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CONTACT INTERACTIONS IN ONE-DIMENSIONAL QUANTUM MECHANICS: A FAMILY OF GENERALIZED δ'-POTENTIALS [1]

A "one-point" approximation is proposed to investigate the transmission of electrons through the extra thin heterostructures composed of two parallel plane layers. The typical example is the bilayer for which the squeezed potential profile is the derivative of Dirac's delta function. The Schrödinger equation with this singular one-dimensional profile produces a family of contact (point) interactions each of which (called a "distributional" δ'-potential) depends on the way of regularization. The discrepancies widely discussed so far in the literature regarding the family of delta derivative potentials are eliminated using a two-scale power-connecting parametrization of the bilayer potential that enables one to extend the family of distributional δ'-potentials to a whole class of "generalized" δ'-potentials. In a squeezed limit of the bilayer structure to zero thickness, the resonant tunneling through this structure is shown to occur in the form of sharp peaks located on the sets of Lebesgue's measure zero (called resonance sets). A four-dimensional parameter space is introduced for the representation of these sets. The transmission on the complement sets in the parameter space is shown to be completely opaque.

Keywords: point interactions, transmission, resonant tunneling, heterostructures.

1. Introduction

Starting with the pioneering work by Berezin and Faddeev [1], various exactly solvable models described by the Schrödinger operators with singular zero-range potentials have been studied within the theory of self-adjoint extensions of symmetric operators. These models are specified by the potentials defined on the sets consisting of isolated points. Therefore in the literature, they are usually referred to as "contact" or "point" interactions (see books [2–4] for details and references). A whole body of works (see, e.g., [5–11], a few to mention), including the very recent studies [12–19] with references therein, has been published. There, the one-dimensional Schrödinger operators were defined via distributions and corresponding two-sided boundary conditions (BCs) at the points of singularity. Alternatively, besides this "point" approach, one can realize various families of point interactions (PIs) from the Schrödinger equation with regular finite-range potentials in a squeezed limit [20–35]. We refer the "squeezing" approach as a "point" approximation of realistic finite-range systems (e.g., ultrathin layered sheets). The advantage of both the distributional and squeezing approaches is the possibility to get the resolvents of these operators in an explicit form, to find their spectra, and to compute scattering coefficients.

In the present paper, we are dealing with the stationary Schrödinger equation in one dimension

$$-\psi''(x) + V(x)\psi(x) = E\psi(x), \qquad (1)$$

where $V(x)$ is a real-valued function defined on the line $-\infty < x < \infty$, being either a regular function or a distribution. There exists a one-to-one correspondence between the full set of self-adjoint extensions of the one-dimensional kinetic energy operator and the two families of BCs: *non-separated* and *separated.* The non-separated extensions describe non-trivial four-parameter PIs subject to the BCs at $x = \pm 0$ on the wave function $\psi(x)$ and its derivative

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$\psi'(x)$ given by the connection matrix Λ of the form [7]

$$\begin{pmatrix}\psi(+0)\\ \psi'(+0)\end{pmatrix} = \Lambda \begin{pmatrix}\psi(-0)\\ \psi'(-0)\end{pmatrix}, \quad \Lambda = \mathrm{e}^{\mathrm{i}\chi}\begin{pmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{21} & \lambda_{22}\end{pmatrix}. \qquad (2)$$

Here, $\chi \in [0, \pi)$ and the λ-elements are finite real parameters fulfilling the condition $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = 1$. In the case where some of the λ-elements are infinite, the corresponding PI is separated, being completely opaque for an incident particle. For instance, if the diagonal elements λ_{11} and λ_{22} are finite, but one of the off-diagonal elements is infinite, we have either the Dirichlet BCs $\psi(\pm 0) = 0$ (if λ_{21} is infinite and $\lambda_{12} = 0$) or the Neumann BCs $\psi'(\pm 0) = 0$ (if λ_{12} is infinite and $\lambda_{21} = 0$).

