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OPERATOR FORMULATION FOR CENTERED OPTICAL SYSTEMS

Currently, there are many mathematical methods in use in geometric optics. This paper presents a new mathematical apparatus: an operator formalism, which describes centered optical systems in the paraxial approximation. This work is an ideological continuation of author's previous research. The refraction and reflection operators of spherical surfaces are defined here. The mathematical properties of the operators are studied, and their physical interpretations are established. In addition, the relations between the lensing operator and the refraction operators of a spherical surface are determined. The behavior of rays is also considered, which helped to establish the injectivity and nondegeneracy for points with infinite coordinates. The operator formalism is helpful for finding a centered optical system that performs a given transformation. Moreover, the interchangeability of the optical operators is investigated, and it is found that each operator has a unique effect.

Keywords: geometric optics, thin lens, spherical mirror, nonlinear operator, optical system.

1. Introduction

Currently, there are many mathematical methods used to model the propagation of light. Some of the classical methods are ray optics and matrix optics [1]. The former describes the propagation of light as rays, which corresponds to Fermat's Principle [1]. The main equation of ray optics is the eikonal equation. Matrix optics is used for paraxial rays. In this method, light rays are described in terms of vectors with two coordinates – position and angle. To describe the action of some optical system, the ray-transfer matrix is applied. Among the new methods of modelling of optical systems imaging is the use of neural networks [2]. This method helps one to solve the eikonal equation [1]. In addition, in ray tracing technology, the Monte Carlo method can be used to study various applied optics problems [3]. There are several methods related to linear algebra that are used in geometric optics. For instance, bilocal operators are used [4] in geometric optics in the general theory of relativity. Some of these methods are used to describe the light polarization and light scattering. For instance, the Jones calculus, where the polarized light is described in terms of the Jones vector and linear optical elements are represented by Jones matrices [5]. To describe the light scattering, the Mueller calculus can be used. Here, the Mueller matrix acts on the Stokes vector that also describes a polarization of light [5]. This work aims to create an operator formalism for modelling the centered optical systems in the paraxial approximation. The idea for this work is based on the previous paper of the author [6], in which the lens operator describing a thin lens was considered in detail. This work generalizes the operator formalism to a wider class of systems, namely, to the centered systems consisting of thin lenses, spherical mirrors, and spherical surfaces between two optically transparent media. The following tasks were set and solved: the determination of refraction and reflection operators of a spherical surface, determination of their mathematical properties, establishment of their relation to the lens operator, as well as the discovery of the physical interpretation of the properties of certain operators. The behavior of rays is also considered, which helps us to establish the injectivity and nondegeneracy for points with infinite coor-

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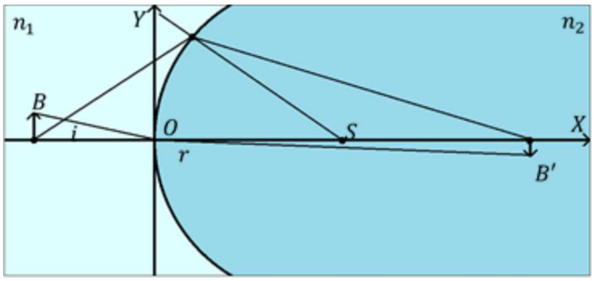

Fig. 1. An image of a point raised above the OX axis in a refracting spherical surface. Point S is the center of the spherical surface

dinates. In addition, the use of the transformation of translation to the study of centered optical systems is investigated. This opens wide opportunities to describe various centered optical systems.

2. Refraction and Reflection Operators on a Spherical Surface

2.1. Refraction operator

Let the spherical surface σ separate two optically transparent media with refraction indices n_1 and n_2 (1). Let us choose a Cartesian coordinate system as shown in Fig. 1. The surface maps the point source $B(x,y)$ to its image $B'(x',y')$. Point B can be described by the vector $\mathbf{r}=(x,y)$, and point B' can be described by the vector $\mathbf{r}'=(x',y')$.

In this paper, we assume that the paraxial approximation [7] is valid. We now write the equation for a spherical surface [7]:

$$\frac{(|x|+R)|x'|}{(|x'|-R)|x|}=\frac{n_2}{n_1}. \tag{1}$$

Let us use the rule of signs [7]. From expression (1) , the formula for the coordinate of the image x' can be obtained:

$$x'=x\frac{n_2}{n_1}\frac{1}{1+\frac{n_2-n_1}{Rn_1}x}. \tag{2}$$

We use the relative refraction index $\frac{n_2}{n_1}\equiv N$. In addition, to determine the concavity and convexity of the surface, we introduce the concept of radius vector $\mathbf{R}=(\pm R,0)$ of the center of the surface, where the sign depends on the orientation of the surface. Then, expression (2) can be written as:

$$x'=Nx\frac{1}{1+\frac{N-1}{(\mathbf{R},\,\mathbf{e}_x)}x}, \tag{3}$$

where $\mathbf{e}_x$ is the OX axis orth.

