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THE METHOD OF COLLECTIVE VARIABLES IN THE THEORY OF NONLINEAR FLUCTUATIONS WITH ACCOUNT FOR KINETIC PROCESSES

The set of parameters of the Bogolyubov reduced description, which includes collective variables, has been optimized for the consistent description of the kinetics and hydrodynamics of the systems of interacting particles. The contributions from short- and long-range interactions between the particles are separated. The short-range interactions (for example, the hard-sphere model) are described in the coordinate-momentum space, and the long-range ones in the space of collective variables. The short-range component is considered to be basic. Using the method of Zubarev non-equilibrium statistical operator, a system of transport equations for the non-equilibrium one-particle distribution function, the non-equilibrium average value for the density of particle interaction energy, and the non-equilibrium distribution function of collective variables are obtained. The applied method of collective variables allowed both the structural function and the hydrodynamic velocities of collective variables to be calculated in approximations higher than the Gaussian one.

K e y w o r d s: simple fluid, nonlinear fluctuations, non-equilibrium statistical operator, distribution function, Fokker–Planck equation.

1. Introduction

The research of nonlinear kinetic and hydrodynamic fluctuations in dense gases, plasma, and liquids in such fields as turbulence phenomena, dynamics of phase transitions, chemical reactions, and self-organizing processes remains a challenging task in the statistical theory of non-equilibrium processes at both the kinetic and hydrodynamic description levels [1–29]. Non-equilibrium states of those systems are far from equilibrium, so important are the studies of both the processes of establishing stationary states with characteristic lifetimes and the processes of relaxation to already known non-equilibrium states, in particular, those described in the framework of molecular hydrodynamics [2, 30–32] in the case of liquids and dense gases. It is important to note that a specific feature of the research of non-equilibrium phenomena in dense gases, liquids, and dense plasma (dust plasma) consists in the consistent description of kinetic and hydrodynamic processes [33–37], as well as in the account for characteristic short- and long-range interactions between the particles in the system.

The development of kinetic equations making allowance for nonlinear hydrodynamic fluctuations [38–41] is an important problem in the theory of transport processes in dense gases and liquids. In particular, this problem arises, when describing low-frequency

anomalies in the kinetic equations and the related "long tails" of correlation functions [1, 42, 43]. In works [33, 44, 45], a consistent description of kinetic and hydrodynamic processes in dense gases and liquids based on the non-equilibrium statistical method of Zubarev operator [8,9] was proposed. In particular, this approach was applied to obtain, from the chain of BBGKY equations, the kinetic equation of the revised Enskog theory [45, 46] for the system of hard spheres and the Enskog–Landau kinetic equation for the one-component system of charged hard spheres. In work [33], non-Markovian transport equations were obtained for the non-equilibrium one-particle distribution function and the non-equilibrium value of the average potential energy of particle interaction. Later [34, 36], those equations were used to study the temporal correlation functions and the spectrum of collective excitations for weakly non-equilibrium processes in liquids. The approach of works [33, 44, 45] can evidently be applied to describe both weakly and strongly non-equilibrium systems.

At the same time, for the description of kinetic processes and nonlinear hydrodynamic fluctuations to be consistent, it is convenient to reformulate the theory in a way that would make it possible to obtain a set of equations for the non-equilibrium one-particle distribution function and the distribution functional for the hydrodynamic variables (particle concentrations, momenta, and energies). The idea of such an approach was formulated in works [47, 48]. In works [49, 50], we developed this approach and used the method of collective variables [51] to consistently describe the kinetic and hydrodynamic processes that are characterized by nonlinear fluctuations of particle concentrations, their momenta, and total energy

In this work, unlike works [49, 50], we introduce only the Fourier component of the particle concentration as a collective variable for the description of collective dynamic processes in the system, because it is related to the momentum density via a continuity equation, and the long-range part of the potential energy of particle interaction is also expressed in its term. In this case, the densities of the kinetic energy and the short-range part of the particle interaction potential are described in the coordinate-momentum space.

In Section 2, we obtain the non-equilibrium statistical operator for a non-equilibrium system state, when the parameters of the reduced description are the non-equilibrium one-particle distribution function, the non-equilibrium average value of the interaction potential energy density, and the non-equilibrium distribution function of the particle concentration (the collective variable).

In Section 3, we consider one of the methods to calculate the structural functions for the distributions of particle concentration and hydrodynamic particle velocities (in the approximation higher than the Gaussian one), which enter the generalized Fokker–Planck equation for the non-equilibrium distribution function of collective variables. In so doing, we separate the contributions from short- and long-range interactions between the particles. As a result, short-range interactions (for example, the hard-sphere model) can be described in the coordinate-momentum space, whereas long-range ones in the space of collective variables. Furthermore, the short-range component is considered as the basic one, which can be described using a chain of BBGKY equations for non-equilibrium distribution functions, in particular, in the case of the hard-sphere model [35].

2. Non-Equilibrium Distribution Function in the Method of Zubarev Non-Equilibrium Statistical Operator

For a consistent description of kinetic and hydrodynamic fluctuations in classical dense gases and liquids, it is necessary to select parameters for the reduced description of one-particle and collective processes. Unlike to what was done in works [49,50], now we choose the non-equilibrium one-particle distribution function $f_1(x;t) = \langle \hat{n}_1(x) \rangle^t$, the non-equilibrium average value of the particle interaction energy $H^{\rm int}(\mathbf{r},t) = \langle \hat{H}^{\rm int}(\mathbf{r}) \rangle^t$, and the non-equilibrium distribution function $f(\rho;t) = \langle \delta(\hat{\rho} - \rho) \rangle^t$ of collective variables corresponding to the particle concentration to play the role of such parameters. The microscopic phase density of the particle number, $\hat{n}_1(x)$, and the microscopic density of the potential energy of interaction between the particles in the system, $\hat{H}^{\rm int}(\mathbf{r})$, are given by the following expressions:

$$\hat{n}_1(x) = \sum_{j=1}^{N} \delta(x - x_j) = \sum_{j=1}^{N} \delta(\mathbf{r} - \mathbf{r}_j)\, \delta(\mathbf{p} - \mathbf{p}_j), \quad (2.1)$$

$$\hat{H}^{\rm int}(\mathbf{r}) = \frac{1}{2} \sum_{j \neq l=1}^{N} \Phi(|\mathbf{r}_{lj}|)\, \delta(\mathbf{r} - \mathbf{r}_j), \quad (2.2)$$

where $x_j = (\mathbf{r}_j, \mathbf{p}_j)$ are the coordinates and momenta of the particles in the phase space, and N is the total particle number in the system volume V. The microscopic phase distribution of collective variables ρ is written as follows:

$$\hat{f}(\rho) = \delta(\hat{\rho} - \rho) = \prod_{\mathbf{k}} \delta(\hat{\rho}_{\mathbf{k}} - \rho_{\mathbf{k}}), \tag{2.3}$$

where

$$\hat{\rho}_{\mathbf{k}} = \sum_{j=1}^{N} \mathrm{e}^{-i\mathbf{k}\mathbf{r}_j} \tag{2.4}$$

is the Fourier component of the particle concentration, and $\rho_{\mathbf{k}}$ is the corresponding collective variable. In the potential of pair interaction between the particles, $\Phi(|\mathbf{r}_{lj}|) = \Phi(|\mathbf{r}_l - \mathbf{r}_j|)$, let us separate the short-range, $\Phi^{\mathrm{sh}}(|\mathbf{r}_{lj}|)$, and long-range, $\Phi^{\mathrm{long}}(|\mathbf{r}_{lj}|)$, contributions:

$$\Phi(|\mathbf{r}_{lj}|) = \Phi^{\mathrm{sh}}(|\mathbf{r}_{lj}|) + \Phi^{\mathrm{long}}(|\mathbf{r}_{lj}|).$$

Accordingly, the non-equilibrium value of the potential energy density for interacting particles has the form

$$\langle \hat{H}^{\mathrm{int}}(\mathbf{r})\rangle^t = \langle \hat{H}^{\mathrm{sh}}(\mathbf{r})\rangle^t + \frac{1}{2V^2}\sum_{\mathbf{q},\mathbf{k}} \nu(\mathbf{k})\mathrm{e}^{i\mathbf{q}\mathbf{r}} \times$$
$$\times \Big(\langle \hat{\rho}_{\mathbf{q}+\mathbf{k}}\hat{\rho}_{-\mathbf{k}}\rangle^t - \langle \hat{\rho}_{\mathbf{q}}\rangle^t\Big),$$

where $\nu(\mathbf{k})$ is the Fourier component of the long-range part of the particle interaction potential. The non-equilibrium particle scattering function $\langle \hat{\rho}_{\mathbf{q}+\mathbf{k}}\hat{\rho}_{-\mathbf{k}}\rangle^t = F(\mathbf{q},\mathbf{k};t)$ is related to the non-equilibrium dynamic structural factor $S(\mathbf{q},\mathbf{k};\omega)$, which can be directly measured in neutron scattering processes.