Some particular examples of Eq. (1) and the corresponding Λ-matrix (2) are important in applications. One of the representations of this matrix to be considered in the present paper is

$$\Lambda = \begin{pmatrix}\theta & 0\\ \alpha & \theta^{-1}\end{pmatrix}, \quad \theta, \alpha \in \mathbb{R}. \qquad (3)$$

The particular case $\theta = 1$ corresponds to the simplest and most widespread PI called a δ-potential. In this case, the potential in Eq. (1) is defined by Dirac's delta function $\delta(x)$, i.e., $V(x) = \alpha\delta(x)$, where α is a strength constant (or intensity). The wave function $\psi(x)$ for this interaction is continuous at the origin $x = 0$, whereas its derivative undergoes a jump α. As derived in [23–31], if the potential part in Eq. (1) is the derivative of the delta function i.e., $V(x) = \gamma\delta'(x)$, $\delta'(x) := d\delta(x)/dx$, with $\gamma \in \mathbb{R}$ being a strength constant, we have $\theta \neq 1$. Extending a bit the classification suggested by Brasche and Nizhnik [15], we call any PI described by the Λ-matrix of the form (3) with $\theta \neq 1$ (even if $\alpha \neq 0$) a "generalized" δ'-potential. Concerning the particular case of a delta derivative potential $\gamma\delta'(x)$ in Eq. (1), we refer the resulting PI as a "distributional" δ'-potential (also even if $\alpha \neq 0$).

On the other hand, as historically adopted in the literature (see, e.g., [3]), the PI, for which the derivative $\psi'(x)$ is continuous at the origin, but $\psi(x)$ discontinuous, is called a δ'-interaction. Its more generalized (two-parameter) version [36] is described by the connection matrix of the form

$$\Lambda = \begin{pmatrix}\theta & \beta\\ 0 & \theta^{-1}\end{pmatrix}, \quad \theta, \beta \in \mathbb{R}. \qquad (4)$$

Different aspects of the δ'-interaction with $\theta = 1$ in (4) have been investigated in a series of publications (see, e.g., [9–11, 20–22, 37–39]). Note that the term "δ'-interaction" is somewhat misleading, because the form of Λ-matrix (4) differs from representation (3) that really corresponds to a delta derivative potential in Eq. (1). Therefore, the terms "δ'-potential" and "δ'-interaction" describe the two completely different situations (for details, see [15]).

2. Resonant Tunneling Through a Distributional δ'-Potential

Consider Eq. (1) with the delta derivative potential $V(x) = \gamma\delta'(x)$ treated through the regularization $\Delta'_\varepsilon(x) \to \delta'(x)$ in the sense of distributions. According to Seba's theorem [20], for any regular function $\mathcal{V}(\xi)$ such that

$$\Delta'_\varepsilon(x) = \varepsilon^{-2}\mathcal{V}(x/\varepsilon) \to \delta'(x) \ \text{ as } \ \varepsilon \to 0, \qquad (5)$$

the corresponding PI is separated, and the BCs $\psi(\pm 0) = 0$ hold true. This result means that the *zero* transmission occurs for *all* $\gamma \in \mathbb{R}$, asserting that the delta derivative potential acts as a fully reflecting wall.

Later on, Patil [40] computed the scattering coefficients for Eq. (1), using the limits

$$\frac{\delta(x+\varepsilon) - \delta(x-\varepsilon)}{2\varepsilon} \to \delta'(x) \qquad (6)$$

for the potential $V(x) = \gamma\delta'(x)$, $\gamma \in \mathbb{R}$, and

$$\frac{\delta(x+\varepsilon) - 2\delta(x) + \delta(x-\varepsilon)}{\varepsilon^2} \to \delta''(x) \qquad (7)$$

for the potential $V(x) = g\delta''(x)$, $g \in \mathbb{R}$. As a result, he found that both these potentials are fully reflecting, in fact supporting Seba's theorem in the case of the delta derivative potential.

However, using another approximation to the distribution $\delta'(x)$, namely, the piecewise constant function

$$\Delta'_\varepsilon(x) = \begin{cases} \varepsilon^{-2} & \text{for } -\varepsilon < x < 0,\\ -\varepsilon^{-2} & \text{for } 0 < x < \varepsilon,\\ 0 & \text{for } \varepsilon < |x| < \infty, \end{cases} \qquad (8)$$

as the simplest regularization being a particular example of (5), Christiansen *et al.* [23] observed that the distributional δ'-potential is not a fully reflected interaction. It has been found a countable set of values $\gamma \in \mathbb{R}$, where the transmission is *non-zero*. The connection matrix in this case is of the form (3) with the element θ being finite, while the element α diverges in general except for the discrete values of γ, where $\alpha = 0$. These values form one of the resonance sets Γ_0 for the family of distributional δ'-potentials:

$$\Gamma_0 = \{\gamma \mid \tan\sqrt{\gamma} = \tanh\sqrt{\gamma}\}. \tag{9}$$

On this set, the connection matrix (3) takes the discrete values with the elements

$$\alpha = 0 \quad \text{and} \quad \theta = \cosh\sqrt{\gamma}/\cos\sqrt{\gamma}, \quad \gamma \in \Gamma_0. \tag{10}$$

Beyond the set Γ_0 ($\gamma \notin \Gamma_0$), the transmission is zero, and the BCs are of the Dirichlet type: $\psi(\pm 0) = 0$. Since potential (8) is a piecewise constant function, the transmission amplitude $\mathcal{T}_\varepsilon$ can be computed explicitly as a function of γ. In the limit as $\varepsilon \to 0$, its form consists of the countable set of sharp peaks that converge pointwise to a discrete (resonance) set on the γ-line (see Fig. 1 computed numerically in [32]).

