C} = e^{-i\lambda\omega t}\left(1 + \frac{\omega^2}{c^2}x^2\right)^{-\frac{\lambda}{2}} \times$$
$$\times \left(\omega^2 x^2 + c^2\right)^{\frac{2\hbar\lambda\omega+V}{2\omega\hbar}} \operatorname{HeunG}\left(-\frac{c^2}{\omega^2},\right.$$
$$\frac{-\hbar^2\left(r^2-\lambda^2-\lambda\right)\omega^2 - 2\hbar V\left(\lambda+\frac{1}{2}\right)\omega + V^2}{4\hbar^2\omega^2},$$
$$\frac{\sqrt{\left(4r^2+1\right)\hbar^2\omega^2} + \hbar\left(2\lambda+1\right)\omega - 2V}{4\omega\hbar},$$
$$\left.\frac{\begin{bmatrix} -\sqrt{\left(4r^2+1\right)\hbar^2\omega^2}\,\hbar\omega + 4h^2\left(r^2-\lambda^2-\right. \\ \left.-\frac{1}{2}\lambda+\frac{1}{4}\right)\omega^2 + 8\hbar V\left(\lambda+\frac{1}{4}\right)\omega - 4V^2 \end{bmatrix}}{8\hbar\omega\left(-\hbar\lambda\omega + V - \frac{\sqrt{(4r^2+1)\hbar^2\omega^2}}{2}\right)}, \frac{1}{2}, 0, x^2\right), \tag{30}$$

where C is normalization constant.

2.2. The second case: Influence of an external electric field

In the second case, in the presence of an electric field, $V = \kappa x$ with $\kappa = eE$, where e and E represent the electric charge and the electric field, respectively. Differential equation (17) is modified as follows:

$$\left\{\left(1+\frac{\omega^2}{c^2}x^2\right)\frac{d^2}{dx^2}-2\frac{\omega^2}{c^2}(\lambda-1)\,x\frac{d}{dx}+\right.$$
$$\left.+\left[\frac{\omega^2}{c^2}\left(\lambda(\lambda-1)-r^2\right)+\frac{1}{c^2}\frac{\kappa^2x^2-2\lambda\omega\kappa x}{\left(1+\frac{\omega^2}{c^2}x^2\right)\hbar^2}\right]\right\}\times$$
$$\times\,\phi(x)=0. \tag{31}$$

The solution to this differential equation can be expressed in terms of a hypergeometric function [30]

$$\varphi(x)=C_1\left(-\mathrm{I}\,\omega x+c\right)^{\frac{\lambda}{2}-\frac{1}{2}}\left(\mathrm{I}\,\omega x+c\right)^{\frac{\frac{\mathrm{I}}{2}\kappa c}{\hbar\omega^2}}\times$$
$$\times\left(\omega x+\mathrm{I}\,c\right)^{\frac{-h(\lambda-1)\omega^2-\mathrm{I}\,\kappa c}{2\hbar\omega^2}}\times$$
$$\times\,{}_2F_1\left(\frac{\mathrm{I}\sqrt{-4\hbar^2\left(r^2+\frac{1}{4}\right)\omega^4+4\kappa^2c^2}+(-2\lambda+1)\,\hbar\omega^2}{2\hbar\omega^2},\right.$$
$$\frac{-\mathrm{I}\sqrt{-4\hbar^2\left(r^2+\frac{1}{4}\right)\omega^4+4\kappa^2c^2}+(-2\lambda+1)\,\hbar\omega^2}{2\hbar\omega^2};$$
$$\left.\frac{-h(\lambda-1)\,\omega^2-\mathrm{I}\,\kappa c}{\hbar\omega^2};\ \frac{-\mathrm{I}\,\omega x+c}{2c}\right)+$$
$$+\,C_2\left(-\mathrm{I}\,\omega x+c\right)^{\frac{\lambda}{2}-\frac{1}{2}}\times$$
$$\times\left(\mathrm{I}\,\omega x+c\right)^{\frac{\frac{\mathrm{I}}{2}\kappa c}{\hbar\omega^2}}\left(\omega x+\mathrm{I}c\right)^{\frac{\hbar\omega^2(\lambda+1)+\mathrm{I}\kappa c}{2\hbar\omega^2}}\times$$
$$\times\,{}_2F_1\left(\frac{2\mathrm{I}\kappa c-\mathrm{I}\sqrt{-4h^2\left(r^2+\frac{1}{4}\right)\omega^4+4\kappa^2c^2}+\hbar\omega^2}{2\hbar\omega^2},\right.$$
$$\frac{2I\kappa c+\mathrm{I}\sqrt{-4\hbar^2\left(r^2+\frac{1}{4}\right)\omega^4+4\kappa^2c^2}+\hbar\omega^2}{2\hbar\omega^2};$$
$$\left.\frac{\hbar\omega^2(\lambda+1)+\mathrm{I}\kappa c}{\hbar\omega^2};\ \frac{-\mathrm{I}\,\omega x+c}{2c}\right). \tag{32}$$

As in the first scenario, the physically acceptable wave function (which is naturally non-divergent) is obtained by applying the boundary conditions:

$$\varphi(+\infty)=\varphi(-\infty)=0 \tag{33}$$

resulting in:

$$\varphi(x)=C_1\left(-\mathrm{I}\,\omega x+c\right)^{\frac{\lambda}{2}-\frac{1}{2}}\left(\mathrm{I}\,\omega x+c\right)^{\frac{\frac{\mathrm{I}}{2}\kappa c}{\hbar\omega^2}}\times$$
$$\times\left(\omega x+\mathrm{I}\,c\right)^{\frac{-\hbar(\lambda-1)\omega^2-\mathrm{I}\,\kappa c}{2\hbar\omega^2}}\times$$
$$\times\,{}_2F_1\left(\frac{\mathrm{I}\sqrt{-4h^2\left(r^2+\frac{1}{4}\right)\omega^4+4\kappa^2c^2}+(-2\lambda+1)\,\hbar\omega^2}{2\hbar\omega^2},\right.$$
$$\frac{-\mathrm{I}\sqrt{-4\hbar^2\left(r^2+\frac{1}{4}\right)\omega^4+4\kappa^2c^2}+(-2\lambda+1)\,\hbar\omega^2}{2\hbar\omega^2};$$
$$\left.\frac{-\hbar(\lambda-1)\,\omega^2-\mathrm{I}\kappa c}{\hbar\omega^2};\ \frac{-\mathrm{I}\,\omega x+c}{2c}\right). \tag{34}$$