It can be shown that the expression for y' can be written as follows [6]:

$$y'=y\frac{1}{1+x\frac{N-1}{(\mathbf{R},\,\mathbf{e}_x)}}. \tag{4}$$

Since the vector $\mathbf{r}(x,y)$ transforms to the vector $\mathbf{r}'(x',y')$, there is a refraction operator, which we will call $\hat{R}[N,\mathbf{R}]$ (sometimes, the parameters will be omitted). Thus, the refraction operator can express the construction of the image of a point:

$$\hat{R}(x,y)=(Nx,y)\frac{1}{1+x\frac{N-1}{(\mathbf{R},\,\mathbf{e}_x)}}. \tag{5}$$

2.2. Properties of the refraction operator

1. The refraction operator is a nonlinear operator

$$\hat{R}(\alpha\mathbf{a}+\beta\mathbf{b})\neq\alpha\hat{R}\mathbf{a}+\beta\hat{R}\mathbf{b}. \tag{6}$$

2. The refraction operator is a noncommutative operator. Let us prove this property. To simplify the calculations, let us deal with the x coordinate.

For the case $\hat{R}[N_2,\mathbf{R}_2]\hat{R}[N_1,\mathbf{R}_1]$:

$$x'=xN_2N_1\frac{1}{1+\frac{N_1-1}{(\mathbf{R}_1,\,\mathbf{e}_x)}x+N_1\frac{N_2-1}{(\mathbf{R}_2,\,\mathbf{e}_x)}x}. \tag{7}$$

For the case $\hat{R}[N_1,\mathbf{R}_1]\hat{R}[N_2,\mathbf{R}_2]$:

$$x'=xN_1N_2\frac{1}{1+\frac{N_2-1}{(\mathbf{R}_1,\,\mathbf{e}_x)}x+N_2\frac{N_1-1}{(\mathbf{R}_2,\,\mathbf{e}_x)}x}. \tag{8}$$

The commutativity of the operators is identical to the equality of the denominators of expressions (7) and (8). This gives the following statement:

$$(\mathbf{R}_1,\mathbf{e}_x)=(\mathbf{R}_2,\,\mathbf{e}_x). \tag{9}$$

Since, in the general case, formula (9) is not correct, the refraction operator is not commutative. However, if $(\mathbf{R}_1,\,\mathbf{e}_x)=(\mathbf{R}_2,\,\mathbf{e}_x)$, the operator $\hat{R}[N,\mathbf{R}]$ is commutative.

3. The refraction operator is nondegenerate. Let us prove this statement. Suppose that there is a nonzero vector $\mathbf{r}(x,y)$ such that:

$$\hat{R}\mathbf{r}=0, \tag{10}$$

$$\hat{R}(x,y)=(Nx,y)\frac{1}{1+x\frac{N-1}{(\mathbf{R},\,\mathbf{e}_x)}}=0. \tag{11}$$

Since the vector $\mathbf{r}(x, y)$ is nonzero, the vector (Nx, y) is also nonzero, and it follows that the factor before the vector (Nx, y) must be zero. This is impossible, if the coordinates are finite (the case of infinite coordinates is considered below). Therefore, the refraction index is nondegenerate.

4. The refraction operator is an injective mapping. Let us prove this statement. Consider two different vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ such that:

$$\hat{R}\mathbf{r}_1 = \hat{R}\mathbf{r}_2. \tag{12}$$

After all simplifications, we get the following system:

$$\begin{cases} x_1 = x_2, \\ y_1 = y_2. \end{cases} \tag{13}$$

Contradictions with the initial statement are obtained. Therefore, the refraction operator is an injective mapping.

5. The refraction operator is a surjective mapping. From expression (5), any restrictions on the values of coordinates do not follow. However, the limiting case of infinite coordinates will be discussed later in Section 4. Therefore, the refraction operator is a bijective mapping.