The existence of a resonance set has rigorously been proven by Golovaty and Man'ko [26] for the whole class of regularizing functions $\Delta'_\varepsilon(x)$ defined by regularization (5). Moreover, the boundary-value problem for finding this set and the algorithm for computing the element θ have been formulated. The gap in the proof of Seba's theorem has been found by Golovaty and Hryniv [27]. It has been shown that the resonance set Γ_0 depends on the regularizing sequence $\Delta'_\varepsilon(x)$. A similar dependence has been established in the case of the potential $V(x) = g\delta''(x)$ [41]. Therefore, the family of regularizing sequences serves as a *hidden* parameter in Eq. (1) with the derivatives of $\delta(x)$. Thus, contrary to the case with the potential $V(x) = \alpha\delta(x)$, the differential equations with coefficients in the form of the derivatives of $\delta(x)$ do not make a physical sense, if they are used without any additional information.

The existence of the non-empty resonance sets on which the transmission is non-zero contradicts Patil's result [40] appeared to be correct. Using approximation (6), he obtained zero transmission for *all* $\gamma \in \mathbb{R}$. This mismatch can be explained, if, for the regularization of $\delta'(x)$, we use the same barrier as well as defined by Eqs. (8), but separated by a distance r. One can calculate then the transmission $\mathcal{T}_\varepsilon(r)$ as a function of ε and r and compare both the repeated limits which appear to be not the same. Indeed, $\lim_{\varepsilon\to 0}\lim_{r\to 0}\mathcal{T}_\varepsilon(r) \to 0$ *almost everywhere*, while $\lim_{r\to 0}\lim_{\varepsilon\to 0}\mathcal{T}_\varepsilon(r) \to 0$ *everywhere*. This means that the relative rate of squeezing the barrier-well thickness ε and the distance r plays the crucial role in realizing PIs.

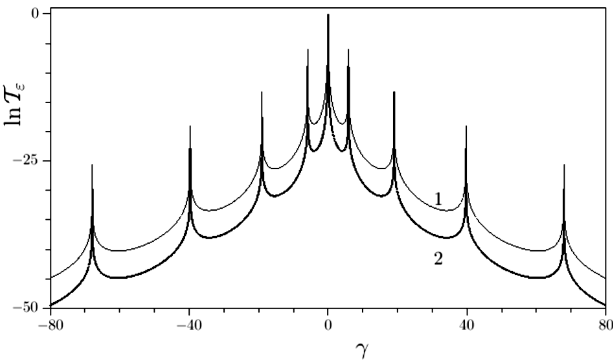

Fig. 1. Transmission amplitude $\mathcal{T}_\varepsilon$ as a function of the strength constant γ calculated at two values: $\varepsilon = 0.01$ (1), $\varepsilon = 0.001$ (2). The calculation has been done at $E = 0.01$ eV and $m^* = 0.1m_e$ (m_e, the electron mass) in the system, where $\hbar^2/2m^* = 1$

Thus, starting from the same regular potential $V(x) = \gamma\Delta'_\varepsilon(x)$, two families: (i) the non-resonant PIs with $\Gamma_0 = \emptyset$ and (ii) the resonant-tunneling PIs with non-empty sets $\Gamma_0 = \{\gamma_n\}_{n\in\mathbb{Z}}$ on which the limit transmission $\mathcal{T}$ is non-zero can be realized [31]. This is an important result saying that *different regularizations* of the δ'-distribution produce *different solutions* of Eq. (1) with the potentials $V(x) = \gamma\delta'(x)$ or $V(x) = g\delta''(x)$, contrary to the delta potential $V(x) = \alpha\delta(x)$.

3. Resonant Tunneling vs Kurasov's Theory

Another mismatch appears, if we note that the wave function $\psi(x)$ in Eq. (1) with the potential $V(x) = \gamma\delta'(x)$ must be discontinuous at the origin $x = 0$. In this case, the product $\delta'(x)\psi(x)$ is ambiguous and should be defined properly. To this end, Griffiths [5] and Kurasov [6] suggested to generalize the product $\delta'(x)\psi(x) = \psi(0)\delta'(x) - \psi'(0)\delta(x)$, valid for any continuous function $\psi(x)$ and its continuous derivative, using the following "symmetrically averaged" representation:

$$\delta'(x)\psi(x) = \frac{\psi(-0) + \psi(+0)}{2}\delta'(x) - \\ - \frac{\psi'(-0) + \psi'(+0)}{2}\delta(x). \tag{11}$$

This representation can be generalized to an "asymmetric" one-parameter form as suggested in [28].