The hypergeometric function can be expressed as follows [30–32]:

$$G(x)={}_2F_1\left(-n,n+\alpha+\beta+1,\alpha+1;x\right), \tag{35}$$

where $n = 0, 1, 2, \ldots$. From (34), the quantization condition takes the form

$$\frac{i\sqrt{-4\hbar^2\left(r^2+\frac{1}{4}\right)\omega^4+4\kappa^2c^2}+(-2\lambda+1)\,\hbar\omega^2}{2\hbar\omega^2}=-n. \tag{36}$$

The function (35) reduces to a polynomial when n is a nonnegative integer. In that case, the Pochhammer symbol $(-n)_k$ vanishes for $k > n$, so the hypergeometric series terminates and defines a polynomial of degree n in x. Consequently, there is no restriction on the radius of convergence, and $G(x)$ is well-defined for all x.

The positive parameter λ is then given by

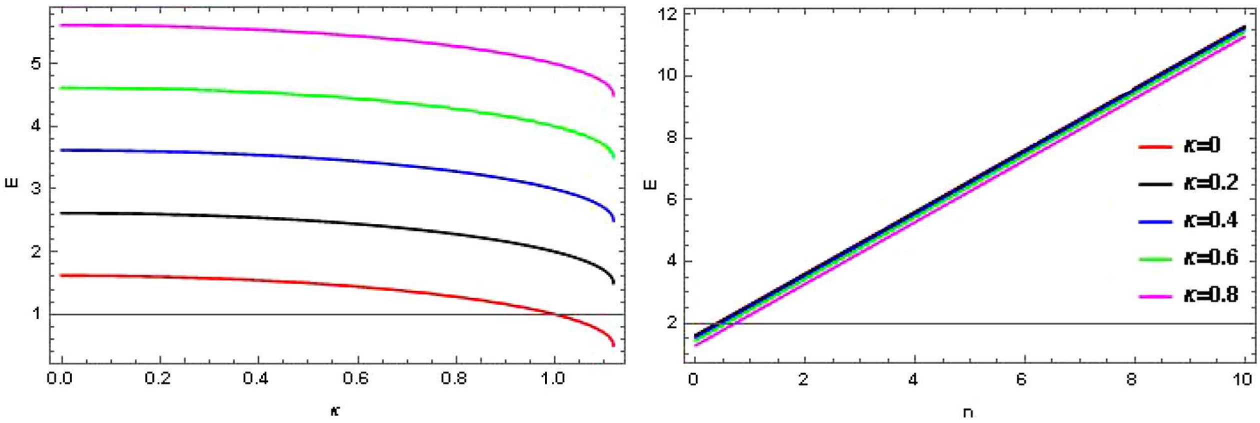
$$\lambda=n+\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4r^2-\frac{4\kappa^2c^2}{\hbar\omega^4}}. \tag{37}$$

The energy is then given by

$$E_n=\left(n+\frac{1}{2}+\frac{1}{2}\sqrt{1+4r^2-\frac{4\kappa^2c^2}{\hbar\omega^4}}\right)\hbar\omega. \tag{38}$$

In this case, the uniform, stationary electric field affects the zero-point energy. However, constraints must be imposed on the electric field to ensure physically acceptable solutions:

$$1+4r^2>\frac{4\kappa^2c^2}{\hbar\omega^4}. \tag{39}$$



a *b*

Fig. 1. Energy E as a function of n and κ: E versus the electric-field parameter κ for $n = 1, 2, 3, 4, 5$ (a); E versus the quantum number n for $\kappa = 0, 0.2, 0.4, 0.6, 0.8$ (b). Panel (a) contains five curves, including $n = 5$

For instance, if we set $r = 1$, the condition simplifies to $0 < \kappa < 1.118$. Under this condition, the results are presented in Fig. 1.

In Fig. 1, which depicts the energy E as a function of the electric field κ and the energy E as a function of the quantum number n for $\kappa = 0$, 0.2, 0.4, 0.6, 0.8, we observe that, as κ approaches 1, the energy decreases. Additionally, for an electric field strength close to 10^{18} V/m, significant physical effects occur. Specifically, extremely strong electric fields can generate electron-positron pairs through the Schwinger effect – a phenomenon predicted by quantum field theory. This effect involves the creation of matter from a strong electric field. When the field strength exceeds the Schwinger limit (approximately 10^{18} V/m), pair creation depletes the electric field, causing it to "decay" into charge pairs and become unstable [21, 33–35].

The total wave function is given by

$$\frac{\varphi(x,t)}{\mathcal{C}} = e^{-i\lambda\omega t}\left(1+\frac{\omega^2}{c^2}x^2\right)^{-\frac{\lambda}{2}}(-\mathrm{I}\,\omega x + c)^{\frac{\lambda}{2}-\frac{1}{2}}\times$$
$$\times\,(\mathrm{I}\,\omega x + c)^{\frac{\frac{\mathrm{I}}{2}\kappa c}{\hbar\omega^2}}\,(\omega x + \mathrm{I}\,c)^{\frac{-h(\lambda-1)\omega^2-\mathrm{I}\,\kappa c}{2\hbar\omega^2}}\times$$
$$\times\,{}_2F_1\left(\frac{\mathrm{I}\sqrt{-4\hbar^2\left(r^2+\frac{1}{4}\right)\omega^4+4\kappa^2c^2}+(-2\lambda+1)\,\hbar\omega^2}{2\hbar\omega^2},\right.$$
$$\frac{-\mathrm{I}\sqrt{-4\hbar^2\left(r^2+\frac{1}{4}\right)\omega^4+4\kappa^2c^2}+(-2\lambda+1)\,\hbar\omega^2}{2\hbar\omega^2};$$
$$\left.\frac{-h(\lambda-1)\,\omega^2-\mathrm{I}\,\kappa c}{\hbar\omega^2};\frac{-\mathrm{I}\,\omega x + c}{2c}\right), \tag{40}$$

where C is the normalization constant.