6. Inverse operator. The existence of the inverse operator is a consequence of the refraction operator bijectivity [8]. We need to find the expression for the inverse operator. Let us check that there is only one possible inverse operator. Let operators $\hat{A}$ and $\hat{B}$ be inverse operators for the operator $\hat{R}$. Therefore, the following equality can be written:

$$\hat{A}\hat{R}\mathbf{r} = \hat{B}\hat{R}\mathbf{r} = \mathbf{r} \Longrightarrow (\hat{A} - \hat{B})\hat{R}\mathbf{r} = 0. \tag{14}$$

Since the vector $\mathbf{r}$ is arbitrary, and the operator $\hat{R}$ is nondegenerate, this implies that $\hat{A} - \hat{B} = 0$. Therefore, there is only one inverse operator. This fact makes inverse operator's finding easier. For instance, its form can be obtained from physical considerations. Let us make the following assumption:

$$\hat{R}^{-1}[N, \mathbf{R}] = \hat{R}\left[\frac{1}{N}, \mathbf{R}\right]. \tag{15}$$

We now check whether expression (15) is true. To do this, we check whether expression (7) is equal to

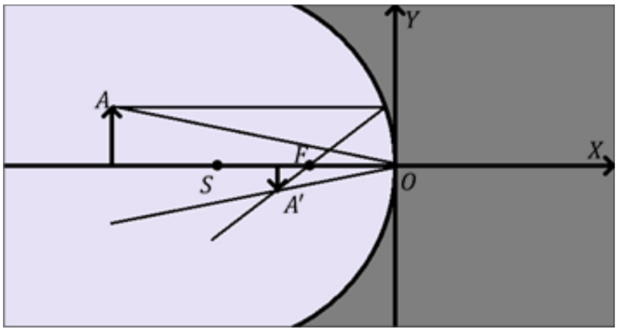

Fig. 2. Point's image formed by a concave spherical mirror. Point S is the center of the spherical surface

x. Let us make the following replacement: $N_1 = N$ and $N_2 = \frac{1}{N}$

$$x' = x\frac{1}{1 + \frac{N-1}{(\mathbf{R}, \mathbf{e}_x)}x + \frac{1-N}{(\mathbf{R}, \mathbf{e}_x)}x}. \tag{16}$$

Therefore, formula (14) is correct.

7. Commutativity of the operators $\hat{R}[N_1, \mathbf{R}]$ *and* $\hat{R}[N_2, \mathbf{R}]$. When proving the absence of commutativity, it was determined how the operators $\hat{R}[N_1, \mathbf{R}]$ and $\hat{R}[N_2, \mathbf{R}]$ behave themselves. Let us investigate how the argument N changes in this case.

Let us return to expression (7). We have

$$x' = xN_2N_1\frac{1}{1 + \frac{N_2N_1-1}{(\mathbf{R}, \mathbf{e}_x)}x}. \tag{17}$$

Therefore, we can specify the following property:

$$\hat{R}[N_1, \mathbf{R}]\,\hat{R}[N_2, \mathbf{R}] = \hat{R}[N_1N_2, \mathbf{R}]. \tag{18}$$

2.3. Reflection operator

Consider the image of a luminous point in a concave spherical mirror. Let us choose a Cartesian coordinate system as shown in Fig. 2. The mirror maps the point source $A(x, y)$ to its image $A'(x', y')$.

Let us write the equation of a spherical mirror [7]:

$$\frac{1}{|x|} + \frac{1}{|x'|} = \frac{1}{F}. \tag{19}$$

Accounting for expression (19) and the rule of signs, we can write an expression for the x' coordinate of the image:

$$x' = -x\frac{1}{1 + Dx}. \tag{20}$$

The similarity of triangles gives the following relation:

$$\frac{|y|}{|x|} = \frac{|y'|}{|x'|}. \quad (21)$$

Expressions (20)–(21) lead to the expression for the y' coordinate of the image:

$$y' = y\frac{1}{1+Dx}. \quad (22)$$

where $D = \frac{1}{F}$ is the optical power of the mirror.

Since the vector $\mathbf{r}(x, y)$ transforms to the vector $\mathbf{r}'(x', y')$, there is a reflection operator for the concave spherical mirror, which is called $\hat{M}_+[D]$. Thus, the reflection operator expresses the construction of point's image:

$$\hat{M}_+(x, y) = (-x, y)\frac{1}{1+Dx}. \quad (23)$$

For concave and convex spherical mirrors, the operator $\hat{M}_+[D]$ can be generalized using the sign of the optical power D, since, in all previous considerations, the optical power was positive. Thus, we introduce the reflection operator $\hat{M}[D]$ for a spherical mirror. Thus, the reflection operator expresses the construction of point's image:

$$\hat{M}(x, y) = (-x, y)\frac{1}{1+Dx}. \quad (24)$$

2.4. Properties of the reflection operator

1. The refraction operator is a nonlinear operator

$$\hat{M}(\alpha\mathbf{a} + \beta\mathbf{b}) \neq \alpha\hat{M}\mathbf{a} + \beta\hat{M}\mathbf{b}. \quad (25)$$