Differentiating twice the wave function written in the form $\psi(x) = \psi(-0)\exp(-\mathrm{i}kx)\Theta(-x) + \psi(+0)\exp(\mathrm{i}kx)\Theta(x)$, $k = \sqrt{E}$, where $\Theta(x)$ is the step function, and taking the relation $\exp(\pm\mathrm{i}kx)\delta'(x) = = \delta'(x) \mp \mathrm{i}k\delta(x)$ into account, one finds [12, 28]

$$\psi''(x) = -k^2\psi(x) + [\psi'(+0) - \psi'(-0)]\,\delta(x) + \\ + [\psi(+0) - \psi(-0)]\,\delta'(x). \tag{12}$$

Using next both relations (11) and (12) in Eq. (1), we find that the connection matrix for this equation with the potential $V(x) = \gamma\delta'(x)$ takes the form (3) with

$$\alpha = 0 \quad \text{and} \quad \theta = \frac{2+\gamma}{2-\gamma}, \quad \gamma \neq \pm 2. \tag{13}$$

This is a particular result of the general theory of distributions developed by Kurasov [6] on test functions discontinuous at $x = 0$.

Thus, the Λ-matrix derived within the approach based on representation (11) *continuosly* depends on the strength constant γ as shown in (13), while the "resonant-tunneling" Λ-matrix with elements (10) takes *discrete* values in the γ-space. Both these representations will be treated below within a unique scheme using a two-scale squeezing procedure for a two-layer structure.

4. Squeezed Limit of a Bilayer Structure

It is fascinating that both the controversial representations (10) and (13) can be obtained within a unique procedure exploiting the very simple physical example of a planar heterostructure consisting of two layers separated by a distance, where the thicknesses of the layers and the distance between them squeeze simultaneously to zero. The electron motion in the systems of this type is usually confined in the longitudinal direction (say, along the x-axis); the latter is perpendicular to the transverse planes, where the electron motion is free. The three-dimensional Schrödinger equation of such a structure can be separated into longitudinal and transverse parts and finally reduced to the one-dimensional form (1).

4.1. Bilayer potential and its two-scale power-connecting parametrization

Let us consider the potential in Eq. (1) in the form

$$\bar{V}(x) = \begin{cases} V_1 & \text{for } 0 < x < l_1, \\ 0 & \text{for } l_1 < x < l_1 + r, \\ V_2 & \text{for } l_1 + r < x < l_1 + r + l_2, \end{cases} \tag{14}$$

where $V_j \in \mathbb{R}$, $j = 1, 2$, and $r > 0$. This potential is a piecewise constant function. Therefore, Eq. (1) admits an explicit solution which can be represented via the transfer matrix $\bar{\Lambda}$ as follows:

$$\begin{pmatrix} \psi(x_2) \\ \psi'(x_2) \end{pmatrix} = \bar{\Lambda} \begin{pmatrix} \psi(x_1) \\ \psi'(x_1) \end{pmatrix}, \quad \bar{\Lambda} = \begin{pmatrix} \bar{\lambda}_{11} & \bar{\lambda}_{12} \\ \bar{\lambda}_{21} & \bar{\lambda}_{22} \end{pmatrix}, \tag{15}$$

where $x_1 = 0$ and $x_2 = l_1 + r + l_2$. Its elements are

$$\bar{\lambda}_{11} = \big[\cos(k_1l_1)\cos(k_2l_2) - \\ - (k_1/k_2)\sin(k_1l_1)\sin(k_2l_2)\big]\cos(kr) - \\ - \big(k_1/k)\sin(k_1l_1)\cos(k_2l_2) + \\ + (k/k_2)\cos(k_1l_1)\sin(k_2l_2)\big]\sin(kr), \tag{16}$$

$$\bar{\lambda}_{12} = \big[(1/k_1)\sin(k_1l_1)\cos(k_2l_2) + \\ + (1/k_2)\cos(k_1l_1)\sin(k_2l_2)\big]\cos(kr) + \\ + \big[(1/k)\cos(k_1l_1)\cos(k_2l_2) - \\ - (k/k_1k_2)\sin(k_1l_1)\sin(k_2l_2)\big]\sin(kr), \tag{17}$$

