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V.V. KUZMICHEV, V.E. KUZMICHEV

Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine
(14b, Metrolohichna Str., Kyiv 03143, Ukraine; e-mail: vkuzmichev@bitp.kiev.ua)

QUANTUM CORRECTIONS TO THE DYNAMICS OF A GRAVITATIONAL SYSTEM[1]

A short introduction into the theory of quantum gravitational systems with a finite number of degrees of freedom is given. The theory is based on the method of quantization of constrained systems. The state vector of the system satisfies a set of wave equations which describes the time evolution of the system in the space of quantum fields. The state vector in such an approach can be normalized to unity. The theory permits a generalization to negative values of the scale factor and, being applied to cosmology, leads to the new understanding of the evolution of the universe. It gives an insight into the reasons why the regime of the expansion may change from acceleration to deceleration or vice versa, revealing a new type of quantum forces acting like dark matter and dark energy in the universe.

Keywords: quantum gravity, quantum geometrodynamics, cosmology.

1. Introduction

The method of quantization of constrained systems can be taken as a basis of the quantum theory of gravity suitable for the investigation of cosmological and other quantum gravitational systems. The canonical approach to the quantization successful in constructing the nonrelativistic quantum mechanics and quantum field theories in the flat spacetime encounters the well-known difficulties, when applied to gravity, such as the understanding of the time evolution, divergence of the norm of state vectors, measurement problem, and others.

The apparent manifestation of the problem of time is the absence of an explicit time parameter in the Wheeler–DeWitt equation considered as the main dynamical equation of the theory. It was realized that one way to solve this problem could be rewriting the classical constraint equations to obtain a Schrödinger-type equation, as a preliminary stage to the quantization. It was proved that general relativity could not be viewed as a parametrized field theory [1]. The concepts of "matter clocks and reference fluids" [2, 3] go back to DeWitt [4], who studied the coupling of clocks to an elastic media. It can be shown that the perfect fluids are a special case of DeWitt's relativistic elastic media [5]. The notion of reference fluid allows one to define the reference frame as a dynamical system. The spacetime geometry is affected by the reference fluid which is considered as a true material system coupled to gravity. As a result, the constraints and Hamilton equations take a new form.

A model with a finite number of degrees of freedom may provide a reasonable framework for addressing the problems of quantum gravity. The homogeneous minisuperspace models have been proven to be successful – consistent with observations and having predictive power – in classical cosmology. This appears explicable, in view of the fact that the Universe can, to first approximation, be considered as being homogeneous, and gives rise to the hope for that homogeneous models could be useful in quantum cosmology as well.

In the simplest case of the maximally symmetric geometry with the Robertson–Walker metric, the geometric properties of the system are determined by a single variable, namely the cosmic scale factor a. The matter sector of the homogeneous isotropic gravitational system will be taken in the form of a uniform scalar field ϕ. This field can be interpreted as a surrogate of all possible real physical fields of matter averaged with respect to the spin, space, and other degrees of freedom. In addition, it will be accepted that the system contains a reference fluid in the form of a

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relativistic matter (further referred as radiation). Following Dirac's approach to quantum gravity, we do not solve constraints prior to quantization, but convert the second-class constraints into the first-class ones which become constraints on the state vector (wave function) in quantum theory. In this theory, the state vector satisfies the set of wave equations which describes the time evolution of a quantum system in a generalized space of quantum fields.

2. Basic Equations and Quantization

The Hamiltonian for the gravitational system is given by (in Planck units)

$$H = \frac{N}{2}\left\{-\pi_a^2 - \kappa a^2 + a^4[\rho_\phi + \rho_\gamma]\right\} + \\ + \lambda_1\left\{\pi_\Theta - \frac{1}{2}\,a^3\rho_0 s\right\} + \lambda_2\left\{\pi_{\tilde\lambda} + \frac{1}{2}\,a^3\rho_0\right\}, \qquad (1)$$

where π_a, π_Θ, $\pi_{\tilde\lambda}$ are the momenta canonically conjugate with the variables a, Θ, $\tilde\lambda$; ρ_ϕ is the energy density of matter (the field ϕ), $\rho_\gamma(\rho_0, s)$ is the energy density of a perfect fluid, which defines a material reference frame, and it is a function of the density of the rest mass ρ_0 and the specific entropy s; the Θ is the thermasy which defines the temperature, $\mathcal{T} = \Theta_{,\nu}U^\nu$; the U^ν is the four-velocity; the $\tilde\lambda$ is the potential for the specific Gibbs free energy $\mathcal{F}$ taken with the inverse sign, $\mathcal{F} = -\tilde\lambda_{,\nu}U^\nu$. The $\kappa = +1, 0, -1$ is the curvature constant. The N, λ_1, and λ_2 are the Lagrange multipliers [6, 7].

