https://doi.org/10.15407/ujpe64.12.1112

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FRADKIN EQUATION FOR A SPIN-3/2 PARTICLE IN THE PRESENCE OF EXTERNAL ELECTROMAGNETIC AND GRAVITATIONAL FIELDS[1]

Fradkin's model for a spin-3/2 particle in the presence of external fields is investigated. Applying the general Gel'fand–Yaglom formalism, we develop this model on the base of a set of six irreducible representations of the proper Lorentz group, making up a 20-component wave function. Applying the standard requirements such as the relativistic invariance, single nonzero mass, spin $S = 3/2$, P-symmetry, and existence of a Lagrangian for the model, we derive a set of spinor equations, firstly in the absence of external fields. The 20-component wave function consists of a bispinor and a vector-bispinor. In the absence of external fields, the Fradkin model reduces to the minimal Pauli–Fierz (or Rarita–Schwinger) theory. Details of this equivalence are given. Then we take the presence of external electromagnetic fields into account. It turns out that the Fradkin equation in the minimal form contains an additional interaction term governed by electromagnetic tensor $F_{\alpha\beta}$. In addition, we consider the external curved space-time background. In the generally covariant case, the Fradkin equation contains the additional gravitational interaction term governed by the Ricci tensor $R_{\alpha\beta}$. If the electric charge of a particle is zero, the Fradkin model remains correct and describes a neutral Majorana-type spin-3/2 particle interacting additionally with the geometric background through the Ricci tensor.

Keywords: Gel'fand–Yaglom formalism, spin-3/2 particle, Fradkin theory, external electromagnetic field, curved space-time, non-minimal interaction, Majorana particle.

1. Introduction

Within the general theory of relativistic wave equations, in addition to the simplest commonly used equations for particles of spins 0, 1/2, 1, 3/2, and 2, more complicated equations may be proposed. These generalized models are based on the use of extended sets of irreducible representations of the Lorentz group. For instance, there are known equations for particles with several mass parameters $(M_1, M_2, ...)$, with several spin parameters ($S = 0$–1; $1/2$–$3/2$), and with additional intrinsic electromagnetic characteristics such as the anomalous magnetic moment ($S = 1/2$, 1, 2), quadrupole moment ($S = 1$), polarizability ($S = 0$, 1), Darwin–Cox structure ($S = 0$), and others (for more details, see [1–50]). Such additional intrinsic characteristics physically manifest themselves in the presence of external fields, for instance, electromagnetic and gravitational ones.

In the frames of the general theory [11, 47, 50] of the first-order relativistic wave equations $(\Gamma_\mu \partial_\mu + M)\Phi = 0$ with the use of extended sets of Lorentz-group representations, we will examine the approach proposed by Fradkin [12] to describe a particle with spin 3/2. Till the present time, it is not clear which additional structure underlies this generalized wave equation. Below, we investigate this Fradkin model in detail.

[1] This work is based on the results presented at the XI Bolyai–Gauss–Lobachevskii (BGL-2019) Conference: Non–Euclidean, Noncommutative Geometry and Quantum Physics.

2. Gel'fand–Yaglom Basis

In this model, a 20-component wave function Ψ of the particle transforms as the following set of six irreducible representations (enumerated by $1, ..., 6$):

$$\begin{aligned} &(1/2, 0) \sim 1, \quad (0, 1/2) \sim 2, \quad (1, 1/2) \sim 5, \\ &(1/2, 1) \sim 6, \quad (1/2, 0)' \sim 3, \quad (0, 1/2)' \sim 4. \end{aligned} \tag{1}$$

The corresponding linking scheme is

$$\begin{array}{ccccccc} & & 4 & - & 3 & & \\ & & | & & | & & \\ 5 & - & 1 & - & 2 & - & 6 \\ | & & & & & & | \\ 3 & & - & - & - & & 4. \end{array}$$

In accordance with the general theory, first, we should determine the matrix Γ_4. Its spin-blocks may have the structure (for more details, see [47]):

$$C^{(\frac{1}{2})} = \begin{vmatrix} c_{12} & c_{14} & i\sqrt{3}c_{15} \\ c_{32} & c_{34} & i\sqrt{3}c_{35} \\ i\sqrt{3}c_{62} & i\sqrt{3}c_{64} & c_{65} \end{vmatrix} \otimes \gamma_4, \quad C^{(\frac{3}{2})} = 2c_{65} I_2 \otimes \gamma_4. \tag{2}$$