The non-equilibrium average values $\langle \hat{n}_1(x)\rangle^t$, $\langle \hat{H}^{\mathrm{int}}(\mathbf{r})\rangle^t$, and $\langle \delta(\hat{\rho} - \rho)\rangle^t$ are calculated using the non-equilibrium N-particle distribution function $\varrho(x^N;t)$, which satisfies the Liouville equation. In accordance with the concept of reduced description of non-equilibrium state, this function is a functional,

$$\varrho(x^N;t) = \varrho\big(..., f_1(x;t), \quad \langle \hat{H}^{\mathrm{int}}(\mathbf{r})\rangle^t, f(\rho;t), ...\big).$$

To find the non-equilibrium distribution function $\varrho(x^N;t)$, we use the Zubarev method [48, 52], where the general solution of the Liouville equation that takes the projecting procedure into account can be presented in the form

$$\varrho(x^N;t) = \varrho_q(x^N;t) - \int_{-\infty}^{t} \mathrm{d}t'\, \mathrm{e}^{\epsilon(t'-t)} T_q(t,t') \times$$
$$\times \big(1 - P_q(t')\big), iL_N \varrho_q(x^N;t'), \tag{2.5}$$

where $\epsilon \to +0$ after the thermodynamic limit transition, which selects retarded solutions of the Liouville equation with the operator iL_N, and

$$T_q(t,t') = \exp_+\left(-\int_{t'}^{t} dt'\,\big(1 - P_q(t')\big) iL_N\right)$$

is the generalized operator of time evolution, where the Kawasaki–Ganton projection operator $P_q(t)$ is taken into account. The structure of $P_q(t)$ depends on the relevant distribution function $\varrho_q(x^N;t)$. In the Zubarev method, the latter is determined from the extremum of information entropy at the fixed values of the reduced description parameters – in our case, these are $f_1(x;t)$, $\langle \hat{H}^{\mathrm{int}}(\mathbf{r})\rangle^t$, and $f(\rho;t)$ – and provided the preservation of the normalization condition

$$\int \mathrm{d}\Gamma_N \varrho_q(x^N;t) = 1, \tag{2.6}$$

where

$$\mathrm{d}\Gamma_N = \frac{(\mathrm{d}x)^N}{N!} = \frac{(\mathrm{d}x_1, ..., \mathrm{d}x_N)}{N!}, \quad \mathrm{d}x = \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{p}.$$

Hence, the relevant distribution function can be written in the form

$$\varrho_q(x^N;t) = \exp\Big[-\Phi(t) - \int \mathrm{d}\mathbf{r}\beta(\mathbf{r};t)\,\hat{H}^{\mathrm{int}}(\mathbf{r}) -$$
$$- \int \mathrm{d}x\gamma(x;t)\,\hat{n}_1(x) - \int \mathrm{d}\rho\, F(\rho;t)\hat{f}(\rho)\Big], \tag{2.7}$$

where $\mathrm{d}\rho = \prod_{\mathbf{k}} \mathrm{d}\rho_{\mathbf{k}}$, $\Phi(t)$ is the Massier–Planck functional, which is determined from the normalization condition for the relevant distribution function

$$\Phi(t) = \ln \int \mathrm{d}\Gamma_N \exp\Big[-\int \mathrm{d}\mathbf{r}\beta(\mathbf{r};t)\,\hat{H}^{\mathrm{int}}(\mathbf{r}) -$$
$$- \int \mathrm{d}x\gamma(x;t)\,\hat{n}_1(x) - \int \mathrm{d}\rho F(\rho;t)\hat{f}(\rho)\Big].$$

The Lagrange multipliers $\gamma(x;t)$, $\beta(\mathbf{r};t)$ and $F(\rho;t)$ are determined from the self-consistency conditions

$$f_1(x;t) = \langle \hat{n}_1(x)\rangle^t = \langle \hat{n}_1(x)\rangle_q^t,$$
$$\langle \hat{H}^{\mathrm{int}}(\mathbf{r})\rangle^t = \langle\, \hat{H}^{\mathrm{int}}(\mathbf{r})\rangle_q^t, \tag{2.8}$$
$$f(\rho;t) = \langle \delta(\hat{\rho} - \rho)\rangle^t = \langle \delta(\hat{\rho} - \rho)\rangle_q^t,$$

where $\langle ... \rangle_q^t = \int \mathrm{d}\Gamma_N \, ... \, \varrho_q(x^N;t)$. To reveal the internal structure of the non-equilibrium distribution function $\varrho(x^N;t)$, let us exclude the function $F(\rho;t)$ from the relevant distribution function. For this purpose, let us apply the self-consistency conditions (2.8). As a result, we get

$$\varrho_q(x^N;t) = \varrho_q^{\mathrm{kin-hyd}}(x^N;t) \frac{f(\rho;t)}{W(\rho;t)}\Big|_{\rho=\hat{\rho}}, \tag{2.9}$$

where

$$W(\rho;t) = \int \mathrm{d}\Gamma_N \, \mathrm{e}^{-\Phi(t) - \int \mathrm{d}\mathbf{r}\beta(\mathbf{r};t)\hat{H}^{\mathrm{int}}(\mathbf{r}) - \int \mathrm{d}x\gamma(x;t)\hat{n}_1(x)} \times$$
$$\times \hat{f}(\rho) = \int \mathrm{d}\Gamma_N \varrho_q^{\mathrm{kin-hyd}}(x^N;t)\hat{f}(\rho) \tag{2.10}$$

is the non-equilibrium structural function for the distribution of collective variables, which is the Jacobian of the transition $\hat{f}(\rho)$ into the space of collective variables $\rho_{\mathbf{k}}$. In so doing, the averaging in Eq. (2.10) is performed with the relevant distribution function

$$\varrho_q^{\mathrm{kin-hyd}}(x^N;t) = \exp\Big\{ - \Phi(t) -$$
$$- \int \mathrm{d}\mathbf{r}\beta(\mathbf{r};t)\hat{H}^{\mathrm{int}}(\mathbf{r}) - \int \mathrm{d}x\gamma(x;t)\hat{n}_1(x)\Big\}, \tag{2.11}$$

which was constructed in works [33, 34, 36] while consistently describing the kinetic and hydrodynamic processes in systems of interacting particles. The relevant distribution function (2.9) corresponds to the Gibbs entropy

$$S(t) = -\langle \ln \varrho_q(x^N;t)\rangle_q^t = \Phi(t) +$$
$$+ \int \mathrm{d}\mathbf{r}\beta(\mathbf{r};t)\langle \hat{H}^{\mathrm{int}}(\mathbf{r})\rangle^t + \int \mathrm{d}x\gamma(x;t)\langle \hat{n}_1(x)\rangle^t +$$
$$+ \int \mathrm{d}\rho f(\rho;t) \ln \frac{f(\rho;t)}{W(\rho;t)}, \tag{2.12}$$

which, together with self-consistency conditions (2.8), can be considered as the entropy of a non-equilibrium state.

To obtain an explicit form for the non-equilibrium distribution function according to Eq. (2.5), it is necessary to apply the Liouville and Kawasaki–Ganton operators to the function $\varrho_q(x^N;t)$. According to Eq. (2.9), the Kawasaki–Ganton projection operator has the following structure:

$$P_q(t)\varrho' = \varrho_q(x^N;t) \int \mathrm{d}\Gamma_N \varrho' + \int \mathrm{d}x \frac{\partial \varrho_q(x^N;t)}{\partial \langle \hat{n}_1(x)\rangle^t} \times$$
$$\times \Big[\int \mathrm{d}\Gamma_N \hat{n}_1(x)\varrho' - \langle \hat{n}_1(x)\rangle^t \int \mathrm{d}\Gamma_N \varrho' \Big] +$$
$$+ \int \mathrm{d}\mathbf{r} \frac{\partial \varrho_q(x^N;t)}{\partial \langle \hat{H}^{\mathrm{int}}(\mathbf{r})\rangle^t} \Big[\int \mathrm{d}\Gamma_N \hat{H}^{\mathrm{int}}(\mathbf{r})\varrho' -$$
$$- \langle \hat{H}^{\mathrm{int}}(\mathbf{r})\rangle^t \int \mathrm{d}\Gamma_N \varrho' \Big] +$$
$$+ \int \mathrm{d}\rho \frac{\partial \varrho_q(x^N;t)}{\partial\big(\frac{f(\rho;t)}{W(\rho;t)}\big)} \frac{1}{W(\rho;t)} \Big[\int \mathrm{d}\Gamma_N \hat{f}(\rho)\varrho' -$$
$$- f(a;t) \int \mathrm{d}\Gamma_N \varrho' \Big] +$$
$$+ \int \mathrm{d}x \int \mathrm{d}\rho \frac{\partial \varrho_q(x^N;t)}{\partial\big(\frac{f(\rho;t)}{W(\rho;t)}\big)} \frac{f(\rho;t)}{W(\rho;t)} \frac{\partial \ln W(\rho;t)}{\partial \langle \hat{n}_1(x)\rangle^t} \times$$
$$\times \Big[\int \mathrm{d}\Gamma_N \hat{n}_1(x)\varrho' - \langle \hat{n}_1(x)\rangle^t \int \mathrm{d}\Gamma_N \varrho' \Big] +$$
$$+ \int \mathrm{d}\mathbf{r} \int \mathrm{d}\rho \frac{\partial \varrho_q(x^N;t)}{\partial\big(\frac{f(\rho;t)}{W(\rho;t)}\big)} \frac{f(\rho;t)}{W(\rho;t)} \frac{\partial \ln W(\rho;t)}{\partial \langle \hat{H}^{\mathrm{int}}(\mathbf{r})\rangle^t} \times$$
$$\times \Big[\int \mathrm{d}\Gamma_N \hat{H}^{\mathrm{int}}(\mathbf{r})\varrho' - \langle \hat{H}^{\mathrm{int}}(\mathbf{r})\rangle^t \int \mathrm{d}\Gamma_N \varrho' \Big]. \tag{2.13}$$

First, let us apply the Liouville operator to the relevant distribution function (2.9). As a result, we obtain

$$iL_N \varrho_q(x^N;t) = - \int \mathrm{d}x\gamma(x;t)\dot{\hat{n}}_1(x)\varrho_q(x^N;t) -$$
$$- \int \mathrm{d}\mathbf{r}\beta(\mathbf{r};t)\dot{\hat{H}}^{\mathrm{int}}(\mathbf{r})\varrho_q(x^N;t) +$$
$$+ \Big[iL_N \frac{f(\rho;t)}{W(\rho;t)}\Big|_{\rho=\hat{\rho}} \Big] \varrho_q^{\mathrm{kin-hyd}}(x^N;t), \tag{2.14}$$

where $\dot{\hat{n}}_1(x) = iL_N\hat{n}_1(x)$ and $\dot{\hat{H}}^{\mathrm{int}}(\mathbf{r}) = iL_N\hat{H}^{\mathrm{int}}(\mathbf{r})$. Using the relation

$$iL_N\hat{f}(\rho) = iL_N\hat{f}(\rho_{\mathbf{k}}) = \sum_{\mathbf{k}} \Big[\frac{\partial}{\partial \rho_{\mathbf{k}}} \hat{f}(\rho)\dot{\hat{\rho}}_{\mathbf{k}} \Big],$$

where $\dot{\hat{\rho}}_{\mathbf{k}} = iL_N\hat{\rho}_{\mathbf{k}}$, the last term in Eq. (2.14) can be rewritten in the form

$$\Big[iL_N \frac{f(\rho;t)}{W(\rho;t)}\Big|_{\rho=\hat{\rho}} \Big] \varrho_q^{\mathrm{kin-hyd}}(x^N;t) =$$
$$= \int \mathrm{d}\rho \sum_{\mathbf{k}} \Big[\dot{\hat{\rho}}_{\mathbf{k}} W(a;t) \Big(\frac{\partial}{\partial \rho_{\mathbf{k}}} \frac{f(\rho;t)}{W(\rho;t)} \Big) \Big] \varrho_L(x^N, \rho;t). \tag{2.15}$$