The Hamiltonian is a linear combination of constraints and thus weakly vanishes, $H \approx 0$. The variations of the Hamiltonian with respect to N, λ_1, and λ_2 give three constraint equations.

In quantum theory, the first-class constraint equations become constraints on the state vector Ψ and define the space of physical states. Passing from classical variables to corresponding operators, using the conservation laws, and introducing the noncoordinate co-frame $h d\tau = s d\Theta - d\tilde\lambda$, $h dy = s d\Theta + d\tilde\lambda$, where h is the specific enthalpy, τ is the proper time, and y is a supplementary variable, we obtain [6, 7]

$$\left(-i\partial_T - \frac{1}{2}E\right)\Psi = 0, \quad \partial_y\Psi = 0, \\ \left(-\partial_a^2 + \kappa a^2 - 2a\hat H_\phi - E\right)\Psi = 0, \qquad (2)$$

where $\hat H_\phi$ is the Hamiltonian operator of the matter field, $d\tau = a dT$, and $E = \rho_\gamma a^4 = \text{const}$.

3. General Solution

The Hamiltonian of matter $\hat H_\phi$ can be diagonalized by means of some state vectors $\langle\chi|u_k\rangle$ in the representation of the generalized field variable $\chi = \chi(a, \phi)$,

$$\langle u_k|\hat H_\phi|u_{k'}\rangle = M_k(a)\,\delta_{kk'}, \qquad (3)$$

which determines the proper energy $M_k(a) = \frac{1}{2}a^3\rho_m$ of a barotropic fluid in the discrete and/or continuous kth state in the comoving volume $\frac{1}{2}a^3$ with energy density ρ_m.

The simple model of matter in the form of a scalar field with the potential $V(\phi) = \lambda_\alpha\phi^\alpha$, where λ_α is the coupling constant, and α takes arbitrary non-negative values, $\alpha \geq 0$, allows one to describe different epochs in the evolution of the universe taken as the quantum system. In the case of the model ϕ^0, the field ϕ averaged over its quantum states reproduces vacuum (dark energy) in the kth state (accelerating expansion epoch). The model ϕ^1 describes the strings in the kth state (formation of a cosmological cellular structure). In the model ϕ^2, the scalar field, after averaging over quantum states, turns into dust with the total mass $M_k = \sqrt{2\lambda_2}(k + \frac{1}{2})$, where k is the number of dust particles (non-relativistic matter epoch). The model ϕ^4 leads to the relativistic matter (epoch before the recombination). In the case $\alpha = \infty$, the field ϕ averaged over the states $|u_k\rangle$ reduces to the stiff Zel'dovich matter (vanishing coupling constant).

The state vector Ψ in the (a, χ)-representation can be represented in the form of a superposition of all possible kth states of the barotropic fluid $\Psi = \sum_k |u_k\rangle\langle u_k|\Psi\rangle$, where $\langle u_k|\Psi\rangle \equiv \psi_k(a, T)$ satisfies the differential equations

$$\left(-i\partial_T - \frac{1}{2}E\right)\psi_k(a, T) = 0, \\ \left(-\partial_a^2 + \kappa a^2 - 2aM_k(a) - E\right)\psi_k(a, T) = 0. \qquad (4)$$

The general solution of set (4) has the form

$$\psi_k(a, T) = \sum_n c_{nk}(T) f_{nk}(a) \qquad (5)$$

with

$$c_{nk}(T) = c_{nk}(T_0)\exp\left\{i\frac{1}{2}E_n(T - T_0)\right\}, \qquad (6)$$

where the summation with respect to discrete values of n and the integration with respect to continuous ones are assumed.

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The wave functions $f(a) \equiv f(a, T_0)$ satisfy the equation

$$\left(-\partial_a^2 + \kappa a^2 - 2aM_k(a) - E\right) f(a) = 0, \tag{7}$$

where $f(a) = f_{nk}(a)$ and $E = E_n$ for a discrete nth state of radiation and $f(a) = f_{Ek}(a)$ for a continuous Eth state of radiation, T_0 is an arbitrary constant taken as a time reference point. The coefficient $c_{nk}(T_0)$ gives the probability $|c_{nk}(T_0)|^2$ to find the system in the nth and kth states at the instant T_0.

The state vector Ψ appears be normalized, $\langle\Psi|\Psi\rangle = 1$, under the condition that the probability summed over all possible quantum states of radiation and the barotropic fluid equals unity.

Equations (2) in the case of matter in the form of a scalar field with $\hat{H}_\phi(-a) = -\hat{H}_\phi(a)$ are invariant under the inversion $a \to -a$. The physical meaning of the solutions of Eq. (7) in the domain $a < 0$ is clarified from the expression

$$T(\tau) = T_0 + \int\limits_0^\tau \frac{d\tau'}{a(\tau')}, \quad \text{at } T(0) = T_0. \tag{8}$$

It gives $T(\tau) = T(-\tau)$ at $a(-\tau) = -a(\tau)$.