2. The reflection operator is not a commutative operator. Let us prove this property.

For the case $\hat{M}[D_2]\hat{M}[D_1]$:

$$(x', y') = \left(x\frac{1}{1+(D_1 - D_2)x}, y\frac{1}{1+(D_1 - D_2)x}\right). \quad (26)$$

For the case $\hat{M}[D_1]\hat{M}[D_2]$:

$$(x', y') = \left(x\frac{1}{1+(D_2 - D_1)x}, y\frac{1}{1+(D_2 - D_1)x}\right). \quad (27)$$

Thus, the operator is noncommutative. However, the analysis of expressions (26) and (27) gives the following equality:

$$\hat{M}[D_2]\hat{M}[D_1] = \hat{L}[D_1 - D_2], \quad (28)$$

where $\hat{L}[D]$ is the lensing operator [6].

3. The reflection operator is nondegenerate. Let us prove this property. Suppose that there is a nonzero vector $\mathbf{r}(x, y)$ such that:

$$\hat{M}\mathbf{r} = 0. \quad (29)$$

$$\hat{M}(x, y) = (-x, y)\frac{1}{1+Dx} = 0. \quad (30)$$

Since the vector $\mathbf{r}(x, y)$ is nonzero, the vector $(-x, y)$ is also nonzero, and it follows that the factor before the vector $(-x, y)$ must be zero. This is impossible for finite coordinates (the case of infinite coordinates is considered below). Therefore, the reflection operator is nondegenerate.

4. The reflection operator is an injective mapping. Let us prove this statement. Suppose that there are two different vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ such that:

$$\hat{M}\mathbf{r}_1 = \hat{M}\mathbf{r}_2. \quad (31)$$

After all simplifications, we get the following system:

$$\begin{cases} x_1 = x_2, \\ y_1 = y_2. \end{cases} \quad (32)$$

Contradictions with the initial statement are obtained. Therefore, the reflection operator is an injective mapping.

5. The reflection operator is a surjective mapping. From expression (24), any restrictions on the values of coordinates do not follow. However, the limiting case of infinite coordinates will be discussed later in Section 4. Therefore, the reflection operator is a bijective mapping.

6. Inverse operator. The existence of the inverse operator is a consequence of the reflection operator bijectivity [8]. Let us investigate expression (28). The following fact is known from the properties of the lensing operator [6]:

$$\hat{L}[0] = \hat{E}. \quad (33)$$

The operator $\hat{E}$ is a unity matrix.

Therefore, we can conclude the following equality:

$$\hat{M}[D]\hat{M}[D] = \hat{L}[D - D] = \hat{E}. \quad (34)$$

From formula (34), it follows that

$$\hat{M}^{-1}[D] = \hat{M}[D]. \quad (35)$$

Thus, we found the inverse operator to the reflection operator $\hat{M}[D]$.

2.5. Lensing operator

The lensing operator was introduced in our work [5] on the basis of the lens formula, and it is now necessary to investigate the interrelation of the lensing operator with the refraction operator.

Recall the definition of the lens operator:

$$\hat{L}[D](x,y) = (x,y)\frac{1}{1+Dx}, \tag{36}$$

where $D = (N-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ is the optical power of the lens [7].

Let us consider a thin lens. A which is a transparent body bounded by two spherical surfaces, and the distance between them can be neglected. Therefore, let us consider the sequential application of two refraction operators of the form $\hat{R}[N, \mathbf{R}_1]$ and $\hat{R}[\frac{1}{N}, \mathbf{R}_2]$. Substitute these parameters into expression (7):

$$x' = x\frac{1}{1+(N-1)\left(\frac{1}{(\mathbf{R}_1, \mathbf{e}_x)} - \frac{1}{(\mathbf{R}_2, \mathbf{e}_x)}\right)x}. \tag{37}$$

The optical power involves the orientation of the surface, and expression (37) can be rewritten as follows:

$$x' = x\frac{1}{1+(N-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)x}. \tag{38}$$

$$x' = x\frac{1}{1+Dx}. \tag{39}$$

Similarly, for the expression for the y' coordinate, we can write:

$$y' = y\frac{1}{1+Dx}. \tag{40}$$

Therefore, we have the following equality:

$$\hat{R}[N,\mathbf{R}_1]\hat{R}\left[\frac{1}{N}, \mathbf{R}_2\right] = \hat{L}\left[(N-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right]. \tag{41}$$

Thus, we have obtained the lensing operator as the sequential application of two refraction operators.

2.6. Physical interpretation of the properties of the refraction and reflection operators

An important issue that requires attention is the physical interpretation of the properties of the refraction and reflection operators. Let us consider them separately.