Due to the uniqueness of the spin and mass of the particle, we have restriction, $c_{65} = 1/2$, and further simplify the spin-blocks as $C^{(\frac{1}{2})} = \beta^{(\frac{1}{2})} \otimes \gamma_4, C^{(\frac{3}{2})} = I_2 \otimes \gamma_4$. The minimal polynomial of the matrix

$$\beta^{(\frac{1}{2})} = \begin{vmatrix} c_{12} & c_{14} & i\sqrt{3}c_{15} \\ c_{32} & c_{34} & i\sqrt{3}c_{35} \\ i\sqrt{3}c_{62} & i\sqrt{3}c_{64} & 1/2 \end{vmatrix} \tag{3}$$

may be of two forms:$[\beta^{(\frac{1}{2})}]^2 = 0$ or $[\beta^{(\frac{1}{2})}]^3 = 0$. The first variant leads to the simpler well-known Pauli–Fierz model [3, 4]. In what follows, we examine the second variant. After performing the needed calculations, from the cubic minimal polynomial, we derive the following restrictions for coefficients: c_{kn}:

$$\begin{aligned} &c_{12} + c_{34} = -1/2, \\ &c_{12}c_{34} - c_{32}c_{14} + 3(c_{15}c_{62} + c_{35}c_{64}) = 1/4, \\ &(c_{12}c_{34} - c_{32}c_{14}) + 6(c_{62}c_{34}c_{15} + c_{64}c_{35}c_{12} - \\ &- c_{32}c_{64}c_{15} - c_{14}c_{35}c_{62}) = 0. \end{aligned}$$

Now, we consider the consequences of the Lagrangian description of the model. A possible structure of the relevant matrix of the bilinear form is $(a, b = \pm 1)$

$$\eta^{(\frac{1}{2})} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -1 \end{vmatrix} \otimes \gamma_4, \quad \eta^{(\frac{3}{2})} = I_2 \otimes \gamma_4, \tag{4}$$

and we get the following restrictions on the coefficients: c_{12} and c_{34} are real-valued, and remaining ones obey the constraints $c_{62} = ac_{15}^*, c_{64} = bc_{35}^*, c_{32} = abc_{14}^*$. Thus, we arrive at the spin-block $\beta^{(\frac{1}{2})}$ in the form

$$\beta^{(\frac{1}{2})} = \begin{vmatrix} c_{12} & c_{14} & i\sqrt{3}c_{15} \\ abc_{14}^* & c_{34} & i\sqrt{3}c_{35} \\ i\sqrt{3}ac_{15}^* & i\sqrt{3}bc_{35}^* & 1/2 \end{vmatrix}, \tag{5}$$

while the coefficients must obey three following constraints:

$$\begin{aligned} &c_{12} + c_{34} = -1/2, \\ &c_{12}c_{34} - abc_{14}c_{14}^* + 3(ac_{15}c_{15}^* + bc_{35}c_{35}^*) = 1/4, \\ &(c_{12}c_{34} - abc_{14}c_{14}^*) + 6(ac_{15}^*c_{34}c_{15} + bc_{35}^*c_{35}c_{12} - \\ &- ac_{14}^*c_{35}^*c_{15} - ac_{14}c_{35}c_{15}^*) = 0. \end{aligned}$$

So, the matrix Γ_4 is constructed.

3. Spin-Vector Equations

We will use the notations

$$(\sigma^4)_{\dot{a}b} = \begin{vmatrix} i & 0 \\ 0 & i \end{vmatrix},$$

$$(\sigma^1)_{\dot{a}b} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, (\sigma^2)_{\dot{a}b} = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, (\sigma^3)_{\dot{a}b} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

and the abridged notations for coefficients, namely,

$$\begin{aligned} &\lambda_1 = c_{12}, \lambda_2 = c_{14}, \lambda_3 = -i\sqrt{2}c_{15}, \lambda_4 = c_{32}, \\ &\lambda_5 = c_{34}, \lambda_6 = -i\sqrt{2}c_{35}, \lambda_7 = i\sqrt{2}c_{62}, \lambda_8 = i\sqrt{2}c_{64}. \end{aligned}$$