$$\bar{\lambda}_{21} = -\big[k_1\sin(k_1l_1)\cos(k_2l_2) + \\ + k_2\cos(k_1l_1)\sin(k_2l_2)\big]\cos(kr) - \\ - \big[k\cos(k_1l_1)\cos(k_2l_2) - \\ - (k_1k_2/k)\sin(k_1l_1)\sin(k_2l_2)\big]\sin(kr), \tag{18}$$

$$\bar{\lambda}_{22} = \big[\cos(k_1l_1)\cos(k_2l_2) - \\ - (k_2/k_1)\sin(k_1l_1)\sin(k_2l_2)\big]\cos(kr) - \\ - \big[(k/k_1)\sin(k_1l_1)\cos(k_2l_2) + \\ + (k_2/k)\cos(k_1l_1)\sin(k_2l_2)\big]\sin(kr), \tag{19}$$

where $k = \sqrt{E}$ and $k_j = \sqrt{E - V_j}$, $j = 1, 2$. Here, $\det\bar{\Lambda} = 1$ and k_j's may be either real or imaginary. The notations with the overhead bars have been introduced for the finite-range quantities.

In order to accomplish explicitly the zero-thickness limit of the bilayer structure specified by elements (16)–(19), we introduce a squeezing parameter $\varepsilon \to 0$ and the power parametrization connected with this parameter as follows:

$$V_j = a_j\varepsilon^{-\nu}, \quad l_1 = l_2 = l = \varepsilon, \quad r = c\,\varepsilon^{\tau}, \tag{20}$$

where $a_j \in \mathbb{R}$, $c, \nu, \tau > 0$ $(j = 1, 2)$. Then the layers are described by the parameters from the four-dimensional space $\mathcal{M} := \{\nu, \tau\} \times \{a_1, a_2\}$. In the following instead of the bars, we provide the quantities by the subscript ε, replacing $\bar{V}(x) \to V_\varepsilon(x)$, $\bar{\Lambda} \to \Lambda_\varepsilon$ and $\bar{\lambda}_{ij} \to \lambda_{ij,\varepsilon}$, $i, j = 1, 2$.

4.2. Existence sets for point interactions

It follows from the explicit form of the matrix elements (16)–(19), in which the parameters are defined by Eqs. (20), that, in the limit as $\varepsilon \to 0$, we have $\lambda_{12,\varepsilon} \to 0$, while the other elements may be finite or even divergent. Note that the connecting-power parametrization (20) differs from that used in many publications (see, e.g., [9–11, 21, 22, 36–39]), where the limit of $\lambda_{12,\varepsilon}$ is finite and non-zero realizing the δ'-interaction. Our following purpose is to find those sets on the quadrant $Q_{++} := \{0 < \nu < \infty,\, 0 < \tau < \infty\}$, where the diagonal elements $\lambda_{11,\varepsilon}$ and $\lambda_{22,\varepsilon}$ are finite and non-zero as $\varepsilon \to 0$. We denote their limiting values by θ and θ^{-1}, respectively. The existence of finite θ and θ^{-1} is a necessary condition for realizing PIs, even if the element $\lambda_{21,\varepsilon}$ diverges. If this off-diagonal element is finite or zero (being θ and θ^{-1} finite), the limiting PIs are non-separated with the Λ-matrix (3). Otherwise, in the case of the divergence, the PIs are separated, obeying the BCs $\psi(\pm 0) = 0$ and acting as fully reflecting walls.

Both the limiting matrix elements θ and α found from the analysis of the $\varepsilon \to 0$ limit of the matrix elements (16) and (18) parametrized by Eqs. (20) appear to be set functions in the $\mathcal{M}$-space. Performing first the limiting procedure at each $\{\nu, \tau\}$-point, we encounter with the following characteristic sets on the Q_{++}-quadrant:

$$
\begin{aligned}
&Q_I := \{a_1, a_2 \mid 0 < \nu < 1,\, 0 < \tau < \infty\},\\
&L_\delta := \{a_1, a_2 \mid \nu = 1,\, 0 < \tau < \infty\},\\
&L_K := \{a_1, a_2 \mid 1 < \nu < 2,\, \tau = \nu - 1\},\\
&P_1 := \{a_1, a_2 \mid \nu = 2,\, \tau = 1\},\\
&L_1 := \{a_1, a_2 \mid \nu = 2,\, 1 < \tau < \infty\},\\
&L_2 := \{a_1, a_2 \mid \nu = 2,\, 2 < \tau < \infty\},\\
&L_S := \{a_1, a_2 \mid 1 < \nu < 2,\, \tau = 2(\nu - 1)\},\\
&P_2 := \{a_1, a_2 \mid \nu = \tau = 2\},\\
&Q_S := \{a_1, a_2 \mid 1 < \nu < 2,\, \nu - 1 < \tau < \infty\},\\
&Q_1 := \{a_1, a_2 \mid 1 < \nu < 2,\, \nu - 1 < \tau < 2(\nu - 1)\},\\
&Q_2 := \{a_1, a_2 \mid 1 < \nu < 2,\, 2(\nu - 1) < \tau < \infty\},
\end{aligned}
\tag{21}
$$