1. The refraction operator. The main properties of the refraction operator are the injectivity, nondegeneracy,, and equalities (15) and (18). The injectivity of an operator means that two different points cannot give the same image. The nondegeneracy means that the image of any point is not constructed at the pole of the spherical surface. Equation (15), which determines the inverse operator, has an easy physical interpretation, namely, the reversibility of light rays [7]. If the beam passes from medium 1 to medium 2, the relative refraction index is N, and if the beam passes from medium 2 to the medium 1 through the same surface, the relative refraction index turns to $\frac{1}{N}$. Equation (18) is essentially the equivalent formulation of the Abbe invariant [7]. Let us describe it in more details.

Suppose that we have k spherical surfaces of the interface of media with the same radii of curvature and such that the i-th surface separates the media with refraction indices n_i and n_{i+1}. Then the Abbe invariant reads:

$$Q = n_1\left(\frac{1}{x_1} - \frac{1}{R}\right) = ... = n_{k+1}\left(\frac{1}{x_k} - \frac{1}{R}\right). \tag{42}$$

If $x_{1\,...\,k} \ll R$, the coordinate of the final image x_k will be determined only by the refraction indices n_1 and n_{k+1}. If we write expression (18) for this system, it will look like this:

$$\hat{R}[N_1, \mathbf{R}]\,...\,\hat{R}[N_k, \mathbf{R}] = \hat{R}[N_1\,...\,N_k, \mathbf{R}], \tag{43}$$

where $N_i = \frac{n_{i+1}}{n_i}$.

As the value of the product $N_1\,...\,N_k$ is

$$N_1 N_2\,...\,N_k = \frac{n_{k+1}}{n_1}, \tag{44}$$

the coordinate of the final image x_k will be determined only by the refraction indices n_1 and n_{k+1}. We have already obtained this assertion by the use of the Abbe invariant.

2. Reflection operator. The main properties of the reflection operator are the nondegeneracy, injectivity, and the expressions for the product of the reflection operators (28) and for the inverse reflection operator (35). The injectivity of the reflection operator means that two different points cannot give the same image. The nondegeneracy means that the image of any point is not constructed at the pole of the spherical mirror. Equation (35), which determines the inverse operator, has a simple physical interpretation, namely, the reversibility of light rays. Equation (28) expresses the hypothetical behavior of two spherical mirrors as one lens, but this is possible, only if the spherical mirrors are able to transmit light without refraction, which is not possible. So, equation (28) is nothing more than a theoretical hypothesis that cannot be realized in practice.

3. Optical operators and translation

3.1. Translation operator

Let us define the operator $\hat{T}[\mathbf{R}]$, which corresponds to the translation of the Cartesian coordinate system into the vector $\mathbf{R}$. Its action reads as follows:

$$\hat{T}\left[\mathbf{R}\right]\mathbf{r} = \mathbf{r} - \mathbf{R}. \tag{45}$$

Let us define its properties.

1. Translation operator is a nonlinear operator

$$\hat{T}(\alpha\mathbf{a} + \beta\mathbf{b}) \neq \alpha\hat{T}\mathbf{a} + \beta\hat{T}\mathbf{b}. \tag{46}$$

2. Translation operators are commutative. Let us prove this property.

$$\hat{T}\left[\mathbf{R}_2\right]\hat{T}\left[\mathbf{R}_1\right]\mathbf{r} = \mathbf{r} - \mathbf{R}_1 - \mathbf{R}_2. \tag{47}$$

$$\hat{T}\left[\mathbf{R}_1\right]\hat{T}\left[\mathbf{R}_2\right]\mathbf{r} = \mathbf{r} - \mathbf{R}_2 - \mathbf{R}_1. \tag{48}$$

Expressions (47) and (48) are the same; so, the translation operators are commutative. It is worth to note the following property:

$$\hat{T}\left[\mathbf{R}_1\right]\hat{T}\left[\mathbf{R}_2\right] = \hat{T}\left[\mathbf{R}_1 + \mathbf{R}_2\right]. \tag{49}$$

3. The translation operator is a bijective mapping. Let us start with the injectivity. Suppose that there are two different vectors $\mathbf{r}_1$ and $\mathbf{r}_1$ such that:

$$\hat{T}\mathbf{r}_1 = \hat{T}\mathbf{r}_2. \tag{50}$$

$$\mathbf{r}_1 - \mathbf{R} = \mathbf{r}_2 - \mathbf{R}. \tag{51}$$

$$\mathbf{r}_1 = \mathbf{r}_2. \tag{52}$$

We have contradictions with the initial statement. Therefore, the translation operator is an injection. The surjectivity is obvious, because the translation operator translates the entire Euclidean space E_2 into the entire manifold M_2 (which is also a linear space).