After a rather long calculation, we arrive at the 2-spinor form of the Fradkin system:

$$\lambda_1 \partial^{\dot{a}b}\psi_b + \lambda_2 \partial^{\dot{a}b}\psi'_b + \lambda_3 \partial^b_{\dot{c}}\psi_b^{(\dot{a}\dot{c})} + M\psi^{\dot{a}} = 0, \tag{6}$$

$$\lambda_1 \partial_{a\dot{b}}\psi^{\dot{b}} + \lambda_2 \partial_{a\dot{b}}\psi'^{\dot{b}} + \lambda_3 \partial^c_{\dot{b}}\psi^{\dot{b}}_{(ac)} + M\psi_a = 0, \tag{7}$$

$$\lambda_4 \partial^{\dot{a}b}\psi_b + \lambda_5 \partial^{\dot{a}b}\psi'_b + \lambda_6 \partial^b_{\dot{c}}\psi_b^{(\dot{a}\dot{c})} + M\psi'^{\dot{a}} = 0, \tag{8}$$

$$\lambda_4 \partial_{a\dot{b}} \psi^{\dot{b}} + \lambda_5 \partial_{a\dot{b}} \psi'^{\dot{b}} + \lambda_6 \partial^{c}_{\dot{b}} \psi^{\dot{b}}_{(ac)} + M\psi'_a = 0, \tag{9}$$

$$\frac{\lambda_7}{2}(\partial^{\dot{c}}_{a}\psi_b + \partial^{\dot{c}}_{b}\psi_a) + \frac{\lambda_8}{2}(\partial^{\dot{c}}_{a}\psi'_b + \partial^{\dot{c}}_{b}\psi'_a) + + \frac{1}{2}(\partial_{\dot{n}a}\psi^{(\dot{n}\dot{c})}_{b} + \partial_{\dot{n}b}\psi^{(\dot{n}\dot{c})}_{a}) + M\psi^{\dot{c}}_{(ab)} = 0, \tag{10}$$

$$\frac{\lambda_7}{2}(\partial^{\dot{a}}_{c}\psi^{\dot{b}} + \partial^{\dot{b}}_{c}\psi^{\dot{a}}) + \frac{\lambda_8}{2}(\partial^{\dot{a}}_{c}\psi'^{\dot{b}} + \partial^{\dot{b}}_{c}\psi'^{\dot{a}}) + + \frac{1}{2}(\partial^{n\dot{a}}\psi^{\dot{b}}_{(nc)} + \partial^{n\dot{b}}\psi^{\dot{a}}_{(nc)}) + M\psi^{(\dot{a}\dot{b})}_{c} = 0, \tag{11}$$

where $\partial_{a\dot{b}} = \frac{1}{i}\partial_\mu \sigma^\mu_{a\dot{b}}$. The coefficients λ_i obey the restrictions

$$\lambda_1 + \lambda_5 = -\frac{1}{2},\ \lambda_1\lambda_5 - \lambda_2\lambda_4 + \frac{3}{2}(\lambda_3\lambda_7 + \lambda_6\lambda_8) = \frac{1}{4},$$
$$\lambda_1\lambda_5 - \lambda_2\lambda_4 = 3[(\lambda_2\lambda_6 - \lambda_3\lambda_5)\lambda_7 + (\lambda_3\lambda_4 - \lambda_1\lambda_6)\lambda_8],$$
$$\lambda_1^* = \lambda_1,\ \lambda_5^* = \lambda_5,\ \lambda_7^* = c_1\lambda_3,\ \lambda_8^* = c_2\lambda_6,\ \lambda_4 = c_1c_2\lambda_2^*,$$
$$c_1 = \pm 1,\ c_2 = \pm 1.$$

Depending on the choice c_1 and c_2, we have four different variants. For definiteness, we will study the variant with $c_1 = +1$ and $c_2 = +1$.

The spinors in Eqs. (6)–(11) can be presented as spin-vectors, according to the formulas

$$\psi^{(\dot{a}\dot{b})}_{c} = \frac{1}{2}(\sigma^{\mu\dot{a}}_{c}\psi^{\dot{b}}_{\mu} + \sigma^{\mu\dot{b}}_{c}\psi^{\dot{a}}_{\mu}),$$
$$\psi^{\dot{c}}_{(ab)} = \frac{1}{2}(\sigma^{\mu\dot{c}}_{a}\psi_{\mu b} + \sigma^{\mu\dot{c}}_{b}\psi_{\mu a}), \tag{12}$$
$$\psi_a = \sigma^{\mu}_{a\dot{b}}\psi^{\dot{b}}_{\mu}, \psi^{\dot{a}} = \sigma^{\mu\dot{a}b}\psi_{\mu b}, \psi'_a = \psi_{0a}, \psi'^{\dot{a}} = \psi^{\dot{a}}_{0}.$$