2.2.1. *Thermal properties*

To analyze the thermodynamic properties of the system, we begin by calculating the partition function, denoted by Z, which is uniquely defined for different statistical systems using the Boltzmann factor $e^{-\beta a}$. The thermodynamic properties of interest include the free energy F, total energy U, entropy S, and specific heat C_V. The partition function is given by [20, 21]:

$$Z = \sum_{n=0}^{\infty} e^{-\beta(E_n - E_0)}, \tag{41}$$

where $\beta = \frac{1}{k_B T}$, k_B is the Boltzmann constant, E_0 is the ground-state energy, and T is the absolute temperature. By setting $r = 1$, the energy levels are given by

$$E_n = \left(n + \frac{1}{2} + \frac{1}{2}\sqrt{5-4\kappa^2}\right)\hbar\omega. \tag{42}$$

Substituting this into the partition function yields

$$Z = \sum_{n=0}^{\infty} e^{-\beta(E_n - E_0)} = e^{-\beta a}\frac{1}{1-e^{-\beta}}, \tag{43}$$

with $a = \frac{1}{2}\sqrt{5-4\kappa^2}$.

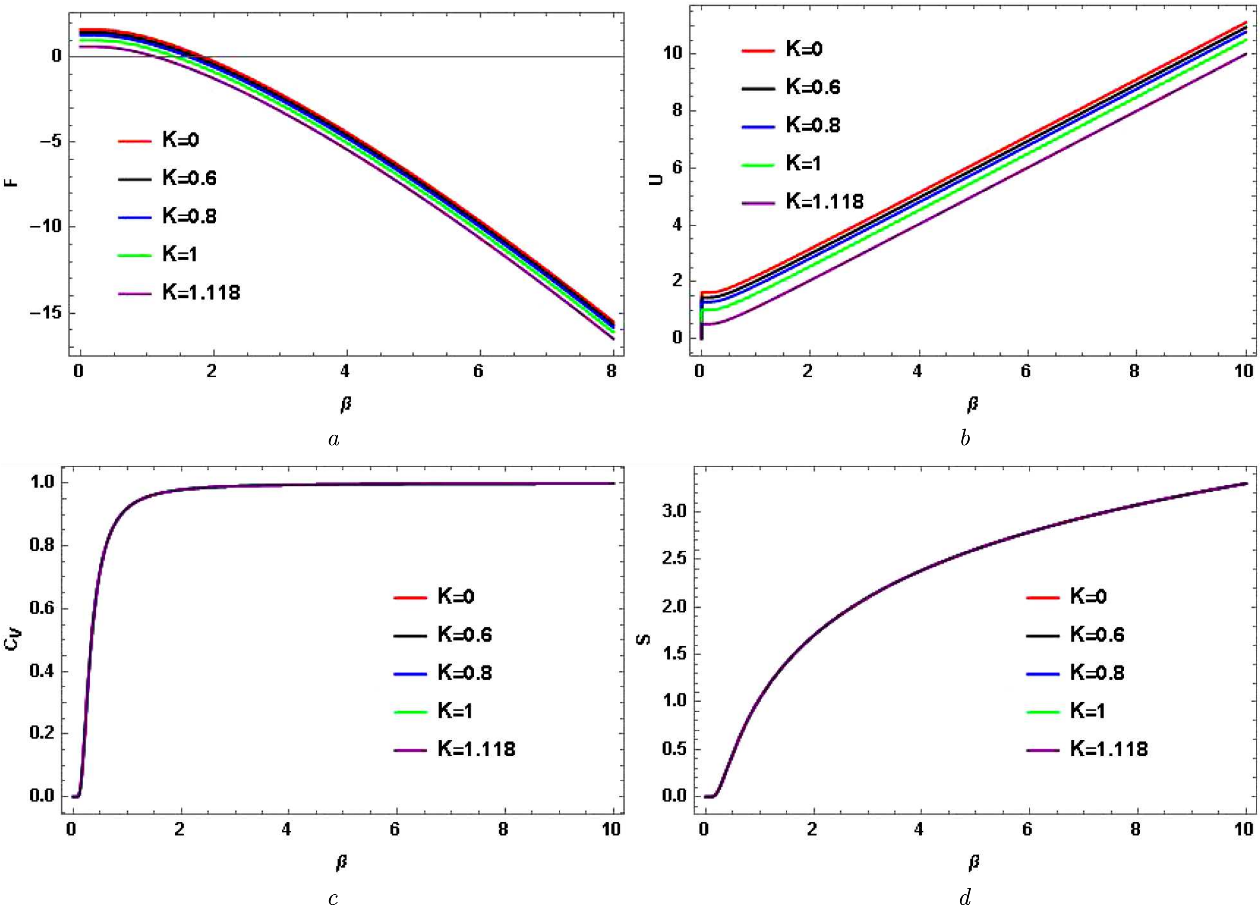


Fig. 2. Thermal properties of the one-dimensional Klein–Gordon oscillator in AdS under the influence of an external electric field: free energy F versus β for $\kappa = 0, 0.6, 0.8, 1, 1.118$ (a); total energy U versus β for $\kappa = 0, 0.6, 0.8, 1, 1.118$ (b); specific heat C_V versus β for $\kappa = 0, 0.6, 0.8, 1, 1.118$ (c); entropy S versus β for $\kappa = 0, 0.6, 0.8, 1, 1.118$ (d)

Here

$$Z_{\mathrm{HO}} = \frac{1}{1 - e^{-\beta}} \tag{44}$$

corresponds to the harmonic oscillator partition function.