The scale factor $a \in (-\infty, 0]$ corresponds to the values $\tau \in (-\infty, 0]$, and the scale factor $a \in [0, +\infty)$ corresponds to the values $\tau \in [0, +\infty)$. As a result for the arrow of time from $\tau = -\infty$ to $\tau = +\infty$, the state vector Ψ describes the quantum gravitational system contracting on the semiaxis $a < 0$, since $|a|$ decreases, and expanding on the semiaxis $a > 0$, because $|a|$ increases.

The instant of time $\tau = 0$ can be interpreted as the instant of the nucleation of the quantum system expanding in time from the point $a = 0$, although any nucleation "from nothing" does not occur physically. What happens at the instant $\tau = 0$ is that the regime of the preceding contraction of the system changes into the subsequent expansion. The state vector contains all information about the system as a whole: the cross-section $|a| = \text{const}$ determines the quantum state of the system at the time instant τ, when such a value of the scale factor holds.

If one applies the above-described scenario to our universe at the Planck epoch, interpreting the passage through the point $a = 0$ at $\tau = 0$ as the nucleation of the expanding universe with $a > 0$ at $\tau > 0$, then the answer to the question "What was with the quantum system before the instant of the nucleation of a universe of our (expanding) type?" can be given: another universe with the same mass-energy $M_k(a)$ and the wave function $f(a)$ characterized by the same quantum numbers for matter and radiation as in the nucleated universe has existed. However, that universe has been contracting up to the state with $a = 0$, which will not necessarily be singular.

The intensity distribution of matter-energy $I(a) = M_k(a)|f_{nk}(a)|^2$ can be calculated for the system, in which the barotropic fluid and radiation are in some definite quantum states [8]. The study of the motion in time of the minimum wave packet for a spatially closed system demonstrates that matter is distributed over a and τ in the form of periodic structures like petals or stretched bubbles and displaced to their edges. These structures are limited by the value $a = 2M_k$ with respect to a, and their number increases with time.

4. Non-Linear Hamilton–Jacobi Equation

The state vector averaged over the states of matter ϕ has the form $f_k(a) \sim \exp(iS_k(a))/\sqrt{\partial_a S_k(a)}$, where the function S_k satisfies the generalized Hamilton–Jacobi equation [9]

$$(\partial_a S_k)^2 + \kappa a^2 - 2aM_k(a) - E = \frac{3}{4}\left(\frac{\partial_a^2 S_k}{\partial_a S_k}\right)^2 - \frac{1}{2}\frac{\partial_a^3 S_k}{\partial_a S_k}. \tag{9}$$

The right-hand side of Eq. (9) is proportional to $\hbar^2$ (in ordinary physical units) and responsible for quantum corrections to the dynamics of the system.

Using the relation between the classical momentum $\pi_a = -\frac{da}{dT}$ and the phase $S_k(a)$ [9], Eq. (9) can be rewritten in the form of the energy conservation law for a test particle with zero energy moving along a coordinate line a

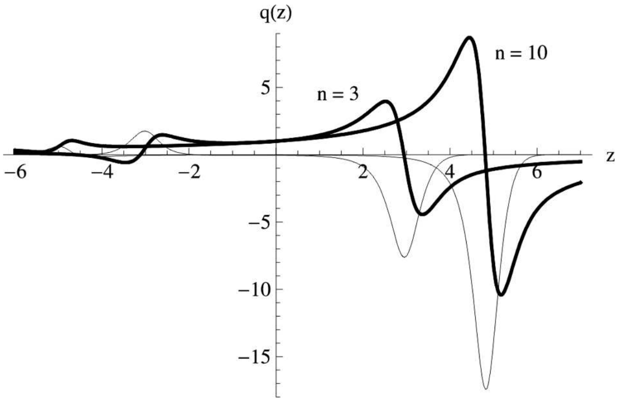
$$\frac{1}{2}\left(\frac{da}{dT}\right)^2 + U(a) = 0 \tag{10}$$

in the potential well

$$U(a) = \frac{1}{2}\left[\kappa a^2 - 2aM_k(a) - Q_k(a) - E\right]. \tag{11}$$

The function

$$Q_k(a) = -\partial_a^2 S_E + \frac{1}{2}\left[\left(\frac{\partial_a^2 S_E}{\partial_a S_E}\right)^2 - \frac{\partial_a^3 S_E}{\partial_a S_E}\right], \tag{12}$$



The real q_R (boldface curve) and imaginary q_I (thin curve) parts of the deceleration parameter (13) *versus* the deviation $z = a - M$ for solution (15) with $n = 10$ and $n = 3$

where $S_E = -iS_k$ is the Euclidean phase, determines the quantum correction ρ_Q to the energy density of matter in the form $\rho_Q = a^{-4} Q_k(a)$.