4. Inverse operator. From the bijectivity of the translation operator, it follows that there is an inverse operator [8]. Its expression can be easily obtained from (49). The following fact follows from the definition:

$$\hat{T}[0] = \hat{E}. \tag{53}$$

Therefore, we have the following equality:

$$\mathbf{R}_1 + \mathbf{R}_2 = 0. \tag{54}$$

The expression for the inverse operator takes the following form:

$$\hat{T}^{-1}[\mathbf{R}] = \hat{T}[-\mathbf{R}]. \tag{55}$$

3.2. Use of translation for optical systems

In the previous srction, we defined the translation operator. Its necessity becomes clear, when we consider a centered optical system consisting of several objects at a distance. As an example, consider a system consisting of a lens and a spherical mirror.

Suppose that we have a centered optical system consisting of a thin lens and a spherical mirror corresponding to the optical operators $\hat{L}[D]$ and $\hat{M}[D']$. Let the optical center of the lens be located at the origin of the coordinate system, and let the pole of the spherical mirror be located at a distance d. Then the image of the point given by the vector $\mathbf{r}$ in the lens is as follows:

$$\mathbf{r}_1 = \hat{L}\left[D\right]\mathbf{r}. \tag{56}$$

Now, we use the fact that the point with the vector $\mathbf{r}_1$ is a source for the spherical mirror. However, it is also necessary to take into account that the optical center of the lens and the pole of the spherical mirror are spatially separated. We use the translation operator to pass to the coordinate system of the spherical mirror. We have

$$\mathbf{r}_2 = \hat{T}\left[\mathbf{R}\right]\mathbf{r}_1, \tag{57}$$

where $\mathbf{R} = (d, 0)$.

Then the, the image of the point $\mathbf{r}_2$ in the spherical mirror is the following point:

$$\mathbf{r}_3 = \hat{M}\left[D'\right]\mathbf{r}_2. \tag{58}$$

On the last step, we return to the original coordinate system. Thus, the action of the studied optical system reads as follows:

$$\mathbf{r}' = \hat{T}^{-1}\hat{M}\hat{T}\hat{L}\mathbf{r}. \tag{59}$$

The translation operator opens up great possibilities for describing the various centered optical systems in the paraxial approximation.

4. Optical Operators and Rays

In previous sections, the mathematical properties of the refraction and reflection operators were discovered. However, when we studied the injectivity and nondegeneracy of the operators, we considered the vectors with finite coordinates. The same assumption was used for the lensing operators in previous studies. Here, we are going to study the imaging of a point located at infinity. Physically, the infinitely distant point is equivalent to a ray. Consider a ray that is directed at an angle α to the main optical axis of the system. Since we work in the paraxial approximation, the angle α should be kept sufficiently small. Angles are measured in the positive direction. The dependence between the x and y coordinates obeys the following equality:

$$\tan\alpha = \frac{y}{x}. \tag{60}$$

Let us find how the operators behave themselves in the case of a light ray.

4.1. Lensing operator

For rays, we should consider the following limits:

$$\begin{cases} x' = \lim\limits_{x\to\infty} \dfrac{x}{1+Dx} = \dfrac{1}{D} = F, \\ y' = \lim\limits_{x\to\infty} \dfrac{y}{1+Dx} = F\tan\alpha. \end{cases} \tag{61}$$

For the y coordinate, equality (60) is used.

4.2. Refraction operator

For rays, we should consider the following limits:

$$\begin{cases} x' = \lim\limits_{x\to\infty} \dfrac{Nx}{1+\frac{N-1}{(\mathbf{R},\,\mathbf{e}_x)}x} = \dfrac{N(\mathbf{R},\,\mathbf{e}_x)}{N-1}, \\ y' = \lim\limits_{x\to\infty} \dfrac{y}{1+\frac{N-1}{(\mathbf{R},\,\mathbf{e}_x)}x} = \dfrac{(\mathbf{R},\,\mathbf{e}_x)}{N-1}\tan\alpha. \end{cases} \tag{62}$$

4.3. Reflection operator

For rays, we should consider the following limits:

$$\begin{cases} x' = \lim\limits_{x\to\infty} \dfrac{-x}{1+Dx} = -F, \\ y' = \lim\limits_{x\to\infty} \dfrac{y}{1+Dx} = F\tan\alpha. \end{cases} \tag{63}$$

If we consider a beam of light rays, it will have small transverse dimensions. Since the light beam diameter is small, we can assume that the beam comes from a single point at infinity. Since there is only one point at infinity for a fixed angle, we can assume that, for points with infinite coordinates, the injectivity of operators holds. From systems (61)-(63), it follows that the nondegeneracy of operators holds for any point with finite and infinite coordinates.