The thermodynamic quantities can now be derived from the partition function. The expressions for the free energy, total energy, entropy, and specific heat are as follows:

$$F = -\frac{1}{\beta} \ln Z_{\mathrm{HO}} + a,\; U = a - \frac{\partial \left(\ln Z_{\mathrm{HO}}\right)}{\partial \beta}, \tag{45}$$

$$\begin{aligned} S &= \ln Z - \beta \frac{\partial \left(\ln Z\right)}{\partial \beta} = S_{\mathrm{HO}}, \\ C_V &= \beta^2 \frac{\partial^2 \left(\ln Z\right)}{\partial \beta^2} = \beta^2 \frac{\partial^2 \left(\ln Z_{\mathrm{HO}}\right)}{\partial \beta^2}. \end{aligned} \tag{46}$$

We observe that the external electric field only affects the total energy and free energy, while the entropy and specific heat remain unchanged.

Figure 2 illustrates the thermal properties of the one-dimensional Klein–Gordon oscillator in AdS space under the influence of an external electric field. The electric field strength is restricted to the range $0 < \kappa < 1.118$ to ensure that the energy spectrum remains real. The thermal quantities are systematically plotted as functions of the inverse temperature parameter β. To analyze the effect of the electric field, comparisons are made with the case of the relativistic oscillator in the absence of an applied electric field ($\kappa = 0$).

• The free energy curves, plotted against β for different values of κ, show that increasing the electric field consistently lowers the free energy compared with the case where $\kappa = 0$.

• The total energy curves, also plotted against β for various values of κ, reveal that as the electric field increases, the total energy remains consistently lower than in the reference case where $\kappa = 0$.

• The specific heat and entropy curves, plotted against β for different values of κ, all coincide. This indicates that these quantities are unaffected by the external electric field.

These results can be explained as follows: as the electric field strength increases, the number of allowed states decreases. As κ increases from zero to one, the energy gap in the spectrum narrows and approaches $E_n = 0$, which corresponds to the vacuum state.

3. The Klein–Gordon Oscillator

In our previous work, we explored how a particle subjected to an external potential could behave as a harmonic oscillator in spacetime. In this study, we introduce the Klein–Gordon oscillator and analyze the particle's behavior under this potential, in which the particle is no longer free. We begin with the d'Alembertian operator of the Klein–Gordon equation in a curved spacetime

$$\square = \frac{1}{c^2}\frac{1}{1+\frac{\omega^2}{c^2}x^2}\frac{\partial^2}{\partial t^2} + \frac{1}{\hbar^2}\left(1+\frac{\omega^2}{c^2}x^2\right)p^2 - 2i\frac{\omega^2}{\hbar c^2}xp. \tag{47}$$

Replacing the momentum p with the expression $p \to \to p - im\omega_1 x$, transforms Eq. (47) into:

$$\square = \frac{1}{c^2}\frac{1}{1+\frac{\omega^2}{c^2}x^2}\frac{\partial^2}{\partial t^2} + \frac{1}{\hbar^2}\left(1+\frac{\omega^2}{c^2}x^2\right)\times$$
$$\times\left(p+im\omega_1 x\right)\left(p-im\omega_1 x\right) - 2i\frac{\omega^2}{\hbar c^2}x\left(p-im\omega_1 x\right) \tag{48}$$

or

$$\square = \frac{1}{c^2}\frac{1}{1+\frac{\omega^2}{c^2}x^2}\frac{\partial^2}{\partial t^2} - \left(1+\frac{\omega^2}{c^2}x^2\right)\frac{\partial^2}{\partial x^2} - 2\frac{\omega^2}{c^2}x\frac{\partial}{\partial x} -$$
$$-\left(1+\frac{\omega^2}{c^2}x^2\right)\left(\frac{m\omega_1}{\hbar} - \frac{m^2\omega_1^2x^2}{\hbar^2}\right) - 2\frac{\omega^2}{c^2}x\left(\frac{m\omega_1 x}{\hbar}\right). \tag{49}$$

As a result of the chosen spacetime metric and the added potential, the modified d'Alembertian contains new terms corresponding to a harmonic oscillator with two distinct angular frequencies, ω_1 and ω. Using this new d'Alembertian, we derive the resulting differential equation

$$\left\{\left(1+\frac{\omega^2}{c^2}x^2\right)\frac{d^2}{dx^2} - 2\frac{\omega^2}{c^2}\left(\lambda-1\right)x\frac{d}{dx} + \right.$$
$$+\frac{\omega^2}{c^2}\left(\lambda\left(\lambda-1\right) - N^2 + 2\xi\right) + \left(1+\frac{\omega^2}{c^2}x^2\right)\times$$
$$\left.\times\left(\frac{m\omega_1}{\hbar} - \frac{m^2\omega_1^2x^2}{\hbar^2}\right) + 2\frac{\omega^2}{c^2}x\left(\frac{m\omega_1 x}{\hbar}\right)\right\}\varphi(x) = 0. \tag{50}$$

The solution to this equation is given by the Heun C function, denoted by Heun C(α, β, γ, δ, η, z) [36–38].

Thus, the wave function is given by

$$\varphi\left(x\right) = C_1\left(\omega^2x^2 + c^2\right)^{\lambda} e^{-\frac{m\omega_1 x^2}{2h}}\times$$
$$\times \operatorname{Heun C}\left(\frac{c^2 m\omega_1}{\omega^2 h}, -\frac{1}{2}, -\frac{3c^2 m\omega_1}{4\omega^2 h}, \lambda,\right.$$
$$\left.\frac{-h\left(N^2-\lambda^2-2\xi-1\right)\omega^2 + c^2 m\omega_1}{4h\omega^2}, -\frac{\omega^2x^2}{c^2}\right) +$$
$$+ C_2\left(\omega^2x^2 + c^2\right)^{\lambda} x e^{-\frac{m\omega_1 x^2}{2h}}\times$$
$$\times \operatorname{Heun C}\left(\frac{c^2 m\omega_1}{\omega^2 h}, \frac{1}{2}, -\frac{3c^2 m\omega_1}{4\omega^2 h}, \lambda,\right.$$
$$\left.\frac{-h\left(N^2-\lambda^2-2\xi-1\right)\omega^2 + c^2 m\omega_1}{4h\omega^2}, -\frac{\omega^2x^2}{c^2}\right). \tag{51}$$