In this expression, we introduced a new relevant distribution function $\varrho_L(x^N,\rho;t)$ with the microscopic distribution of collective variables

$$\varrho_L(x^N,\rho;t)=\varrho_q^{\rm kin-hyd}(x^N;t)\frac{\hat f(\rho)}{W(\rho;t)}, \tag{2.16}$$

which is related to $\varrho_q(x^N;t)$ via the relation

$$\varrho_q(x^N;t)=\int {\rm d}\rho f(\rho;t)\varrho_L(x^N,\rho;t) \tag{2.17}$$

and is normalized,

$$\int {\rm d}\Gamma_N\varrho_L(x^N,\rho;t)=1. \tag{2.18}$$

Using relation (2.16), the average values calculated using the relevant distribution functions, can be expressed as follows:

$$\begin{aligned}&\langle ...\rangle_q^t=\int {\rm d}\rho\langle ...\rangle_L^t f(\rho;t),\\ &\langle ...\rangle_L^t=\int {\rm d}\Gamma_N ...\varrho_L(x^N,\rho;t).\end{aligned} \tag{2.19}$$

Now, taking Eqs. (2.15) and (2.16) into account, the result of the Liouville operator action on $\varrho_q(x^N;t)$ looks like

$$\begin{aligned}&iL_N\varrho_q(x^N;t)=\\ &=-\int {\rm d}\rho\int dx\gamma(x;t)\dot{\hat n}_1(x)\varrho_L(x^N,\rho;t)f(\rho;t)-\\ &-\int {\rm d}\rho\int d{\bf r}\beta({\bf r};t)\dot{\hat H}^{\rm int}({\bf r})\varrho_L(x^N,\rho;t)f(\rho;t)+\\ &+\int {\rm d}\rho\sum_{\bf k}\left[\dot{\hat\rho}_{\bf k}W(\rho;t)\frac{\partial}{\partial\rho_{\bf k}}\frac{f(\rho;t)}{W(\rho;t)}\right]\varrho_L(x^N,\rho;t).\end{aligned} \tag{2.20}$$

Substituting this expression into Eq. (2.5), we obtain the non-equilibrium distribution function in the form

$$\begin{aligned}&\varrho(x^N;t)=\int {\rm d}\rho f(\rho;t)\varrho_L(x^N,\rho;t)+\\ &+\int {\rm d}\rho\int {\rm d}{\bf r}\int\limits_{-\infty}^{t}{\rm d}t' e^{\epsilon(t'-t)}T_q(t,t')(1-P_q(t'))\times\\ &\times\dot{\hat H}^{\rm int}({\bf r})\varrho_L(x^N,\rho;t)f(\rho;t')\beta({\bf r};t')-\\ &-\int {\rm d}\rho\int {\rm d}x\int\limits_{-\infty}^{t}{\rm d}t' e^{\epsilon(t'-t)}T_q(t,t')\times\\ &\times(1-P_q(t'))\dot{\hat n}_1(x)\varrho_L(x^N,\rho;t')f(\rho;t')\gamma(x;t')-\\ &-\int {\rm d}\rho\sum_{\bf k}\int\limits_{-\infty}^{t}{\rm d}t' e^{\epsilon(t'-t)}T_q(t,t')(1-P_q(t'))\times\\ &\times\left[\dot{\hat\rho}_{\bf k}W(\rho;t')\frac{\partial}{\partial\rho_{\bf k}}\frac{f(\rho;t')}{W(\rho;t')}\right]\varrho_L(x^N,\rho;t'),\end{aligned} \tag{2.21}$$

Making use of it, we get the corresponding generalized transport equations for the reduced description parameters:

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\frac{\bf p}{m}\frac{\partial}{\partial{\bf r}}\right]f_1(x;t)-\int {\rm d}x'\frac{\partial}{\partial{\bf r}}\Phi(|{\bf r}-{\bf r}'|)\times\\ &\times\left[\frac{\partial}{\partial{\bf p}}-\frac{\partial}{\partial{\bf p}'}\right]g_2(x,x';t)=-\int {\rm d}{\bf r}'\int d\rho\times\\ &\times\int\limits_{-\infty}^{t}{\rm d}t' e^{\epsilon(t'-t)}\phi_{nH}^{\rm int}(x,{\bf r}',\rho;t,t')f(\rho;t')\beta({\bf r}';t')-\\ &-\int {\rm d}x'\int {\rm d}\rho\int\limits_{-\infty}^{t}{\rm d}t' e^{\epsilon(t'-t)}\phi_{nn}(x,x',\rho;t,t')\times\\ &\times f(\rho;t')\gamma(x';t')-\sum_{\bf k}\int {\rm d}a\int\limits_{-\infty}^{t}{\rm d}t' e^{\epsilon(t'-t)}\times\\ &\times\left\{\phi_{n\rho}(x,{\bf k},\rho;t,t')\frac{\partial}{\partial\rho_{\bf k}}\right\}\frac{f(\rho;t')}{W(\rho;t')},\end{aligned} \tag{2.22}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\langle\hat H^{\rm int}({\bf r})\rangle^t=\langle\dot{\hat H}^{\rm int}({\bf r})\rangle_q^t-\int {\rm d}{\bf r}'\int {\rm d}\rho\times\\ &\times\int\limits_{-\infty}^{t}{\rm d}t' e^{\epsilon(t'-t)}\phi_{HH}^{\rm int}({\bf r},{\bf r}',\rho;t,t')f(\rho;t')\beta({\bf r}';t')-\\ &-\int {\rm d}x'\int {\rm d}\rho\int\limits_{-\infty}^{t}{\rm d}t' e^{\epsilon(t'-t)}\phi_{Hn}^{\rm int}({\bf r},x',\rho;t,t')\times\\ &\times f(\rho;t')\gamma(x';t')-\sum_{\bf k}\int {\rm d}\rho\int\limits_{-\infty}^{t}{\rm d}t' e^{\epsilon(t'-t)}\times\\ &\times\left\{\phi_{H\varepsilon}^{\rm int}({\bf r},{\bf k},\rho;t,t')\frac{\partial}{\partial\rho_{\bf k}}\right\}\frac{f(\rho;t')}{W(\rho;t')},\end{aligned} \tag{2.23}$$

$$\begin{aligned}&\frac{\partial}{\partial t}f(\rho;t)=\sum_{\bf k}\frac{\delta}{\delta\rho_{\bf k}}v_\rho({\bf k};t)f(\rho;t)-\sum_{\bf k}\frac{\delta}{\delta\rho_{\bf k}}\int {\rm d}{\bf r}'\times\\ &\times\int {\rm d}\rho'\int\limits_{-\infty}^{t}{\rm d}t' e^{\epsilon(t'-t)}\phi_{\rho H}({\bf r}',{\bf k},\rho,\rho';t,t')\times\\ &\times f(\rho';t')\beta({\bf r}';t')-\sum_{\bf k}\frac{\delta}{\delta\rho_{\bf k}}\int {\rm d}x'\int {\rm d}\rho'\times\end{aligned}$$

$$\times \int\limits_{-\infty}^{t} \mathrm{d}t' e^{\epsilon(t'-t)} \phi_{\rho n}(x', \mathbf{k}, \rho, \rho'; t, t') f(\rho'; t') \gamma(x'; t') +$$
$$+ \sum_{\mathbf{k},\mathbf{q}} \int \mathrm{d}\rho' \int\limits_{-\infty}^{t} \mathrm{d}t' e^{\epsilon(t'-t)} \frac{\delta}{\delta \rho_{\mathbf{k}}} \phi_{\rho\rho} \times$$
$$\times (\mathbf{k}, \mathbf{q}, \rho, \rho'; t, t') \frac{\delta}{\delta \rho_{\mathbf{q}}} \frac{f(\rho'; t')}{W(\rho'; t')}. \quad (2.24)$$

The generalized transport equations (2.22)–(2.24) contain the relevant particle pair distribution function

$$g_2(x, x'; t) = \int \mathrm{d}\Gamma_{N-2} \varrho(x^N; t) =$$
$$= \int \mathrm{d}\rho g_2^L(x, x'; \rho; t) f(\rho; t), \quad (2.25)$$

where $g_2^L(x, x'; \rho; t) = \int \mathrm{d}\Gamma_{N-2} \varrho_L(x^N, \rho; t)$ is the relevant particle L-pair distribution function.

The generalized transfer kernels

$$\phi_{\alpha\beta}(t, t') = \langle I_\alpha(t) T_q(t, t') I_\beta(t') \rangle_L^{t'}, \ \alpha, \beta = \{n, H, \rho\}, \quad (2.26)$$

entering the transfer equation describe non-Markovian processes; they are non-equilibrium correlation functions. They are built on generalized fluxes

$$\hat{I}_n(x; t) = \big(1 - P(t)\big) \dot{\hat{n}}_1(x), \quad (2.27)$$
$$\hat{I}_H^{\mathrm{int}}(\mathbf{r}; t) = \big(1 - P(t)\big) \dot{\hat{H}}^{\mathrm{int}}(\mathbf{r}), \quad (2.28)$$
$$\hat{I}_\rho(\mathbf{k}; t) = \big(1 - P(t)\big) \dot{\hat{\rho}}_{\mathbf{k}}, \quad (2.29)$$

where $P(t)$ is the generalized Mori operator related to the Kawasaki–Ganton projection operator $P_q(t)$ via the relationship

$$P_q(t) a(x) \varrho_q(x^N; t) = \varrho_q(x^N; t) P(t) a(x).$$

It is important to note that the average values in Eq. (2.26) are calculated using the distribution function $\varrho_L(x^N, \rho; t)$ [Eq. (2.16)]. Therefore, the transfer kernels are functions of the collective variables $\rho_{\mathbf{k}}$.