which are illustrated by Fig. 2. Thus, "moving" on the Q_{++}-quadrant from left to right, we examine that, on the strip Q_I, the PIs are trivial describing the perfect transmission (the Λ-matrix is the identity I). Next, on the line L_δ, the transmission is partial, and the point interactions are of the δ-type. The total strength constant of the resulting δ-potential is the algebraic sum of the layer strengths. On the set $Q_I \cup L_\delta$, there are no constraints on the strengths a_1 and a_2 (see Table). On the Q_{++}-sets displaced to the right from the L_δ, the element $\lambda_{21,\varepsilon}$ diverges in general as $\varepsilon \to 0$. However, on certain subsets of the $\{a_1, a_2\}$-plane (of Lebesgue's measure zero), the *cancellation of divergences* may occur, resulting in the finite limit of $\lambda_{21,\varepsilon}$. The cancellation effect takes place on the open set Q_S including its boundary $L_{\delta'} := L_K \cup P_1 \cup L_1$, where the potential $V_\varepsilon(x)$ converges to $\gamma\delta'(x)$ in the sense of distributions under the condition $a_1 + a_2 = 0$. Thus, the region of existence of the distribution $\gamma\delta'(x)$ is the set $L_{\delta'} \times \Sigma_0 \in \mathcal{M}$, where

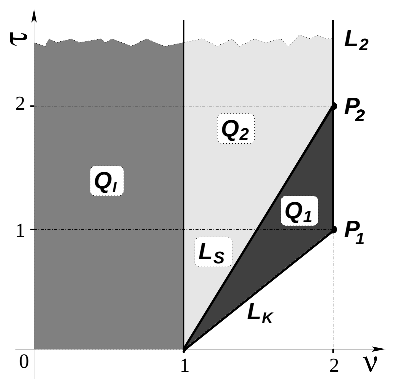

Fig. 2. Set partition defined by Eqs. (21)

$$\Sigma_0 := \{a_1, a_2 \,|\, a_1 + a_2 = 0\}. \tag{22}$$

Elements of Λ-matrix (3) as functions of $\mathcal{M}$-sets

$\mathcal{M}$-sets	θ	α
$Q_0 \times \{a_1, a_2\}$	1	0
$L_\delta \times \{a_1, a_2\}$	1	$a_1 + a_2$
$L_K \times \Sigma_c$	$-a_1/a_2$	0
$P_1 \times \Sigma'_c$	$-\frac{\sqrt{-a_1}\sin\sqrt{-a_1}}{\sqrt{-a_2}\sin\sqrt{-a_2}}$	0
$P_2 \times \Sigma'_0$	$\cos\sqrt{-a_1}/\cos\sqrt{-a_2}$	$c\sqrt{-a_1}\sin\sqrt{-a_1}\times$ $\times\sqrt{-a_2}\sin\sqrt{-a_2}$
$L_S \times \Sigma_0$	1	$-ca_1^2$
$L_2 \times \Sigma'_0$	$\cos\sqrt{-a_1}/\cos\sqrt{-a_2}$	0
$Q_2 \times \Sigma_0$	1	0
$P_1 \times \Gamma_c$	$\sinh\sqrt{\gamma}/\sin\sqrt{\gamma}$	0
$P_2 \times \Gamma_0$	$\cosh\sqrt{\gamma}/\cos\sqrt{\gamma}$	$-c\gamma\sinh\sqrt{\gamma}\sin\sqrt{\gamma}$
$L_2 \times \Gamma_0$	$\cosh\sqrt{\gamma}/\cos\sqrt{\gamma}$	0

Here, the strength $\gamma := a_1 = -\,a_2$ is the set function

$$\gamma = a_1 \begin{cases} c & \text{for } \ L_K, \\ 1+c & \text{for } \ P_1, \\ 1 & \text{for } \ L_1. \end{cases} \tag{23}$$

Consequently, in the region $Q_S \cup L_{\delta'}$, the transmission can be either zero or non-zero depending on some constraints imposed on a_1 and a_2. The L_δ-line appears to be a transient set splitting the regions of the "regular" and "singular" PIs. In its turn, the $L_{\delta'}$-line is a transient set separating the regions of the existence and non-existence of PIs.