Here

$$\alpha = \frac{c^2 m\omega_1}{\omega^2 h},\ \beta = -\frac{1}{2},\ \gamma = -\frac{3c^2 m\omega_1}{4\omega^2 h},\ \delta = \lambda,$$
$$\eta = \frac{-h\left(N^2-\lambda^2-2\xi-1\right)\omega^2 + c^2 m\omega_1}{4h\omega^2}. \tag{52}$$

The parameters μ and ν are defined as follows:

$$\mu = \frac{\alpha}{2}(1+\beta) - \frac{\beta}{2}(1+\gamma) - \frac{\gamma}{2} - \eta =$$
$$= -\frac{1}{4}\lambda + \frac{1}{4}N^2 - \frac{1}{4}\lambda^2 - \frac{1}{2}\xi, \tag{53}$$

$$\nu = \frac{\alpha}{2}(1+\gamma) + \frac{\beta}{2}(1+\gamma) + \frac{\gamma}{2} + \delta + \eta =$$
$$= \frac{h\left(N^2-\lambda^2+\lambda-2\xi-2\right)\omega^2 + 2c^2 m\omega_1\left(\lambda-1\right)}{4h\omega^2}. \tag{54}$$

For the second part of our solution we obtain

$$\beta_2 = \frac{1}{2}, \tag{55}$$

$$\mu_2 = \frac{\alpha}{2}(1+\beta) - \frac{\beta}{2}(1+\gamma) - \frac{\gamma}{2} - \eta = = \frac{h\left(N^2 - \lambda^2 - 3\lambda - 2\xi - 2\right)\omega^2 + 2c^2 m\omega_1}{4\hbar\omega^2}, \tag{56}$$

$$\nu_2 = \frac{\alpha}{2}(1+\gamma) + \frac{\beta}{2}(1+\gamma) + \frac{\gamma}{2} + \delta + \eta = = \frac{h\left(N^2 - \lambda^2 + 3\lambda - 2\xi\right)\omega^2 + 2c^2 m\omega_1(\lambda - 1)}{4\hbar\omega^2}. \tag{57}$$

For the first part, the boundary conditions $\varphi(+\infty) = = \varphi(-\infty) = 0$ require that only the first solution be retained when $\lambda < 0$.

Thus, the solution is given by

$$\varphi(x) = C_1\left(\omega^2 x^2 + c^2\right)^\lambda e^{-\frac{m\omega_1 x^2}{2\hbar}} \times \times \operatorname{Heun C}\left(\frac{c^2 m\omega_1}{\omega^2\hbar}, -\frac{1}{2}, -\frac{3c^2 m\omega_1}{4\omega^2\hbar}, \lambda, \frac{-\hbar(N^2 - \lambda^2 - 2\xi - 1)\omega^2 + c^2 m\omega_1}{4\hbar\omega^2}, -\frac{\omega^2 x^2}{c^2}\right). \tag{58}$$

To determine the system's energy, it is necessary to impose the polynomial α-condition, for the Heun C function, which is given by [39]:

$$\delta = -\left(n + \frac{(\gamma + \beta + 2)}{2}\right)\alpha. \tag{59}$$

We obtain

$$E_n = 2n\hbar\omega. \tag{60}$$

An important observation concerns the behavior induced by the introduced potential: the relativistic effects associated with AdS spacetime appear to vanish, reducing the system to a simple harmonic oscillator. This finding is particularly surprising and requires further analysis. It appears that the inclusion of the potential not only introduces a distinct effect but also effectively eliminates the relativistic effects of the AdS spacetime on the particle. This interplay between the added potential and the relativistic properties of the space is a significant feature of our results and warrants further investigation.

Furthermore, we can follow the interpretation provided by Navarro *et al.* [19], who demonstrated that the single-particle sector of Klein–Gordon theory formulated on the universal covering space of Anti-de Sitter (AdS) spacetime admits a natural interpretation as a special-relativistic oscillator in Minkowski spacetime. They showed that the quantum wavefunctions exhibit behavior markedly different from their nonrelativistic counterparts, despite the fact that the energy spectrum exactly matches that of the nonrelativistic oscillator. This correspondence arises from imposing appropriate nonrelativistic limits.

The wave function for this system is given by

$$\frac{\varphi(x)}{C} = \left(\omega^2 x^2 + c^2\right)^\lambda e^{-\frac{m\omega_1 x^2}{2\hbar}} \times \times \operatorname{Heun C}\left(\frac{c^2 m\omega_1}{\omega^2\hbar}, -\frac{1}{2}, -\frac{3c^2 m\omega_1}{4\omega^2\hbar}, \lambda, \frac{-\hbar(N^2 - \lambda^2 - 2\xi - 1)\omega^2 + c^2 m\omega_1}{4\hbar\omega^2}, -\frac{\omega^2 x^2}{c^2}\right), \tag{61}$$

where C is the normalization constant.

4. Conclusion

This study builds upon a proposal for the relativistic harmonic oscillator, which extends the symmetry algebra of quantum operators in a relativistic free system. From a geometric perspective, we demonstrate that a static $(1+1)$-dimensional anti-de Sitter (AdS) metric can describe a one-dimensional relativistic oscillator.

In the first part of our investigation, we introduced a constant external potential, which, as anticipated, provided a constant contribution to the system's energy. In the second part, we incorporated a stationary external electric field. Our calculations revealed a new term that accounts for the effect of the electric field. In particular, as the electric field approaches 10^{18} V/m (the threshold for the Schwinger effect), the energy decreases. In this context, we also computed thermodynamic properties, observing that both the free and total energy curves are lower than in the case without an electric field.