In Eq. (2.24), the functions $v_\rho(\mathbf{k}; t)$ are fluxes in the space of collective variables. They are called hydrodynamic velocities and defined as follows:

$$v_\rho(\mathbf{k}; t) = \int \mathrm{d}\Gamma_N \dot{\hat{\rho}}_{\mathbf{k}} \varrho_L(x^N, \rho; t) = \langle \dot{\hat{\rho}}_{\mathbf{k}} \rangle_L^t. \quad (2.30)$$

The presented system of transport equations (2.22)–(2.24) gives a consistent description of kinetic and hydrodynamic processes in classical liquids and dense gases that accounts for long-term fluctuations. This system of equations is not closed with respect to the Lagrange parameters $\beta(\mathbf{r}; t)$ and $\gamma(x; t)$, which are determined from the corresponding self-consistency conditions. It should be noted that if the kinetic processes and the contribution of the average potential energy are neglected, we obtain the following generalized (non-Markovian) Fokker–Planck equation for the non-equilibrium distribution function of collective variables (this equation can also be obtained using the Zwanzig projection operator method or the Zubarev method [52]):

$$\frac{\partial}{\partial t} f(\rho; t) = \sum_{\mathbf{k}} \frac{\delta}{\delta \rho_{\mathbf{k}}} v_\rho(\mathbf{k}; t) f(\rho; t) +$$
$$+ \sum_{\mathbf{k},\mathbf{q}} \int \mathrm{d}\rho' \int\limits_{-\infty}^{t} \mathrm{d}t' e^{\epsilon(t'-t)} \frac{\delta}{\delta \rho_{\mathbf{k}}} \times$$
$$\times \phi_{\rho\rho} (\mathbf{k}, \mathbf{q}, \rho, \rho'; t, t') \frac{\delta}{\delta \rho_{\mathbf{q}}} \frac{f(\rho'; t')}{W(\rho'; t')}. \quad (2.31)$$

One of the main problems in the analysis of transport equations (2.22)–(2.24) and transfer kernels (2.26) is the calculation of the structural function of collective variables, $W(\rho; t)$, and the hydrodynamic velocities $v_\rho(\mathbf{k}; t)$.

3. Calculation of Structural Function $W(\rho; t)$ and Hydrodynamic Velocities $v_\rho(\mathbf{k}; t)$ Using the Method of Collective Variables

Let us calculate the structural function $W(\rho; t)$ and the hydrodynamic velocities $v_\rho(\mathbf{k}; t)$ in the framework of the method of collective variables [49–51, 54, 55]. First, we should calculate the structural function $W(\rho; t)$ for collective variables in the case where the interaction between the particles is described by the short-range potential $\Phi^{\mathrm{sh}}(|\mathbf{r}_{lj}|)$ – e.g., the hard-sphere potential – at short distances, and by a certain long-range potential $\Phi^{\mathrm{long}}(|\mathbf{r}_{lj}|)$ beyond them. Accordingly, let us separate short- and long-range interactions between the particles in the Liouville operator,

$$iL_N = iL_N^0 + \hat{T}_N + iL_N^{\mathrm{long}},$$

where iL_N^0 is the Liouville operator for N non-interacting particles, $\hat{T}_N$ is the scattering operator of the system in the case of the hard-sphere model [35, 38, 41, 45], and iL_N^{long} is the potential part of the

Liouville operator with the long-range interaction potential between the particles.

At the next stage, we apply the integral representation for the δ-function and present $\hat{f}(\rho)$ in the form

$$\hat{f}(\rho) = \int d\omega \exp\left\{- i\pi \sum_{\mathbf{k}} \omega_{\mathbf{k}} (\hat{\rho}_{\mathbf{k}} - \rho_{\mathbf{k}})\right\}. \tag{3.1}$$

Using the cumulant expansion [49, 50, 55] for $W(\rho; t)$, we obtain

$$W(\rho; t) = \int d\Gamma_N \varrho_{\rm rel}^{\rm kin-hyd}(x^N; t)\hat{f}(\rho) =$$
$$= \int d\omega \exp\left\{- i\pi \sum_{\mathbf{k}} \omega_{\mathbf{k}}\, \bar{\rho}_{\mathbf{k}} - \right.$$
$$- \frac{1}{2V^2} \sum_{\mathbf{q}} \sum_{\mathbf{k}} \beta_{-\mathbf{q}}(t)\, \nu\,(\mathbf{k})(\rho_{\mathbf{q}+\mathbf{k}}\rho_{-\mathbf{k}} - \rho_{\mathbf{q}}) -$$
$$\left. - \frac{\pi^2}{2} \sum_{\mathbf{k}_1, \mathbf{k}_2} \mathfrak{M}_2(\mathbf{k}_1, \mathbf{k}_2; t)\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2}\right\} \exp\left\{\sum_{n\geq 3} D_n(\omega; t)\right\}, \tag{3.2}$$

where

$$\bar{\rho}_{\mathbf{k}} = \rho_{\mathbf{k}} - \langle\hat{\rho}_{\mathbf{k}}\rangle^t_{\rm kin-sh}, \quad d\omega = \Pi_{\mathbf{k}} d\omega^r_{\mathbf{k}} d\omega^s_{\mathbf{k}},$$
$$\omega_{\mathbf{k}} = \omega^r_{\mathbf{k}} - i\omega^s_{\mathbf{k}}, \omega_{-\mathbf{k}} = \omega^*_{\mathbf{k}},$$

and

$$D_n(\omega; t) = \frac{(-i\pi)^n}{n!} \times$$
$$\times \sum_{\mathbf{k}_1, ..., \mathbf{k}_n} \mathfrak{M}_n(\mathbf{k}_1, ..., \mathbf{k}_n; t)\, \omega_{\mathbf{k}_1} \ldots \omega_{\mathbf{k}_n}. \tag{3.3}$$

The non-equilibrium cumulative averages of the nth order

$$\mathfrak{M}_n(\mathbf{k}_1, ..., \mathbf{k}_n; t) = \langle\hat{\rho}_{\mathbf{k}_1}, ..., \hat{\rho}_{\mathbf{k}_n}\rangle^{t,c}_{\rm kin-sh} \tag{3.4}$$

where the superscript c means the cumulative average, are calculated using the relevant distribution function and the short-range interaction potential between the particles,

$$\varrho_q^{\rm kin-sh}(x^N; t) = \exp\left\{- \Phi(t) - \right.$$
$$\left. - \int d\mathbf{r}\beta(\mathbf{r}; t)\hat{H}^{\rm sh}(\mathbf{r}) - \int dx\gamma(x; t)\hat{n}_1(x)\right\}, \tag{3.5}$$

It is important to note that contributions from short- and long-range interactions in Eq. (3.2) are separated. Short-range interactions are taken into account in the relevant distribution (3.5), which can be considered as basic, and long-range interactions are presented in terms of collective variables,

$$\int d\mathbf{r}\beta(\mathbf{r}; t)\hat{H}^{\rm long}(\mathbf{r}) = \frac{1}{2V^2} \sum_{\mathbf{q}} \sum_{\mathbf{k}} \beta_{-\mathbf{q}}(t)\nu(\mathbf{k}) \times$$
$$\times \left(\rho_{\mathbf{q}+\mathbf{k}}\rho_{-\mathbf{k}} - \rho_{\mathbf{q}}\right).$$

For further calculations, the structural function $W(\rho; t)$ is presented in the form

$$W(\rho; t) = W_\beta(\rho; t) \int d\omega \exp\left\{- i\pi \sum_{\mathbf{k}} \omega_{\mathbf{k}}\bar{\rho}_{\mathbf{k}} - \right.$$
$$\left. - \frac{\pi^2}{2} \sum_{\mathbf{k}_1, \mathbf{k}_2} \mathfrak{M}_2(\mathbf{k}_1, \mathbf{k}_2; t)\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2}\right\} \times$$
$$\times \left(1 + B + \frac{1}{2!}\, B^2 + \frac{1}{3!}\, B^3 + ... + \frac{1}{n!}\, B^n + ...\right), \tag{3.6}$$

where

$$W_\beta(\rho; t) = \exp\left(- \frac{1}{2V^2} \sum_{\mathbf{q}} \sum_{\mathbf{k}} \beta_{-\mathbf{q}}(t)\, \nu\,(\mathbf{k}) \times \right.$$
$$\left. \times \left(\rho_{\mathbf{q}+\mathbf{k}}\rho_{-\mathbf{k}} - \rho_{\mathbf{q}}\right)\right) \tag{3.7}$$

and $B = \sum_{n\geq 3} D_n(\omega; t)$. If the series expansion of the exponent $\exp\{\sum_{n\geq 3} D_n(\omega; t)\}$ in Eq. (3.6) is confined to the first term, which is equal to unity, we obtain the Gaussian approximation for $W(\rho; t)$,

$$W^G(\rho; t) = W_\beta(\rho; t) \int d\omega \exp\left\{i\pi \sum_{\mathbf{k}} \omega_{\mathbf{k}}\bar{\rho}_{\mathbf{k}} - \right.$$
$$\left. - \frac{\pi^2}{2} \sum_{\mathbf{k}_1, \mathbf{k}_2} \mathfrak{M}_2(\mathbf{k}_1, \mathbf{k}_2; t)\omega_{\mathbf{k}_1}\, \omega_{\mathbf{k}_2}\right\}, \tag{3.8}$$

where the non-equilibrium cumulative density-density average has the form

$$\mathfrak{M}_2(\mathbf{k}_1, \mathbf{k}_2; t) = \langle\hat{\rho}_{\mathbf{k}_1}\hat{\rho}_{\mathbf{k}_2}\rangle^{t,c}_{\rm kin-sh} =$$
$$= \langle\hat{\rho}_{\mathbf{k}}\hat{\rho}_{-\mathbf{k}}\rangle^t_{\rm kin-sh} - \langle\hat{\rho}_{\mathbf{k}}\rangle^t_{\rm kin-sh}\langle\hat{\rho}_{-\mathbf{k}}\rangle^t_{\rm kin-sh}.$$