5. Λ-Matrix for Non-Separated Point Interactions

There are two ways of the cancellation of divergences as $\varepsilon \to 0$ in the singular element $\lambda_{21,\varepsilon}$ given by Eq. (18) and parametrized by (20). The first way is to equate the *whole* expression (18) to zero, resulting in $\alpha = 0$ in Λ-matrix (3). The second way is realized, if only the term in front of $\cos(kr)$ in (18) equals zero, retaining the term in front of $\sin(kr)$ to be "free".

5.1. Realizing point interactions as $\boldsymbol{\lambda_{21,\varepsilon} \to 0}$

Imposing the constraint $\lambda_{21,\varepsilon} \to 0$ in Eq. (18), we find that the non-separated PIs can be realized only on the line L_K including its limiting point P_1. This realization occurs only on some sets in the $\{a_1, a_2\}$-space forming a family of curves. Thus, on the L_K-line, the existence set consists of two curves given by

$$\Sigma_c := \{a_1, a_2 \mid a_1 + a_2 + c\, a_1 a_2 = 0\}. \tag{24}$$

Using this relation between a_1 and a_2 in Eqs. (16) and (19) parametrized by (20), we obtain $\theta = -\,a_1/a_2$. Equating this value and expression (13), we get the $\{a_1, a_2\}$-representation of the strength constant γ for Kurasov's δ'_K-potential: $\gamma = 2(a_1 + a_2)/(a_1 - a_2)$. This representation does not coincide with the first formula in Eqs. (23) associated with the distribution $\gamma\delta'(x)$, except for the trivial case: $\Sigma_c \cap \Sigma_0 = \{0\}$. Therefore, no distributional δ'-potentials can exist on the line L_K.

However, while approaching the limiting point P_1, the situation crucially changes because of the appearance of a countable number of curves

$$\Sigma'_c := \{a_1, a_2 \mid A_1 + A_2 = c\, A_1 A_2\} \tag{25}$$

with

$$A_j := \sqrt{-\,a_j}\, \tan\sqrt{-\,a_j}\,, \quad j = 1, 2, \tag{26}$$

on which $\lambda_{21,\varepsilon} \to 0$ as $\varepsilon \to 0$. Similarly to [34], we refer this "furcation" effect as the *splitting* of the δ'_K-potential defined on the *resonance set* $L_K \times \Sigma_c$ into the countable family of generalized δ'-potentials defined on the resonance set $P_1 \times \Sigma'_c$. Beyond the resonance sets Σ_c and Σ'_c, the PIs are separated obeying the BCs $\psi(\pm 0) = 0$. Using the expressions for these resonance sets in Eqs. (16) and (19), in the limit as $\varepsilon \to 0$, we get the explicit values for the element θ in Λ-matrix (3), which are written out in Table. Particularly, as derived in [28], the intersection $\Gamma_c := \Sigma'_c \cap \Sigma_0$ [setting $a_1 = -\,a_2 = \gamma$, $\nu = 2$ with $\tau \in (2, \infty)$] yields a discrete set

$$\Gamma_c = \{\gamma \mid \tan\sqrt{\gamma} = \tanh\sqrt{\gamma}/(1 + c\sqrt{\gamma}\tanh\sqrt{\gamma})\tag{27}$$

in the γ-space. As a result, the elements of Λ-matrix (3) also take the discrete values:

$$\alpha = 0 \quad \text{and} \quad \theta = \sinh\sqrt{\gamma}/\sin\sqrt{\gamma}, \quad \gamma \in \Gamma_c. \tag{28}$$

Therefore, the four-dimensional $\mathcal{M}$-representation of resonance sets allows us to "cover" both the "continuous" and "discrete" representations from a unique point of view.

5.2. Realizing point interactions as $\boldsymbol{\lambda_{21,\varepsilon} \to \alpha} \in \mathbb{R}$

The second way of the cancellation of divergences occurs on the angular domain $L_S \cup P_2 \cup L_2 \cup Q_2$ resulting in the appearance of the resonance sets, where the set Σ_0 defined by Eq. (22) is a constituent. Thus, on the line L_S, the limiting element α in general is non-zero, and $\theta = 1$, so that the corresponding PI is of the delta type. The L_S-line is a transient set (with partial transmission) separating the regions Q_1 (full reflection) and Q_2 (perfect transmission). In the one-dimensional case, the existence of the point $\vartheta = 1/2$ that separates the whole axis $-\infty < \vartheta < \infty$ into a non-transparent half-axis and a half-axis of full transparency has been discovered by Seba in [20]. Therefore, we call this delta PI, realized due to the cancellation of divergences on the resonance set $L_S \times \Sigma_0$, the δ_S-potential.