In the final case concerning the Klein–Gordon oscillator, we obtained a notable result: the relativistic effects of AdS spacetime vanish in the presence of the Klein–Gordon oscillator potential. The Klein–Gordon oscillator potential and the relativistic geometry of AdS spacetime effectively cancel each other,

Five Heun equations

Types of Heun equations	Heun equation	Regular singularities	Irregular singularities	Heun function
Heun G	$\frac{d^2}{dz^2}y(z)+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right)\left(\frac{d}{dz}y(z)\right)+\frac{(\alpha\beta z-q)y(z)}{z(z-1)(z-a)}=0$	$0,1,a,\infty$	Γ	Heun G$(a,q,\alpha,\beta,\gamma,\delta,z)$
Heun C	$\frac{d^2}{dz^2}y(z)+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}-\epsilon\right)\left(\frac{d}{dz}y(z)\right)+\left(\frac{-\alpha\beta+q}{z-1}-\frac{q}{z}\right)y(z)=0$	$0,1$	∞	Heun C$(\alpha,\beta,\gamma,\delta,\eta,z)$
Heun B	$\frac{d^2}{dz^2}y(z)+\left(-2z-\beta+\frac{1+\alpha}{z}\right)\left(\frac{d}{dz}y(z)\right)+$ $+\left(\gamma-\alpha-2-\frac{(1+\alpha)\beta+\delta}{2z}\right)y(z)=0$	0	∞	Heun B$(\alpha,\beta,\gamma,\delta,z)$
Heun D	$\frac{d^2}{dz^2}y(z)-\frac{\left(z^2\alpha-2z^3+\alpha+2z\right)\left(\frac{d}{dz}y(z)\right)}{(z+1)^2(z-1)^2}+\frac{\left(\delta+(2\alpha+\gamma)z+\beta z^2\right)y(z)}{(z-1)^3(z+1)^3}$	$\varnothing$	–1,1	Heun D$(\alpha,\beta,\gamma,\delta,z)$
Heun T	$\frac{d^2}{dz^2}y(z)+\left(-3z^2-\gamma\right)\left(\frac{d}{dz}y(z)\right)+(z\beta+\alpha-3z)\,y(z)=0$	$\varnothing$	∞	Heun T(α,β,γ,z)

reducing the system to a harmonic oscillator. Consequently, we determined the energy of a particle in such an oscillator.

In conclusion, this work provides insight into the dynamics of relativistic systems, highlighting the interplay between relativistic harmonic oscillators, the symmetry algebra of quantum operators, and external fields within AdS spacetime. These findings contribute to a better understanding of the factors that influence the behavior of relativistic oscillators, and the fundamental principles governing relativistic systems. Moreover, the results may be relevant to other areas of physics.

Future work will extend this study to various quantum systems in curved spacetimes to explore the interaction between matter and spacetime geometry in greater detail. This approach aims to clarify the interplay between quantum systems and spacetime geometry, thereby expanding our understanding of their dynamics. Such investigations may also provide a basis for further studies in related areas of physics.

APPENDIX A: Proof of Eq. (6)

The Lagrangian is given by:

$$\mathcal{L}=T-V=-mc^2\sqrt{1+\frac{\omega^2}{c^2}x^2-\frac{1}{1+\frac{\omega^2}{c^2}x^2}\frac{v^2}{c^2}}-V. \tag{A1}$$

To calculate the Hamiltonian, we use:

$$H=\sum_i \dot{q}_i p_i-\mathcal{L}. \tag{A2}$$

In $(1+1)$ dimension, squaring (A2) leads to

$$H^2=(vp-\mathcal{L})^2. \tag{A3}$$

Expanding (A3) gives

$$H^2=v^2p^2+\mathcal{L}^2-2vp\mathcal{L}. \tag{A4}$$

Substituting the expression for $\mathcal{L}$, we derive:

$$H^2=v^2p^2+\left(mc^2\sqrt{1+\frac{\omega^2}{c^2}x^2-\frac{1}{1+\frac{\omega^2}{c^2}x^2}\frac{v^2}{c^2}}+V\right)^2+$$
$$+2vp\left(mc^2\sqrt{1+\frac{\omega^2}{c^2}x^2-\frac{1}{1+\frac{\omega^2}{c^2}x^2}\frac{v_x^2}{c^2}}+V\right). \tag{A5}$$

Next, the conjugate momentum p is calculated as:

$$p=\frac{\partial\mathcal{L}}{\partial\dot{x}}=\frac{\frac{mc^2}{1+\frac{\omega^2}{c^2}x^2}\frac{v}{c^2}}{\sqrt{1+\frac{\omega^2}{c^2}x^2-\frac{1}{1+\frac{\omega^2}{c^2}x^2}\frac{v^2}{c^2}}}. \tag{A6}$$

Rewriting the Hamiltonian squared, we get:

$$H^2=v^2p^2+\left[m^2c^4\left(1+\frac{\omega^2}{c^2}x^2-\frac{1}{1+\frac{\omega^2}{c^2}x^2}\frac{v^2}{c^2}\right)+\right.$$
$$\left.+V^2+2Vmc^2\sqrt{1+\frac{\omega^2}{c^2}x^2-\frac{1}{1+\frac{\omega^2}{c^2}x^2}\frac{v^2}{c^2}}\right]+$$
$$+2\left(\frac{\frac{mc^2}{1+\frac{\omega^2}{c^2}x^2}\frac{v^2}{c^2}}{\sqrt{1+\frac{\omega^2}{c^2}x^2-\frac{1}{1+\frac{\omega^2}{c^2}x^2}\frac{v^2}{c^2}}}\right)\times$$
$$\times\left(mc^2\sqrt{1+\frac{\omega^2}{c^2}x^2-\frac{1}{1+\frac{\omega^2}{c^2}x^2}\frac{v^2}{c^2}}+V\right). \tag{A7}$$

Further simplifications yield:

$$H^2=m^2c^4+m^2c^2\omega^2x^2+p^2c^2+$$
$$+\frac{\omega^4}{c^2}x^4p^2+2\omega^2x^2p^2-V^2+2VH. \tag{A8}$$

APPENDIX B: Heun equation

The Heun function is a special function used in mathematical physics. It represents a solution to the Heun differential equation, a second-order linear differential equation. The function is denoted by Heun C(α, β, γ, δ, η, z), where α, β, δ, η, and z are algebraic expressions. This notation is used to define the Heun function.