5. Case of Flat Surfaces

Let us consider another limiting case, when the radii of spherical surfaces strive for infinity. This is equivalent to the case of flat mirrors and the flat interface of two optically transparent media.

Let us consider the reflection operator. As is known, the optical power of a spherical mirror equals

$$D = \frac{1}{F} = \frac{2}{R}. \tag{64}$$

Let us consider the limits:

$$\begin{cases} x' = \lim\limits_{R\to\infty} \dfrac{-x}{1+\frac{2x}{R}} = -x, \\ y' = \lim\limits_{R\to\infty} \dfrac{y}{1+\frac{2x}{R}} = y. \end{cases} \tag{65}$$

This result has a clear physical interpretation. A flat mirror reflects the point to a point located on the same distance from the mirror; however, this point is located behind the mirror.

Let us consider the refraction operator. Let us consider the limits:

$$\begin{cases} x' = \lim\limits_{R\to\infty} \dfrac{Nx}{1+\frac{N-1}{(\mathbf{R},\,\mathbf{e}_x)}x} = Nx, \\ y' = \lim\limits_{R\to\infty} \dfrac{y}{1+\frac{N-1}{(\mathbf{R},\,\mathbf{e}_x)}x} = y. \end{cases} \tag{66}$$

We obtain an interesting result. The flat interface of two optically transparent media creates the image of the point at the same altitude; however, the x coordinate of the image is the x coordinate of the point, but N times scaled.

6. Modeling of Centered Optical Systems

6.1. Interchangeability of optical operators

A very important task is to analyze whether the sets of parameters exist and are such that two different optical operators act equally. For example, can a spherical mirror act in the same way as a thin lens?

To find the answer, we need to check the following equations:

$$\hat{L}[D]\,\mathbf{r} = \hat{M}[D']\,\mathbf{r}. \tag{67}$$

$$\hat{L}[D]\,\mathbf{r} = \hat{R}[N,\,\mathbf{R}]\,\mathbf{r}. \tag{68}$$

$$\hat{M}[D]\,\mathbf{r} = \hat{R}[N,\,\mathbf{R}]\,\mathbf{r}. \tag{69}$$

Let us check whether equality (67) is possible. We use the definitions of the lensing and reflection operators:

$$\frac{x}{1+Dx}(x,y) = \frac{x}{1+D'x}(-x,y). \tag{70}$$

From the equality of the y-coordinates, it follows that $D = D'$. Then, from expression (70), it follows that $x = 0$, and, for points with $x = 0$, any optical system gives the same point; so, it is not interesting for us. Therefore, we can make the following conclusion:

$$\forall\mathbf{r}\;\hat{L}[D]\,\mathbf{r} \neq \hat{M}[D']\,\mathbf{r}. \tag{71}$$

Let us check whether equality (68) is possible:

$$\frac{x}{1+Dx}(x,y) = \frac{x}{1+\frac{N-1}{(\mathbf{R},\mathbf{e}_x)}x}(Nx,y). \tag{72}$$

From the equality of the y-coordinates, it follows that $D = \frac{N-1}{(\mathbf{R},\mathbf{e}_x)}$. Then, it follows from expression (72) that $N = 1$. This means that the operator $\hat{R}[N,\,\mathbf{R}]$ becomes the identical operator, i.e., we have no refraction on the spherical surface. Therefore, we conclude that:

$$\forall\mathbf{r}\;\hat{L}[D]\,\mathbf{r} \neq \hat{R}[N,\,\mathbf{R}]\,\mathbf{r}. \tag{73}$$

Let us finally check whether equality (69) is possible. We can write

$$\frac{x}{1+Dx}(-x,y) = \frac{x}{1+\frac{N-1}{(\mathbf{R},\mathbf{e}_x)}x}(Nx,y). \tag{74}$$

In this case, we obtain an expression similar to (72), which allows us to conclude that $N = -1$, which has no physical meaning, because the refraction indices are always positive values. Therefore, the following statement holds true:

$$\forall\,\mathbf{r}\;\hat{M}[D]\,\mathbf{r} \neq \hat{R}[N,\,\mathbf{R}]\,\mathbf{r}. \tag{75}$$

Thus, we find that the optical operators are not interchangeable.