In the semiclassical limit, $\rho_Q \sim -a^{-6}$, and Eq. (10) reduces to the equation of the Einstein–Cartan theory of gravity with torsion [9]. In this case, the matter coupled to gravity can be considered as a perfect fluid with spin [10]. Such a fluid often called the Weyssenhoff fluid [11], is a perfect incompressible fluid every element of which is interpreted as a particle with spin.

The deceleration parameter q in the model under consideration is reduced to the expression

$$q = 1 - \frac{a}{2U}\frac{dU}{da}. \tag{13}$$

5. Example

In the case of closed system filled with dust and relativistic matter, Eq. (9) has two solutions [12]

$$\partial_z S_1(z) = i \frac{e^{z^2} H_n^{-2}(z)}{2\int\limits_0^z dx\, e^{x^2} H_n^{-2}(x)}, \tag{14}$$

$$\partial_z S_2(iz) = -\frac{e^{-z^2} H_{-n-1}^{-2}(iz)}{2\int\limits_0^{iz} dx\, e^{x^2} H_{-n-1}^{-2}(x)}, \tag{15}$$

where $H_\nu(y)$ is the Hermitian polynomial, $z = a - M$, and $E + M^2 = 2n + 1$. Here, M is the total mass of k non-interacting identical particles with the masses $\sqrt{2\lambda_2}$.

Usually, the second solution is discarded as unphysical. However, in quantum cosmology both solutions should be considered, since, only in such an approach, one can obtain nontrivial results about topological properties of the universe as an essentially quantum system and clarify the nature of dark matter and dark energy.

In Figure, the real q_R and imaginary q_I parts of the deceleration parameter (13) are shown as functions of the deviation z for the potential well (11) with solutions (15) at $n = 10$ and $n = 3$. In the region $|z| \le M$, where $|q_R| \gg |q_I|$ (i.e., $|q_I/q_R|_{z=0} \approx 0.02$), the contribution from q_I can be neglected. In this stage, the universe expands with deceleration, since the antigravitational action of the forces performing the positive work is not enough to overcome the attraction of ordinary and dark matters. The value $q_R(z = 0) = 1$ reproduces the results of general relativity. At the point $z = 0$, we have $a = M$. In the region $a \approx 2M$, the redistribution of energy takes place in the universe, as demonstrated by the peaks on the curves q_R and q_I in Figure. The forces of attraction and repulsion compete with each other at $a < 2M$, where $q_R > 0$ and $q_I < 0$. At reaching the region $z > M$, where $a > 2M$, both parts of the deceleration parameter become negative, demonstrating that the expansion of the universe is accelerating. Starting from the point $z \simeq 1.5M$ ($z = 6$ for $n = 10$), the parameter q_I vanishes and the rate of expansion is described only by the real part $q_R < 0$. In the limit $z \to +\infty$, the forces of attraction and repulsion will exactly compensate each other.

6. Concluding Remarks

In this note, we present some results of our studies of the influence of the quantum nature of gravity on properties of systems with a finite number of degrees of freedom. In particular, on the basis of the wave equations of quantum cosmology for the exactly solvable model, one can explain the accelerating expansion in the early universe (the domain of comparatively small values of quantum numbers) and a later transition from the decelerating expansion to the accelerating expansion of the universe (the domain of the very large values of quantum numbers) from a single approach. Another result worth mentioned here is that Hamilton–Jacobi equations of the theory can be reduced to the equations of the Einstein–Cartan theory of gravity with torsion. These equations can be considered as describing the homogeneous, isotropic, and spatially closed universe filled with the substance in the form of a perfect fluid with spin (Weyssenhoff fluid).

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В.В. Кузьмичов, В.Є. Кузьмичов

КВАНТОВІ ПОПРАВКИ
ДО ДИНАМІКИ ГРАВІТАЦІЙНОЇ СИСТЕМИ

Р е з ю м е

Дано короткий вступ до теорії квантових гравітаційних систем зі скінченною кількістю ступенів вільності. Теорія заснована на методі квантування систем із в'язями. Вектор стану системи задовольняє набору хвильових рівнянь, який описує еволюцію системи у часі в просторі квантових полів. У такому підході вектор стану можна нормувати на одиницю. Теорія дозволяє зробити узагальнення на область від'ємних значень масштабного фактора і, при застосуванні до космології, веде до нового розуміння еволюції всесвіту. Теорія дає розуміння причин, через які режим розширення може змінюватися від прискорення до уповільнення або навпаки, виявляючи новий тип квантових сил, що діють у всесвіті подібно до темної матерії та темної енергії.