There are five types of Heun equations, each associated with a corresponding function that can be expressed in the corresponding Heun form (see Table).

B1. The general Heun function

The general Heun function is a solution of the differential equation with four regular singular points, known as the general Heun equation. This equation can be expressed in canonical form as follows [26, 27]:

$$\frac{d^2H(z)}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right)\frac{dH(z)}{dz}+$$
$$+\frac{(\alpha\beta z-q)\,H(z)}{z(z-1)(z-a)}=0, \tag{B1}$$

where the parameters satisfy the Fuchsian relation [1]:

$$1+\alpha+\beta=\gamma+\delta+\epsilon. \tag{B2}$$

The general Heun function is given by

$$H(x)=\mathrm{Heun\,G}(b,q_1,\alpha_1,\beta_1,\gamma_1,\delta;z)+x^{1-\gamma}\times$$
$$\times\,\mathrm{Heun\,G}(b,q_2,\alpha_2,\beta_2,\gamma_2,\delta;z), \tag{B3}$$

where the parameters α_2, β_2, γ_2, q_2, corresponding to the second solution, are defined as follows:

$$\alpha_2=\alpha+1-\gamma,\quad \beta_2=\beta+1-\gamma,\quad \gamma_2=2-\gamma,$$
$$q_2=q+(\alpha\delta+\epsilon)(1-\gamma). \tag{B4}$$

The polynomial condition for the general Heun function, known as the "polynomial α-condition", arises from the Frobenius method and is given by

$$\alpha=-n, \tag{B5}$$

where n is a nonnegative integer.

B2. Confluent Heun equation

The confluent Heun equation, which has two regular singular points, is written in its standard form as follows [36–38]:

$$\frac{d^2H(z)}{dz^2}+\left(\alpha+\frac{\beta+1}{z}+\frac{\gamma+1}{z-1}\right)\times$$
$$\times\frac{dH(z)}{dz}+\left(\frac{\mu}{z}+\frac{\nu}{z-1}\right)H(z)=0,$$

[1] The Fuchsian relation is a condition that must be satisfied by the parameters of the general Heun equation.

where $H(z)=\mathrm{Heun\,C}(\alpha,\beta,\gamma,\delta,\eta;z)$ denotes the confluent Heun function. The parameters μ and ν are defined as follows:

$$\mu=\frac{\alpha}{2}(1+\beta)-\frac{\beta}{2}(1+\gamma)-\frac{\gamma}{2}-\eta,$$
$$\nu=\frac{\alpha}{2}(1+\gamma)+\frac{\beta}{2}(1+\gamma)+\frac{\gamma}{2}+\delta+\eta.$$

A condition known as the "polynomial condition" for the confluent Heun function, related to the Frobenius method, is given by

$$\delta=-\alpha\left[n+\frac{1}{2}(2+\beta+\gamma)\right],$$

where n is a nonnegative integer.

1. W. Greiner. *Relativistic Quantum Mechanics: Wave Equations*, 3rd edition (Springer, 2000) [ISBN: 978-3-540-67457-3].
2. *Advances in Algebraic Quantum Field Theory*. Edited by R. Brunetti, C. Dappiaggi, K. Fredenhagen, J. Yngvason (Springer, 2015) [ISBN: 978-3-319-21353-8].
3. M. Cariglia. Quantum mechanics of Yano tensors: Dirac equation in curved spacetime. *Class. Quantum Gravity* **21**, 1051 (2004).
4. S.A. Franchino-Viñas, J.J. Relancio. Geometrizing the Klein–Gordon and Dirac equations in doubly special relativity. *Class. Quantum Gravity* **40**, 054001 (2023).
5. S.A. Fulling. *Aspects of Quantum Field Theory in Curved Spacetime* (Cambridge University Press, 1989) [ISBN: 978-0-521-34400-5].
6. M. Cvetič, H. Lü, C.N. Pope, J.F. Vázquez-Poritz. AdS spacetime in warped spacetimes. *Phys. Rev. D* **62**, 122003 (2000).
7. A. Buchel, S.L. Liebling, L. Lehner. Boson stars in AdS spacetime. *Phys. Rev. D* **87**, 123006 (2013).
8. N. Beisert, C. Ahn, L.F. Alday, Z. Bajnok, J.M. Drummond, L. Freyhult, N. Gromov, R.A. Janik, V. Kazakov, T. Klose, G.P. Korchemsky, C. Kristjansen, M. Magro, T. McLoughlin, J.A. Minahan *et al.* Review of AdS/CFT integrability: An overview. *Lett. Math. Phys.* **99**, 3 (2012).
9. H. Nastase. Introduction to AdS-CFT. arXiv:0712.0689 [hep-th].
10. S.S. Gubser. AdS/CFT and gravity. *Phys. Rev. D* **63**, 084017 (2001).
11. S.J. Avis, C.J. Isham, D. Storey. Quantum field theory in anti-de Sitter space-time. *Phys. Rev. D* **18**, 3565 (1978).
12. H. Friedrich. Einstein equations and conformal structure: Existence of anti-de Sitter-type space-times. *J. Geom. Phys.* **17**, 125 (1995).
13. H. Yukawa. Structure and mass spectrum of elementary particles. II. Oscillator model. *Phys. Rev.* **91**, 416 (1953).
14. R.P. Feynman, M. Kislinger, F. Ravndal. Current matrix elements from a relativistic quark model. *Phys. Rev. D* **3**, 2706 (1971).
15. D. Itô, K. Mori, E. Carriere. An example of dynamical systems with linear trajectory. *Nuovo Cimento A* **51**, 1119 (1967).