To integrate over ω in Eq. (3.8), it is necessary to transform the expression in the exponent into the quadratic diagonal form in $\omega_{\mathbf{k}}$. In this regard, it is necessary to determine the eigenvalues by solving the equation

$$\det\left|\tilde{\mathfrak{M}}_2(\mathbf{k}_1, \mathbf{k}_2; t) - \tilde{E}(\mathbf{k}; t)\right| = 0, \tag{3.9}$$

where $\tilde{E}(\mathbf{k};t)$ is a diagonal matrix. Taking this circumstance into account, we get

$$W^G(\rho;t) = W_\beta(\rho;t)\int d\tilde{\omega}\,\exp\Big\{-i\pi\sum_{\mathbf{k}}\tilde{\rho}_{\mathbf{k}}\tilde{\omega}_{\mathbf{k}} - \\ -\frac{\pi^2}{2}\sum_{\mathbf{k}}E(\mathbf{k};t)\tilde{\omega}_{\mathbf{k}}\tilde{\omega}_{-\mathbf{k}}\Big\}. \quad (3.10)$$

The integrand in Eq. (3.10) is a quadratic function of $\tilde{\omega}_{\mathbf{k}}$. Therefore, integrating over $\omega_{\mathbf{k}}$, we obtain the following expression for the structural function in the Gaussian approximation, $W^G(\rho;t)$:

$$W^G(\rho;t) = W_\beta(\rho;t)\times \\ \times\exp\Big\{-\frac{1}{2}\sum_{\mathbf{k}}E^{-1}(\mathbf{k};t)\tilde{\rho}_{\mathbf{k}}\tilde{\rho}_{-\mathbf{k}}\Big\}\times \\ \times\exp\Big\{-\frac{1}{2}\sum_{\mathbf{k}}\ln\,\pi\,\det\tilde{E}(\mathbf{k};t)\Big\}, \quad (3.11)$$

or, in terms of the variables $\bar{\rho}_{\mathbf{k}}$,

$$W^G(\rho;t) = Z(t)W_\beta(\rho;t)\times \\ \times\exp\Big\{-\frac{1}{2}\sum_{\mathbf{k}}\bar{E}(\mathbf{k};t)\,\bar{\rho}_{\mathbf{k}}\,\bar{\rho}_{-\mathbf{k}}\Big\}, \quad (3.12)$$

where

$$Z(t) = \exp\Big\{-\frac{1}{2}\sum_{\mathbf{k}}\ln\,\pi\,\det\tilde{E}(\mathbf{k};t)\Big\}.$$

The structural function in the Gaussian approximation, $W^G(\rho;t)$, makes it possible to calculate the complete structural function (3.2) in higher approximations in the Gaussian momenta [49, 50]:

$$W(\rho;t) = W_\beta(\rho;t)\bar{W}^G(\rho;t)\times \\ \times\exp\Big\{\sum_{n\geq 3}\langle\tilde{D}_n(\rho;t)\rangle_G\Big\}, \quad (3.13)$$

here

$$\bar{W}^G(\rho;t) = Z(t)\,\exp\Big\{-\frac{1}{2}\sum_{\mathbf{k}}\bar{E}(\mathbf{k};t)\bar{\rho}_{\mathbf{k}}\bar{\rho}_{-\mathbf{k}}\Big\} \quad (3.14)$$

and $\langle\tilde{D}_n(\rho;t)\rangle_G$ can be approximately represented as follows:

$$\langle\tilde{D}_3(\rho;t)\rangle_G = \langle\bar{D}_3(\rho;t)\rangle_G,$$
$$\langle\tilde{D}_4(\rho;t)\rangle_G = \langle\bar{D}_4(\rho;t)\rangle_G,$$
$$\langle\tilde{D}_6(\rho;t)\rangle_G = \langle\bar{D}_6(\rho;t)\rangle_G - \frac{1}{2}\langle\bar{D}_3(\rho;t)\rangle_G^2,$$
$$\langle\tilde{D}_8(\rho;t)\rangle_G = \langle\bar{D}_8(\rho;t)\rangle_G - \langle\bar{D}_3(\rho;t)\rangle_G\langle\bar{D}_5(\rho;t)\rangle_G - \\ -\frac{1}{2}\langle\bar{D}_4(\rho;t)\rangle_G^2,$$
$$\langle\tilde{D}_n(\rho;t)\rangle_G = \frac{1}{\bar{W}^G(\rho;t)}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n}\bar{\mathfrak{M}}_n(\mathbf{k}_1,\ldots,\mathbf{k}_n;t)\times \\ \times\frac{1}{(i\pi)^n}\,\frac{\delta^n}{\delta\bar{\rho}_{\mathbf{k}_1}\ldots\delta\bar{\rho}_{\mathbf{k}_n}}\,\bar{W}^G(\rho;t),$$

where $\langle\tilde{D}_n(\rho;t)\rangle_G$ are the nth renormalized nonequilibrium cumulative averages for the variables $\bar{\rho}_{\mathbf{k}}$ of higher orders.

The method used for the calculation of the structural function $W(\rho;t)$ can be applied for approximate calculations of the hydrodynamic velocities $v_\rho(\mathbf{k};t)$. According to their definition (2.30), the general formula for the hydrodynamic velocities can be written in the form

$$v_\rho(\mathbf{k};t) = \int d\Gamma_N\dot{\hat{\rho}}_{\mathbf{k}}\varrho_{\rm rel}^{\rm kin-hyd}(x^N;t)\hat{f}(\rho). \quad (3.15)$$

Additionally, let us introduce the function

$$W(\rho,\lambda;t) = \int d\Gamma_N\;{\rm e}^{-i\pi\sum_{\mathbf{k}}\lambda_{\mathbf{k}}\dot{\hat{\rho}}_{\mathbf{k}}}\times \\ \times\varrho_{\rm rel}^{\rm kin-hyd}(x^N;t)\hat{f}(\rho), \quad (3.16)$$

so that

$$v_\rho(\mathbf{k};t) = \frac{\partial}{\partial(-i\pi\lambda_{\mathbf{k}})}\,\ln W(\rho,\lambda;t)\Big|_{\lambda_{\mathbf{k}}=0}. \quad (3.17)$$

We will calculate the function $W(\rho,\lambda;t)$ using the calculation results obtained for the structural function $W(\rho;t)$. For this purpose, $W(\rho,\lambda;t)$ should be rewritten as follows:

$$W(\rho,\lambda;t) = \int d\Gamma_N\int d\omega\,\exp\Big\{-i\pi\sum_{\mathbf{k}}\lambda_{\mathbf{k}}\dot{\hat{\rho}}_{\mathbf{k}}\Big\}\times \\ \times\exp\Big\{-i\pi\sum_{\mathbf{k}}\omega_{\mathbf{k}}(\hat{\rho}_{\mathbf{k}}-\rho_{\mathbf{k}})\Big\}\varrho_{\rm rel}^{\rm kin-hyd}(x^N;t). \quad (3.18)$$

Now, let us take averaging (3.18) with $\varrho_q^{\rm kin-sh}(x^N;t)$ into account and use the cumulative expansion. Then,

$$W(\rho,\lambda;t) = W_\beta(\rho,\lambda;t)\int d\omega\,\exp\Big\{-i\pi\sum_{\mathbf{k}}\omega_{\mathbf{k}}\bar{\rho}_{\mathbf{k}} + \\ +\sum_{n\geq 1}\big[D_n(\omega;t)+D_n(\lambda;t)+D_n(\omega,\lambda;t)\big]\Big\}, \quad (3.19)$$

where

$$W_\beta(\rho,\lambda;t)=W_\beta(\rho;t)\exp\Big\{-i\pi\sum_{\mathbf{k}}\lambda_{\mathbf{k}}\dot{\bar\rho}_{\mathbf{k}}\Big\},$$

$$D_n(\omega;t)=\frac{(-i\pi)^n}{n!}\times$$

$$\times\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n}\mathfrak{M}_n(\mathbf{k}_1,\ldots,\mathbf{k}_n;t)\omega_{\mathbf{k}_1}\ldots\omega_{\mathbf{k}_n},$$

$$D_n(\lambda;t)=\frac{(-i\pi)^n}{n!}\times$$

$$\times\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n}\mathfrak{M}^{(1)}_n(\mathbf{k}_1,\ldots,\mathbf{k}_n;t)\lambda_{\mathbf{k}_1}\ldots\lambda_{\mathbf{k}_n},$$

$$D_n(\omega,\lambda;t)=\frac{(-i\pi)^n}{n!}\times$$

$$\times\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n}\mathfrak{M}^{(2)}_n(\mathbf{k}_1,\ldots,\mathbf{k}_n;t)\omega_{\mathbf{k}_1}\ldots\omega_{\mathbf{k}_{n-1}}\ldots\lambda_{\mathbf{k}_n},\quad(3.20)$$

and the cumulants have the following structure:

$$\mathfrak{M}_n(\mathbf{k}_1,\ldots,\mathbf{k}_n;t)=\langle\hat\rho_{\mathbf{k}_1},\ldots,\hat\rho_{\mathbf{k}_n}\rangle^{t,c}_{\text{kin}-\text{sh}},$$

$$\mathfrak{M}^{(1)}_n(\mathbf{k}_1,\ldots,\mathbf{k}_n;t)=\langle\dot{\hat\rho}_{\mathbf{k}_1},\ldots,\dot{\hat\rho}_{\mathbf{k}_n}\rangle^{t,c}_{\text{kin}-\text{sh}},$$

$$\mathfrak{M}^{(2)}_n(\mathbf{k}_1,\ldots,\mathbf{k}_n;t)=n\big[(n-j)+(j-n+1)\delta_{\ldots,l_{n-1}}\big]\times$$

$$\times\langle\hat\rho_{\mathbf{k}_1},\ldots,\hat\rho_{\mathbf{k}_{n-j}},\ldots,\dot{\hat\rho}_{\mathbf{k}_{n-j+1}},\ldots,\dot{\hat\rho}_{\mathbf{k}_n}\rangle^{t,c}_{\text{kin}-\text{sh}}.$$