While approaching the limiting set $P_2 \cup L_2$, the set Σ_0 splits into the countable set

$$\Sigma'_0 := \{a_1, a_2 \mid A_1 + A_2 = 0\}, \tag{29}$$

where A_1 and A_2 are defined by Eqs. (26). Correspondingly, on the resonance sets $P_2 \times \Sigma_0'$ and $L_2 \times \Sigma_0'$, the Λ-matrix elements θ and α can be computed explicitly, and their values are pointed out in Table. Similarly, the resonance set for the distributional δ'-potential is the intersection $\Gamma_0 := \Sigma_0' \cap \Sigma_0$. Setting $a_1 = -a_2 = \gamma$ and $\nu = 2$ with $\tau \to \infty$, we obtain expressions (9) and (10).

6. Concluding Remarks

Thus, the whole family of the singular PIs realized on the set $Q_S \cup L_{\delta'}$ can be interpreted as the objects with *resonant tunneling* through a single-point potential. This phenomenon emerges from the cancellation of divergences in the most singular element $\lambda_{21,\varepsilon}$ as $\varepsilon \to 0$. The two-scale parametrization (20) allows us to resolve the controversy (existing so far in the literature) between the discrete [see Eqs. (9), (10), (27), (28)] and continuous [see Eqs. (13)] presentations of Λ-matrix (3). It is convenient to present the resonance sets in the four-dimensional space $\mathcal{M}$. They are written out in the Table together with the elements θ and α of the Λ-matrix as functions of the resonance sets. The limiting transmission amplitude $\mathcal{T}$ is given in terms of these elements according to the following formula: $4\,\mathcal{T}^{-1} = (\theta + \theta^{-1})^2 + (\alpha/k)^2$ (for details, see, e.g., [31]).

Another key point is the existence of boundary cluster sets, where two types of splitting the resonance sets occur: $\Sigma_0 \implies \Sigma_0'$ and $\Sigma_c \implies \Sigma_c'$. On these sets, three types of splitting the PIs are singled out: $\delta_K' \,(L_K \times \Sigma_c) \implies \delta'(P_1 \times \Sigma_c')$, $\delta_S \,(L_S \times \Sigma_0) \implies \delta'(P_2 \times \Sigma_0',)$ and $I_P \,(Q_2 \times \Sigma_0) \implies \implies \delta'(L_2 \times \Sigma_0')$, where I_P denotes the family of PIs with perfect transmission. On the set $[Q_1 \cup (L_1 \backslash L_2)] \times \{a_1, a_2\}$ and beyond the resonance sets listed in the Table from the third line to the sixth one, the PIs are fully non-transparent fulfilling the Dirichlet BCs $\psi(\pm 0) = 0$.

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КОНТАКТНІ ВЗАЄМОДІЇ В ОДНОВИМІРНІЙ КВАНТОВІЙ МЕХАНІЦІ: СІМ'Я УЗАГАЛЬНЕНИХ δ'-ПОТЕНЦІАЛІВ

Р е з ю м е

Для дослідження проходження електронів через надзвичайно тонкі гетероструктури, що складаються з двох паралельних плоских шарів, пропонується використовувати "одноточкове" наближення. Типовим прикладом такої структури є подвійний шар, що описується потенціалом, який у границі стиснення до нульової товщини має вигляд похідної дельта-функції Дірака. Рівняння Шредінґера з цим синґулярним одновимірним потенціальним профілем породжує сім'ю контактних (точкових) взаємодій, кожна з яких (названа "потенціалом δ'-розподілу") залежить від способу реґуляризації. Використовуючи двомасштабну степенево-пов'язувану параметризацію потенціалу, що описує подвійний шар, усунуто всі розбіжності, які досі широко дискутувались у літературі стосовно взаємодії із потенціалом вигляду похідної дельта-функції Дірака. При застосовуванні даної параметризації, стало можливим розширити сім'ю потенціалів δ'-розподілу до цілого класу "узагальнених" δ'-потенціалів. Показано, що в границі стиснення подвійного шару до нульової товщини резонансне тунелювання проявляється у вигляді гострих піків, які локалізуються на множинах нульової міри Лебеґа (названі резонансними множинами). Для представлення цих множин введено чотиривимірний простір параметрів. Показано, що проходження електронів на комплементарних множинах у цьому просторі є абсолютно відбиваючим.