16. V. Aldaya, J. Bisquert, J. Navarro-Salas. The quantum relativistic harmonic oscillator: Generalized Hermite polynomials. *Phys. Lett. A* **156**, 381 (1991).
17. V. Aldaya, J. Guerrero. Canonical coherent states for the relativistic harmonic oscillator. *J. Math. Phys.* **36**, 3191 (1995).
18. V. Aldaya, J. Bisquert, J. Guerrero, J. Navarro-Salas. Group-theoretical construction of the quantum relativistic harmonic oscillator. *Rep. Math. Phys.* **37**, 387 (1996).
19. D.J. Navarro, J. Navarro-Salas. Special-relativistic harmonic oscillator modeled by Klein–Gordon theory in anti-de Sitter space. *J. Math. Phys.* **37**, 6060 (1996).
20. A. Bouzenada, A. Boumali, F. Serdouk. Thermal properties of the 2D Klein–Gordon oscillator in a cosmic string space–time. *Theor. Math. Phys.* **216**, 1055 (2023).
21. A. Boumali, R. Allouani, A. Bouzenada, F. Serdouk. Effect of the applied electric field on the thermal properties of the relativistic harmonic oscillator in one dimension. *Ukr. J. Phys.* **68**, 235 (2023).
22. M. Moshinsky, A. Szczepaniak. The Dirac oscillator. *J. Phys. A Math. Gen.* **22**, L817 (1989).
23. S. Bruce, P. Minning. The Klein–Gordon oscillator. *Nuovo Cimento A* **106**, 711 (1993).
24. L.H. Ryder. *Quantum Field Theory.* 2nd edition (Cambridge University Press, 1996) [ISBN: 978-0-521-47814-4].
25. G. Sterman. *An Introduction to Quantum Field Theory* (Cambridge University Press, 1993) [ISBN: 978-0-521-31132-8].
26. T.A. Ishkhanyan, T.A. Shahverdyan, A.M. Ishkhanyan. Expansions of the solutions of the general Heun equation governed by two-term recurrence relations for coefficients. *Adv. High Energy Phys.* **2018**, 4263678 (2018).
27. M. Abu-Saleem, H.S. Vieira. Some aspects of the five-dimensional Lovelock black hole spacetime: Strong homotopy retract, perihelion precession and quasistationary levels. *Ann. Phys.* **433**, 168583 (2021).
28. *Heun's Differential Equations.* Edited by A. Ronveaux (Oxford University Press, 1995) [ISBN: 978-0-19-859695-0].
29. H.S. Vieira. Quasibound states, stability and wave functions of the test fields in the consistent 4D Einstein–Gauss–Bonnet gravity. *Universe* **9**, 205 (2023).
30. M. Saito, B. Sturmfels, N. Takayama. *Gröbner Deformations of Hypergeometric Differential Equations* (Springer, 2000), Vol. 6 [ISBN: 978-3-540-66065-1].
31. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Edited by M. Abramowitz, I.A. Stegun (Dover, 1965) [ISBN: 978-0-486-61272-0].
32. N. Virchenko, S.L. Kalla, A. Al-Zamel. Some results on a generalized hypergeometric function. *Integral Transforms Spec. Funct.* **12**, 89 (2001).
33. G.W. Semenoff, K. Zarembo. Holographic Schwinger effect. *Phys. Rev. Lett.* **107**, 171601 (2011).
34. H. Kitamoto. Schwinger effect in inflaton-driven electric field. *Phys. Rev. D* **98**, 103512 (2018).
35. F. Hebenstreit, R. Alkofer, H. Gies. Particle self-bunching in the Schwinger effect in spacetime-dependent electric fields. *Phys. Rev. Lett.* **107**, 180403 (2011).
36. P.P. Fiziev. Novel relations and new properties of confluent Heun's functions and their derivatives of arbitrary order. *J. Phys. A Math. Theor.* **43**, 035203 (2010).
37. A.M. Ishkhanyan, A.E. Grigoryan. Fifteen classes of solutions of the quantum two-state problem in terms of the confluent Heun function. *J. Phys. A Math. Theor.* **47**, 465205 (2014).
38. V.A. Shahnazaryan, T.A. Ishkhanyan, T.A. Shahverdyan, A.M. Ishkhanyan. New relations for the derivative of the confluent Heun function. *Armen. J. Phys.* **5**, 146 (2012).
39. W.C.F. da Silva, K. Bakke, R.L.L. Vitória. Non-relativistic quantum effects on the harmonic oscillator in a spacetime with a distortion of a vertical line into a vertical spiral. *Eur. Phys. J. C* **79**, 657 (2019).

Received 10.10.24

С. Гарах, А. Бумалі, С. Заєм

ДИНАМІКА ОСЦИЛЯТОРА
КЛЯЙНА–ҐОРДОНА В ПРОСТОРІ
АНТИ-ДЕ СІТТЕРА

У роботі досліджено квантову динаміку бозонних частинок у просторі Анти-де Сіттера (AdS), у межах двох взаємодоповнювальних підходів. Спочатку отримано точні розв'язки для бозонної частинки, на яку діє стаціонарне електричне поле, завдяки чому пояснено вплив поля на термодинамічні характеристики квантових осциляторів. Далі аналіз розширено введенням лінійного члена взаємодії в рівняння Кляйна–Ґордона, унаслідок чого отримано осцилятор Кляйна–Ґордона (KGO). Для цієї модифікованої системи отримано та всебічно інтерпретовано точні аналітичні розв'язки. Теоретичною основою дослідження є квантова алгебра операторної симетрії, застосування якої дає змогу докладно описати складну поведінку бозонних частинок у викривленому просторі-часі. Запропонований підхід дає комплексне розуміння як мікроскопічних квантових станів, так і макроскопічної термодинамічної поведінки бозонних систем у викривлених геометріях AdS.

Ключові слова: рівняння Кляйна–Ґордона, викривлений простір-час, простір-час AdS, релятивістичний гармонічний осцилятор, термодинамічні властивості, функція Гойна.