Let us firstly consider the Gaussian approximation for $W(\rho,\lambda;t)$, i.e., we retain only the terms with $n=2$, which are linear in $\lambda_{\mathbf{k}}$, in the exponent in the integrand,

$$W^G(\rho,\lambda;t)=W_\beta(\rho,\lambda;t)\int d\omega\exp\Big\{i\pi\sum_{\mathbf{k}}\omega_{\mathbf{k}}\bar\rho_{\mathbf{k}}-$$

$$-\frac{\pi^2}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\mathfrak{M}_2(\mathbf{k}_1,\mathbf{k}_2;t)\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2}-$$

$$-\frac{\pi^2}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\mathfrak{M}^{(2)}_2(\mathbf{k}_1,\mathbf{k}_2;t)\,\omega_{\mathbf{k}_1}\lambda_{\mathbf{k}_2}\Big\}.\quad(3.21)$$

Transforming the expression in the exponent to a diagonal quadratic form in the variables $\omega_{\mathbf{k}}$, as was done for $W(\rho;t)$, and integrating over the new variables $\bar\omega_{\mathbf{k}}$, we obtain

$$W^G(\rho,\lambda;t)=W_\beta(\rho,\lambda;t)\times$$

$$\times\exp\Big\{-\frac{\pi^2}{2}\sum_{\mathbf{k}}E^{-1}(\mathbf{k};t)b_{\mathbf{k}}b_{-\mathbf{k}}-$$

$$-\frac{1}{2}\sum_{\mathbf{k}}\ln\pi\det\tilde E(\mathbf{k};t)\Big\},\quad(3.22)$$

where

$$b_{\mathbf{k}}=\bar\rho_{\mathbf{k}}+\frac{i\pi}{2}\,\mathfrak{M}^{(2)}_2(\mathbf{k};t)\lambda_{\mathbf{k}}.$$

Here, the cumulants $\mathfrak{M}^{(2)}_2(\mathbf{k};t)$ have the following structure:

$$\mathfrak{M}^{(2)}_2(\mathbf{k};t)=\langle\dot{\hat\rho}_{\mathbf{k}}\hat\rho_{-\mathbf{k}}\rangle^t_{\text{kin}-\text{sh}}-\langle\dot{\hat\rho}_{\mathbf{k}}\rangle^t_{\text{kin}-\text{sh}}\langle\hat\rho_{-\mathbf{k}}\rangle^t_{\text{kin}-\text{sh}}.\quad(3.23)$$

Since $\dot{\hat{\mathbf{v}}}_{\mathbf{k}}=-i\mathbf{k}\cdot\hat{\mathbf{v}}_{\mathbf{k}}$, where $\hat{\mathbf{v}}_{\mathbf{k}}=\sum_{j=1}^N\mathbf{v}_je^{-i\mathbf{k}\cdot\mathbf{r}}$ is the Fourier component of the microscopic particle velocity density, then

$$\mathfrak{M}^{(2)}_2(\mathbf{k};t)=$$

$$=-i\mathbf{k}\cdot(\langle\hat{\mathbf{v}}_{\mathbf{k}}\hat\rho_{-\mathbf{k}}\rangle^t_{\text{kin}-\text{sh}}-\langle\hat{\mathbf{v}}_{\mathbf{k}}\rangle^t_{\text{kin}-\text{sh}}\langle\hat\rho_{-\mathbf{k}}\rangle^t_{\text{kin}-\text{sh}})$$

is the velocity-density cross-correlation function for the system with the short-range interaction $\varrho^{\text{kin}-\text{sh}}_q(x^N;t)$.

Next, we calculate the hydrodynamic velocities $v_\rho(\mathbf{k};t)$ in the Gaussian approximation according to the formula

$$v_\rho(\mathbf{k};t)=\frac{\partial}{\partial(-i\pi\lambda_{\mathbf{k}})}\ln W^G(\rho,\lambda;t)\Big|_{\lambda_{\mathbf{k}}=0}=$$

$$=\langle\dot{\hat\rho}_{\mathbf{k}}\rangle^t_{\text{kin}-\text{sh}}-\frac{\pi^2}{2}\,E^{-1}(\mathbf{k};t)\,\mathfrak{M}^{(2)}_2(\mathbf{k};t)\,\bar\rho_{\mathbf{k}}.\quad(3.24)$$

This expression contains two terms. The former is related to the Fourier component of the particle velocity density averaged using the distribution function with short-range interaction $\varrho^{\text{kin}-\text{sh}}_q(x^N;t)$: $\langle\dot{\hat\rho}_{\mathbf{k}}\rangle^t_{\text{kin}-\text{sh}}=-i\mathbf{k}\cdot\langle\hat{\mathbf{v}}_{\mathbf{k}}\rangle^t_{\text{kin}-\text{sh}}$. The latter is related to the relationship between the correlation functions $E^{-1}(\mathbf{k};t)$ and $\mathfrak{M}^{(2)}_2(\mathbf{k};t)$, and is linear in the collective variables of particle concentration $\bar\rho_{\mathbf{k}}(t)=\rho_{\mathbf{k}}-\langle\hat\rho_{\mathbf{k}}\rangle^t_{\text{kin}-\text{sh}}$. According to Eqs. (3.17) and (3.19), in approximations higher than the Gaussian one, $v_\rho(\mathbf{k};t)$ is a function of the collective variables $\rho_{\mathbf{k}}$ of the second, third, and higher orders, which is important from the viewpoint of calculating the contributions of fluctuations to the generalized transfer coefficients and the temporal correlation functions [56, 57]. In particular, going beyond the Gaussian approximation can be done by taking the correlations $\mathfrak{M}^{(2)}_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;t)$ into account. Then, for $W(\rho,\lambda;t)$, we obtain

$$W(\rho,\lambda;t)=W_\beta(\rho;t)\int d\omega\times$$

$$\times \exp\Big\{i\pi\sum_{\mathbf{k}}\omega_{\mathbf{k}}\bar{\rho}_{\mathbf{k}} - i\pi\sum_{\mathbf{k}}\langle\dot{\hat{\rho}}_{\mathbf{k}}\rangle^{t,c}_{\mathrm{kin-sh}}\lambda_{\mathbf{k}} -$$
$$-\frac{\pi^2}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\mathfrak{M}_2(\mathbf{k}_1,\mathbf{k}_2;t)\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2} -$$
$$-\frac{\pi^2}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\mathfrak{M}^{(2)}_2(\mathbf{k}_1,\mathbf{k}_2;t)\,\omega_{\mathbf{k}_1}\lambda_{\mathbf{k}_2} -$$
$$-\frac{\pi^2}{2}\left(\frac{-i\pi}{3}\right)\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\mathfrak{M}^{(2)}_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;t)\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2}\lambda_{\mathbf{k}_3}\Big\}. \tag{3.25}$$

In the exponent on the right-hand side of this expression, we have a quadratic dependence on $\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2}$. This expression can be rewritten in the form

$$W(\rho,\lambda;t) = W_\beta(\rho;t)\exp\Big\{-i\pi\sum_{\mathbf{k}}\langle\dot{\hat{\rho}}_{\mathbf{k}}\rangle^{t,c}_{\mathrm{kin-sh}}\lambda_{\mathbf{k}}\Big\}\times$$
$$\times\int d\omega\exp\Big\{i\pi\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}} -$$
$$-\frac{\pi^2}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\bar{\mathfrak{M}}_2(\mathbf{k}_1,\mathbf{k}_2;t)\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2}\Big\}, \tag{3.26}$$

where the notation

$$\bar{\mathfrak{M}}_2(\mathbf{k}_1,\mathbf{k}_2;t) = \mathfrak{M}_2(\mathbf{k}_1,\mathbf{k}_2;t) +$$
$$+\left(\frac{-i\pi}{3}\right)\sum_{\mathbf{k}_3}\mathfrak{M}^{(2)}_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;t)\,\lambda_{\mathbf{k}_3} \tag{3.27}$$

was introduced.

Next, after diagonalizing $W(\rho,\lambda;t)$ in accordance with

$$\det\Big|\tilde{\bar{\mathfrak{M}}}_2(\mathbf{k}_1,\mathbf{k}_2,\lambda;t) - \tilde{E}(\mathbf{k},\lambda;t)\Big| = 0, \tag{3.28}$$

and integrating over ω, we finally get

$$W(\rho,\lambda;t) = W_\beta(\rho;t)\times$$
$$\times\exp\Big\{-i\pi\sum_{\mathbf{k}}\langle\dot{\hat{\rho}}_{\mathbf{k}}\rangle^{t}_{\mathrm{kin-sh}}\lambda_{\mathbf{k}} -$$
$$-\frac{\pi^2}{2}\sum_{\mathbf{k}}E^{-1}(\mathbf{k},\lambda;t)b_{\mathbf{k}}b_{-\mathbf{k}} -$$
$$-\frac{1}{2}\sum_{\mathbf{k}}\ln\pi\det\tilde{E}(\mathbf{k},\lambda;t)\Big\}. \tag{3.29}$$

Note that $E^{-1}(\mathbf{k},\lambda;t)$ depends on the parameter $\lambda_{\mathbf{k}}$. Now, we can calculate the hydrodynamic velocity $v_\rho(\mathbf{k};t)$ taking into account the third-order correlations $\mathfrak{M}^{(2)}_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;t)$. As a result, we obtain

$$v_\rho(\mathbf{k};t) = \frac{\partial}{\partial(-i\pi\lambda_{\mathbf{k}})}\ln W(\rho,\lambda;t)\Big|_{\lambda_{\mathbf{k}}=0} =$$
$$= \langle\dot{\hat{\rho}}_{\mathbf{k}}\rangle^{t}_{\mathrm{kin-sh}} - \frac{\pi^2}{2}E^{-1}(\mathbf{k};t)\,\mathfrak{M}^{(2)}_2(\mathbf{k};t)\,\bar{\rho}_{\mathbf{k}} -$$
$$-\frac{\pi^2}{2}\frac{\partial}{\partial(-i\pi\lambda_{\mathbf{k}})}E^{-1}(\mathbf{k},\lambda;t)|_{\lambda_{\mathbf{k}}=0}\bar{\rho}_{\mathbf{k}}\,\bar{\rho}_{-\mathbf{k}} -$$
$$-\frac{1}{2}\frac{\partial}{\partial(-i\pi\lambda_{\mathbf{k}})}\ln\pi\det\tilde{E}(\mathbf{k},\lambda;t)|_{\lambda_{\mathbf{k}}=0}. \tag{3.30}$$

As one can see, $v_\rho(\mathbf{k};t)$ is a quadratic function of the collective variables $\bar{\rho}_{\mathbf{k}}\bar{\rho}_{-\mathbf{k}}$ in this approximation.