6.2. Modeling of centered optical systems

Let us consider the following problem:

Suppose that we have a point with the radius vector $\mathbf{r}_0(x_0,y_0)$ and its image $\mathbf{r}_1(x_1,y_1)$. It is necessary to find all possible centered optical systems that perform such a mapping. Let the OX axis determine the direction of the optical axis of the system. Let us use the operator approach. Suppose that we have an unknown optical operator $\hat{O}$. Since we do not know whether the origin of the coordinate system coincides with the center of the optical system, we should use an additional translation operator $\hat{R}[\mathbf{R}]$, where the vector $\mathbf{R}$ has the coordinates $(d,0)$. The translation operator translates vectors into the coordinate system of the optical operator. Then, we can write the following equation:

$$\hat{O}\hat{T}\,\mathbf{r}_0 = \hat{T}\,\mathbf{r}_1. \tag{76}$$

Now, let us apply (76) to each optical operator.

1. Lensing operator. Suppose that our optical operator is the lensing operator $\hat{L}[D]$. Then Eq. (76) reads

$$\hat{L}\,[D]\,\hat{T}\,[\mathbf{R}]\,\mathbf{r}_0 = \hat{T}\,[\mathbf{R}]\,\mathbf{r}_1. \tag{77}$$

We have one vector equation that splits into two scalar ones:

$$\begin{cases} \dfrac{1}{1+D\,(x_0-d)}(x_0-d) = x_1 - d, \\ \dfrac{1}{1+D\,(x_0-d)}y_0 = y_1, \end{cases} \tag{78}$$

The following expressions can be obtained from this set:

$$d = \frac{y_1x_0 - y_0x_1}{y_1 - y_0}. \tag{79}$$

$$D = \frac{(y_0-y_1)^2}{(x_0-x_1)y_1y_0}. \tag{80}$$

Thus, we obtain two unknown parameters of this system, which are expressed in terms of the known coordinates of the source and image vectors.

2. Reflection operator. Suppose that our optical operator is the reflection operator $\hat{M}[D]$; so, Eq. (76) can be written as follows:

$$\hat{M}[D]\,\hat{T}\,[\mathbf{R}]\,\mathbf{r}_0 = \hat{T}\,[\mathbf{R}]\,\mathbf{r}_1. \tag{81}$$

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Similarly to the previous case, the parameters of this system can be found as follows:

$$d = \frac{y_0 x_1 - y_1 x_0}{y_1 - y_0}. \tag{82}$$

$$D = \frac{(y_1 - y_0)^2}{(x_0 + x_1) y_1 y_0}. \tag{83}$$

Thus, we obtain two unknown parameters of this system, which are expressed in terms of the known coordinates of the source and image vectors.

3. Refraction operator on a spherical surface. Let our optical operator be the refraction operator $\hat{R}\,[N, \mathbf{R}]$. Then equation (76) can be written as

$$\hat{R}\,[N,\, \mathbf{R}_0]\, \hat{T}\,[\mathbf{R}]\, \mathbf{r}_0 = \hat{T}\,[\mathbf{R}]\, \mathbf{r}_1. \tag{84}$$

We have one vector equation that splits into two scalar ones. Unlike the previous cases, there is a problem, because we have two equations and three unknown variables. Theoretically, this problem cannot be solved.

7. Conclusions

The refraction and reflection operators on a spherical surface are defined, and their properties are investigated. A lens operator is obtained from the refraction operators on spherical surfaces, which is the continuation of our previous research. The physical interpretation of the mathematical properties of the refraction and reflection operators are established. In addition, the behavior of rays is considered, which helped us to discover the injectivity and nondegeneracy for points with infinite coordinates. The use of translation operators is considered, which allows the modeling of a large number of centered optical systems in the paraxial approximation.

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ОПЕРАТОРНИЙ ФОРМАЛІЗМ
ДЛЯ ЦЕНТРОВАНИХ ОПТИЧНИХ СИСТЕМ

На даний момент існує багато математичних методів, що використовуються в геометричній оптиці. Ця робота пропонує новий математичний апарат: операторний формалізм, який описує центровані оптичні системи в параксіальному наближенні. Ця робота є ідеологічним продовженням попередніх досліджень автора. У роботі визначено оператори заломлення та відбиття на сферичних поверхнях. Досліджено їхні математичні властивості та встановлено відповідну фізичну інтерпретацію. Додатково, було встановлено зв'язок між оператором лінзування і операторами заломлення та відбиття на сферичних поверхнях. Було досліджено поведінку променів, що допомогло встановити ін'єктивність та невиродженість для точок з нескінченними координатами. Операторний формалізм є зручним для знаходження оптичних систем, що виконують задану побудову зображення. Додатково було визначено невзаємозамінність оптичних операторів, що означає, що кожний оператор має унікальний вплив.

К л ю ч о в і с л о в а: геометрична оптика, тонка лінза, сферичне дзеркало, нелінійний оператор, оптичні системи.