In the general case, by applying the method used for calculating $W(\rho;t)$, we obtain the following result for $W(\rho,\lambda;t)$:

$$W(\rho,\lambda;t) = W_\beta(\rho,\lambda;t)\,\bar{W}^G(\rho,\lambda;t)\times$$
$$\times\exp\Big\{\sum_{n\geq3}\big(\langle\tilde{D}_n(\rho;t)\rangle_G + \langle\tilde{D}^{(2)}_n(\rho,\lambda;t)\rangle_G\big)\Big\}, \tag{3.31}$$

where

$$\bar{W}^G(\rho,\lambda;t) = \exp\Big\{-\frac{\pi^2}{2}\sum_{\mathbf{k}}E^{-1}(\mathbf{k};t)\,b_{\mathbf{k}}b_{-\mathbf{k}} -$$
$$-\frac{1}{2}\sum_{\mathbf{k}}\ln\pi\det\tilde{E}(\mathbf{k};t)\Big\}. \tag{3.32}$$

$\langle\tilde{D}_n(\rho;t)\rangle_G$ can be approximately given as follows:

$$\langle\tilde{D}_3(\rho;t)\rangle_G = \langle\bar{D}_3(\rho;t)\rangle_G,$$
$$\langle\tilde{D}_4(\rho;t)\rangle_G = \langle\bar{D}_4(\rho;t)\rangle_G,$$
$$\langle\tilde{D}_6(\rho;t)\rangle_G = \langle\bar{D}_6(\rho;t)\rangle_G - \frac{1}{2}\langle\bar{D}_3(\rho;t)\rangle^2_G,$$
$$\langle\tilde{D}_8(\rho;t)\rangle_G = \langle\bar{D}_8(\rho;t)\rangle_G -$$
$$-\langle\bar{D}_3(\rho;t)\rangle_G\langle\bar{D}_5(\rho;t)\rangle_G - \frac{1}{2}\langle\bar{D}_4(\rho;t)\rangle^2_G,$$
$$\langle\tilde{D}_n(\rho;t)\rangle_G = \frac{1}{\bar{W}^G(\rho;t)}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n}\bar{\mathfrak{M}}_n(\mathbf{k}_1,\ldots,\mathbf{k}_n;t)\times$$
$$\times\frac{1}{(i\pi)^n}\frac{\delta^n}{\delta b_{\mathbf{k}_1}\ldots\delta b_{\mathbf{k}_n}}\bar{W}^G(\rho;t)$$

and $\langle\tilde{D}^{(2)}_n(\rho;t)\rangle_G$ can also be approximately given as follows:

$$\langle\tilde{D}^{(2)}_3(\rho;t)\rangle_G = \langle\bar{D}^{(2)}_3(\rho;t)\rangle_G,$$

$$\langle\tilde{D}_4^{(2)}(\rho;t)\rangle_G = \langle\bar{D}_4^{(2)}(\rho;t)\rangle_G,$$

$$\langle\tilde{D}_6^{(2)}(\rho;t)\rangle_G = \langle\bar{D}_6^{(2)}(\rho;t)\rangle_G - \frac{1}{2}\langle\bar{D}_3^{(2)}(\rho;t)\rangle_G^2,$$

$$\langle\tilde{D}_8^{(2)}(\rho;t)\rangle_G = \langle\bar{D}_8^{(2)}(\rho;t)\rangle_G -$$

$$- \langle\bar{D}_3^{(2)}(\rho;t)\rangle_G\langle\bar{D}_5^{(2)}(\rho;t)\rangle_G - \frac{1}{2}\langle\bar{D}_4^{(2)}(\rho;t)\rangle_G^2,$$

$$\langle\tilde{D}_n^{(2)}(\rho;t)\rangle_G = \frac{1}{\bar{W}^G(\rho;t)} \times$$

$$\times \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n} \bar{\mathfrak{M}}_n^{(2)}(\mathbf{k}_1,\ldots,\mathbf{k}_n;t)\lambda_{\mathbf{k}_1}) \times$$

$$\times \frac{1}{(i\pi)^n}\frac{\delta^n}{\delta b_{\mathbf{k}_1}\ldots\delta b_{\mathbf{k}_n}}\bar{W}^G(\rho;t),$$

where $\langle\tilde{D}_n(\rho;t)\rangle_G$, $\langle\tilde{D}_n^{(2)}(\rho;t)\rangle_G$ are the nth renormalized non-equilibrium cumulant averages for the variables $\hat{\rho}_{\mathbf{k}}$ of higher orders.

4. Conclusions

In the framework of the Zubarev non-equilibrium statistical operator method, a system of transfer equations that consistently describe kinetic and hydrodynamic fluctuations in a system of interacting particles is obtained for the non-equilibrium one-particle distribution function $f_1(x;t) = \langle\hat{n}_1(x)\rangle^t$, the non-equilibrium average value of the particle interaction energy density $H^{\rm int}(\mathbf{r},t) = \langle\hat{H}^{\rm int}(\mathbf{r})\rangle^t$, and the non-equilibrium distribution function of collective variables $f(\rho;t) = \langle\delta(\hat{\rho}-\rho)\rangle^t$. The separation of contributions from short- and long-range interactions between the particles brought about the situation, when short-range interactions (for example, the model of hard spheres) are described in the coordinate-momentum space, and long-range ones in the space of collective variables of particle concentration. In this case, the short-range component is considered as a basic one with the distribution $\varrho_q^{\rm kin-sh}(x^N;t)$, and it is described by a chain of BBGKY equations for non-equilibrium particle distribution functions, e.g., in the model of hard spheres [35].

The applied method of collective variables [46, 54, 55] made it possible to calculate both the structural function and the hydrodynamic velocities in terms of collective variables in approximations higher than the Gaussian one. In particular, on the basis of Eq. (3.19), in the approximation next after the Gaussian one, the hydrodynamic velocities (3.17) are proportional to $\bar{\rho}_{\mathbf{k}}\bar{\rho}_{\mathbf{k}'}$ and $\bar{\rho}_{\mathbf{k}}\bar{\rho}_{\mathbf{k}'}\bar{\rho}_{\mathbf{k}''}$, and the transport kernels in the Fokker–Planck equation are fourth-order correlation functions of the variables $\bar{\rho}_{\mathbf{k}}$.

It is worth to note that, in the Gaussian approximation for $\tilde{W}^G(\mathbf{k};t)$ and $v_\rho(\mathbf{k};t)$, the Fokker–Planck equation gives rise to transport equations for $\langle\hat{\rho}_{\mathbf{k}}\rangle^t$ whose structure is the same as in the case of generalized diffusion, but the averaging is carried out using the distribution function

$$\varrho_L(x^N,\rho;t) = \varrho_q^{\rm kin-hyd}(x^N;t)\frac{\hat{f}(\rho)}{W^G(\rho;t)}.$$

The proposed approach makes it possible to go beyond the Gaussian approximation for $\tilde{W}(\mathbf{k};t)$ and $v_\rho(\mathbf{k};t)$, and therefore for the transfer kernels in the Fokker–Planck equation. This circumstance allows one to obtain a nonlinear system of equations for $\langle\hat{\rho}_{\mathbf{k}}\rangle^t$. It is important to note that the kinetic equation (2.22) [49, 50] contains a generalized integral of the Fokker–Planck type with the generalized coefficients of diffusion and friction in the phase space $(\mathbf{r},\mathbf{p},t)$, where the region of $|\mathbf{r}|$ variation is limited by the values $|\mathbf{k}|_{hydr}^{-1}$ corresponding to collective nonlinear hydrodynamic processes. This means that, in the regions limited by $|\mathbf{k}|_{hydr}^{-1}$, the processes are described by the generalized diffusion and friction coefficients, and, at small $|\mathbf{k}|_{hydr}^{-1}$, they are described by the generalized coefficients in the space of collective variables. In the following works, we will investigate the transport equations (2.22)–(2.24) in fluctuation approximations that are higher than the Gaussian one.

1. P. Résibois, M. de Leener, *Classical Kinetic Theory of fluids* (John Wiley and Sons, 1977).
2. J. Boon, S. Yip, *Molecular Hydrodynamics* (McGraw-Hill Inc., 1980).
3. G. Röpke, *Nonequilibrium Statistical Physics* (Wiley-VCH Verlag GmbH and Co., 2013).
4. Yu.L. Klimontovich. *Turbulent Motion and the Structure of Chaos. A New Approach to the Statistical Theory of Open Systems* (Kluwer Academic, 1991).
5. U. Balucani, M. Zoppi. *Dynamics of the Liquid State* (Clarendon Press, 1994).
6. R. Balescu. *Statistical Dynamics: Matter out of Equilibrium* (World Scientific, 1997).
7. R. Zwanzig. *Nonequilibrium Statistical Mechanics* (Oxford University Press, 2001).
8. D. Zubarev, V. Morozov, G. Röpke. *Statistical Mechanics of Nonequilibrium Processes* (Akademie, 1996), Vol. 1.
9. D. Zubarev, V. Morozov, G. Röpke. *Statistical Mechanics of Nonequilibrium Processes* (Akademie, 1996), Vol. 2.
10. G. Mazenko. *Nonequilibrium Statistical Mechanics* (Wiley-VCH Verlag GmbH. and Co., 2006).

11. Yu.L. Klimontovich, H. Wilhelmsson, A.G. Zagorodnii, I.P. Yakimenko. *Statistical Theory of Confined Plasma-Molecular Systems* (Moscow State University, 1990) (in Russian).
12. Y.L. Klimontovich, D. Kremp, W.D Kraeft. *Advances in Chemistry and Physics* (Wiley, 2007).
13. L.A. Bulavin. *Neutron Diagnostics of Liquid Matter State* (Institute for Safety Problems of Nuclear Power Plants, 2012) (in Ukrainian).
14. M. Bonitz, J. Lopez, K. Becker, H. Thomsen., *Complex Plasmas. Scientific Challenges and Technological Opportunities* (Springer, 2014).
15. S.V. Peletminskii, Yu.V. Slyusarenko, A.I. Sokolovsky. Kinetics and hydrodynamics of long-wave fluctuations under external random force. *Physica A* **326**, 412 (2003).
16. S.O. Nikolayenko, Yu.V. Slyusarenko. Microscopic theory of relaxation processes in systems of particles interacting with the hydrodynamic medium. *J. Math. Phys.* **50**, 083305 (2009).
17. O.Yu. Slyusarenko, A.V. Chechkin, Yu.V. Slyusarenko. The Bogolyubov–Born–Green–Kirkwood–Yvon hierarchy and Fokker-Planck equation for many-body dissipative randomly driven systems. *J. Math. Phys.* **56**, 043302 (2015).
18. Y.A. Humenyuk, M.V. Tokarchuk. Extension of hydrodynamic balance equations for simple fluids. *J. Stat. Phys.* **142**, 1052 (2011).
19. K. Yoshida, T. Arimitsu. Inertial-range structure of Gross–Pitaevskii turbulence within a spectral closure approximation. *J. Phys. A: Math. Theor.* **46**, 335501 (2013).
20. P. Mendoza-Mendez, L. Lopez-Flores, A. Vizcarra-Redon, L.F. Sanchez-Diaz, M. Medina-Noyola. Generalized Langevin equation for tracer diffusion in atomic liquids. *Physica A* **394**, 1 (2014).
21. J.P. Boon, J.F. Lutsko, C. Lutsko. Microscopic approach to nonlinear reaction-diffusion: The case of morphogen gradient formation. *Phys. Rev. E* **85**, 021126 (2012).
22. G.F. Mazenko. Fundamental theory of statistical particle dynamics. *Phys. Rev. E* **81**, 061102 (2010).
23. G.F. Mazenko. Smoluchowski dynamics and the ergodic-nonergodic transition. *Phys. Rev. E* **83**, 041125 (2011).
24. P. Kostrobij, R. Tokarchuk, M. Tokarchuk, V. Markiv. Zubarev nonequilibrium statistical operator method in Renyi statistics. Reaction-diffusion processes. *Condens. Matter Phys.* **17**, 33005 (2014).
25. P.A. Hlushak, M.V. Tokarchuk. Quantum transport equations for Bose systems taking into account nonlinear hydrodynamic processes. *Condens. Matter Phys.* **17**, 23606 (2014).
26. C.A.B. Silva, A.R. Vasconcellos, J.G. Ramos, R. Luzzi. Generalized kinetic equation for far-from-equilibrium many-body systems. *J. Stat. Phys.* **143**, 1020 (2011).
27. V.N. Tsytovich, U. de Andelis. Kinetic theory of dusty plasmas. V. The hydrodynamic equations. *Phys. Plasmas* **11**, 496 (2004).
28. A.I. Olemskoi. *Theory of structure transformations in non-equilibrium condensed matter*. Horizons in World Physics Series. Vol. 231. (NOVA Science Publishers, 1999).
29. B. Markiv, R. Tokarchuk, P. Kostrobij, M. Tokarchuk. Nonequilibrium statistical operator method in Renyi statistics. *Physica A* **390**, 785 (2011).
30. I.M. Mryglod, M.V. Tokarchuk. On statistical hydrodynamics of simple liquds. In: *Problems of Atomic Science and Technique. Series: Nuclear Physics Investigations (Theory and Experiment)* (Kharkov Physico-Technical Institute, 1992), **3** (24), p. 134.
31. I.M. Mryglod, I.P. Omelyan, M.V. Tokarchuk. Generalized collective modes for the Lennard–Jones fluid. *Mol. Phys.* **84**, 235 (1995).
32. B.B. Markiv, I.P. Omelyan, M.V. Tokarchuk. Relaxation to the state of molecular hydrodynamics in the generalized hydrodynamics of liquids. *Phys. Rev. E* **82**, 041202 (2010).
33. D.N. Zubarev, V.G. Morozov, I.P. Omelyan, M.V. Tokarchuk. Unification of the kinetic and hydrodynamic approaches in the theory of dense gases and liquids. *Theor. Math. Phys.* **96**, 997 (1993).
34. M.V. Tokarchuk, I.P. Omelyan, A.E. Kobryn. A consistent description of kinetics and hydrodynamics of systems of interacting particles by means of the nonequilibrium statistical operator method. *Condens. Matter Phys.* **1**, 687 (1998).
35. A.E. Kobryn, I.P. Omelyan, M.V. Tokarchuk. The modified group expansions for construction of solutions to the BBGKY hierarchy. *J. Stat. Phys.* **92**, 973 (1998).
36. B. Markiv, I. Omelyan, M. Tokarchuk. Consistent description of kinetics and hydrodynamics of weakly nonequilibrium processes in simple liquids. *J. Stat. Phys.* **155**, 843 (2014).
37. B. Markiv, M. Tokarchuk. Consistent description of kinetics and hydrodynamics of dusty plasma. *Phys. Plasmas* **21**, 023707 (2014).
38. J.R. Dorfman. Advances and challenges in the kinetic theory of gases. *Physica A* **106**, 77 (1981).
39. Yu.L. Klimontovich. On the need for and the possibility of a unified description of kinetic and hydrodynamic processes. *Theor. Math. Phys.* **92**, 909 (1992).
40. Yu.L. Klimontovich. The unified description of kinetic and hydrodynamic processes in gases and plasmas. *Phys. Lett. A* **170**, 434 (1992).
41. E.G.D. Cohen. Fifty years of kinetic theory. *Physica A* **194**, 229 (1993).
42. S.K. Schnyder, F. Hofling, T. Franosch, Th. Voigtmann. Long-wavelength anomalies in the asymptotic behavior of mode-coupling theory. *J Phys.: Condens. Matter.* **23**, 234121 (2011).
43. T. Franosch. Long-time limit of correlation functions. *J. Phys. A: Math. Theor.* **47**, 325004 (2014).
44. D.N. Zubarev, V.G. Morozov. Formulation of boundary conditions for the BBGKY hierarchy with allowance for local conservation laws. *Theor. Math. Phys.* **60**, 814 (1984).

45. D.N. Zubarev, V.G. Morozov, I.P. Omelyan, M.V. Tokarchuk. Kinetic equations for dense gases and liquids. *Theor. and Math. Phys.* **87**, 412 (1991).
46. A.E. Kobryn, V.G. Morozov, I.P. Omelyan, M.V. Tokarchuk. Enskog-Landau kinetic equation. Calculation of the transport coefficients for charged hard spheres. *Physica A* **230**, 189 (1996).
47. D.N. Zubarev, V.G. Morozov. Nonequilibrium statistical ensembles in kinetic theory and hydrodynamics. In: *Collection of Scientific Works of Mathematical Institute of USSR Academy of Sciences* (Nauka, 1989), **191**, p. 140 (in Russian).
48. V.G. Morozov, A.E. Kobryn, M.V. Tokarchuk. Modified kinetic theory with consideration for slow hydrodynamical processes. *Condens. Matter Phys.* **4**, 117 (1994).
49. P. Hlushak, M. Tokarchuk. Chain of kinetic equations for the distribution functions of particles in simple liquid taking into account nonlinear hydrodynamic fluctuations. *Physica A* **443**, 231 (2016).
50. I.R. Yukhnovskii, P.A. Hlushak, M.V. Tokarchuk. BBGKY chain of kinetic equations, non-equilibrium statistical operator method and collective variable method in the statistical theory of non-equilibrium liquids. *Condens. Matter Phys.* **19**, 43705 (2016).
51. I.R. Yukhnovskii, M.F. Holovko. *Statistical Theory of Classical Equilibrium Systems (*Naukova Dumka, 1980) (in Russian).
52. D.N. Zubarev, A.M. Khazanov. Generalized Fokker–Planck equation and construction of projection operators for different methods of reduced description of nonequilibrium states. *Theor. Math. Phys.* **34**, 43 (1978).
53. K. Kawasaki. *In Phase Transition and Critical Phenomena.* Edited by C. Domb, M.S. Green (Acad. Press, 1976), Vol. 5A, p. 165.
54. D.N. Zubarev. Statistical thermodynammics of turbulent transport processes. *Theor. Math. Phys.* **53**, 1004 (1982).
55. I.M. Idzyk, V.V. Ighatyuk, M.V. Tokarchuk. Fokker–Planck equation for nonequilibrium distribution function of collective variables. I. Calculation of the statistical weight, entropy, hydrodynamic velocities. *Ukr. J. Phys.* **41**, 596 (1996).
56. M.V. Ignatiuk, V.V. Tokarchuk. Statistical theory of nonlinear hydrodynamic fluctuations in ionic systems. *Theor. Math. Phys.* **108**, 1208 (1996).
57. V.G. Morozov. Low-frequency correlation functions in case of nonlinear dynamics of fluctuations. *Physica A* **110**, 201 (1982).

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МЕТОД КОЛЕКТИВНИХ ЗМІННИХ
В ТЕОРІЇ НЕЛІНІЙНИХ ФЛУКТУАЦІЙ
З УРАХУВАННЯМ КІНЕТИЧНИХ ПРОЦЕСІВ

Для узгодженого опису кінетики та гідродинаміки систем взаємодіючих частинок оптимізовано набір параметрів скороченого опису згідно з Боголюбовим, що передбачає залучення колективних змінних. При цьому розділяються внески від короткосяжних і далекосяжних взаємодій між частинками. Короткосяжні взаємодії (наприклад, модель твердих сфер) описуються в координатно-імпульсному просторі, а далекосяжні – у просторі колективних змінних. Короткосяжна складова розглядається як базисна. Використовуючи метод нерівноважного статистичного оператора Зубарєва, ми отримали систему рівнянь переносу для нерівноважної одночастинкової функції розподілу, нерівноважного середнього значення густини енергії взаємодії частинок та нерівноважної функції розподілу колективних змінних. Застосований метод колективних змінних дав можливість розрахувати у вищих наближеннях, ніж гаусове, як структурну функцію, так і гідродинамічні швидкості колективних змінних.

Ключові слова: проста рідина, нелінійні флуктуації, нерівноважний статистичний оператор, функція розподілу, рівняння Фоккера–Планка.