Further, the above spinor equations can be transformed to the vector-bispinor form:

$$\psi_\mu = \begin{vmatrix} \psi^{\dot{a}}_{\mu} \\ \psi_{\mu b} \end{vmatrix},\ \psi_0 = \begin{vmatrix} \psi^{\dot{a}}_{0} \\ \psi_{0b} \end{vmatrix},\ \gamma_\mu = \frac{1}{i}\begin{vmatrix} 0 & \sigma^{\mu\dot{a}b} \\ \sigma^{\mu}_{a\dot{b}} & 0 \end{vmatrix}, \tag{13}$$

$$\lambda_1\hat{\partial}\gamma_\mu\psi_\mu - i\lambda_2\hat{\partial}\psi_0 + + \frac{\lambda_3}{2}\{\hat{\partial}\gamma_\mu\psi_\mu - 4\partial_\mu\psi_\mu\} + M\gamma_\mu\psi_\mu = 0,$$
$$i\lambda_4\hat{\partial}(\gamma_\mu\psi_\mu) + \lambda_5\hat{\partial}\psi_0 - 2i\lambda_6\left\{(\partial_\mu\psi_\mu) - - \frac{1}{4}\hat{\partial}(\gamma_\mu\psi_\mu)\right\} + M\psi_0 = 0, \tag{14}$$
$$\lambda_7\left\{\partial_\lambda\gamma_\mu\psi_\mu - \frac{1}{4}\gamma_\lambda\hat{\partial}\gamma_\mu\psi_\mu\right\} - i\lambda_8\left\{\partial_\lambda\psi_0 - \frac{1}{4}\gamma_\lambda\hat{\partial}\psi_0\right\} + + \frac{1}{8}\{3\gamma_\lambda\hat{\partial}\gamma_\mu\psi_\mu - 4\partial_\lambda\gamma_\mu\psi_\mu + 8\hat{\partial}\psi_\lambda - 4\gamma_\lambda\partial_\mu\psi_\mu\} + + M\left\{\psi_\lambda - \frac{1}{4}\gamma_\lambda\gamma_\mu\psi_\mu\right\} = 0.$$

It should be noted that convoluting the last equation with γ_λ gives four identities $0 = 0$. This means that, in fact, system (14) has only 20 independent equations among 24 ones. This correlates with the total number of components in one bispinor and one vector-bispinor.

System (14) can be transformed to other (equivalent) form, when one equation contains the terms $M\psi_0$ and $M\psi_\lambda$:

$$i\lambda_4\hat{\partial}(\gamma_\mu\psi_\mu) + \lambda_5\hat{\partial}\psi_0 - - 2i\lambda_6\left\{(\partial_\mu\psi_\mu) - \frac{1}{4}\hat{\partial}(\gamma_\mu\psi_\mu)\right\} + M\psi_0 = 0,$$
$$\left(\lambda_7 - \frac{1}{2}\right)\partial_\lambda(\gamma_\mu\psi_\mu) + + \frac{1}{4}\left(\lambda_1 + \frac{\lambda_3}{2} - \lambda_7 + \frac{3}{2}\right)\gamma_\lambda\hat{\partial}(\gamma_\mu\psi_\mu) - - i\lambda_8\partial_\lambda\psi_0 + \frac{i}{4}(\lambda_8 - \lambda_2)\gamma_\lambda\hat{\partial}\psi_0 + \hat{\partial}\psi_\lambda - - \frac{1}{2}(1 + \lambda_3)\gamma_\lambda(\partial_\mu\psi_\mu) + M\psi_\lambda = 0. \tag{15}$$

4. Fradkin and Pauli–Fierz Equations

Now, we consider the relationships between the Fradkin and Pauli–Fierz equations. We start with Eqs. (14). Let us multiply the first equation in (14) by λ_7 and the second one by $(-i\lambda_8)$. After summing two results, we get

$$\left(-\frac{1}{2}\hat{\partial} + M\right)[\lambda_7(\gamma_\mu\psi_\mu) - i\lambda_8\psi_0] + + \hat{\partial}[(\lambda_4\lambda_8 - \lambda_5\lambda_7)(\gamma_\mu\psi_\mu) - i(\lambda_2\lambda_7 - \lambda_1\lambda_8)\psi_0] - - 2(\lambda_3\lambda_7 + \lambda_6\lambda_8)[(\partial_\mu\psi_\mu) - \frac{1}{4}\hat{\partial}(\gamma_\mu\psi_\mu)] = 0.$$

Similarly, multiplying the first equation in (14) by $(\lambda_4\lambda_8 - \lambda_5\lambda_7)$, and the second one by $[-i(\lambda_2\lambda_7 - \lambda_1\lambda_8)]$ and summing two results, we obtain

$$M[(\lambda_4\lambda_8 - \lambda_5\lambda_7)(\gamma_\mu\psi_\mu) - i(\lambda_2\lambda_7 - \lambda_1\lambda_8)\psi_0] = = (\lambda_1\lambda_5 - \lambda_2\lambda_4\left\{\frac{2}{3}\left[(\partial_\mu\psi_\mu) - \frac{1}{4}\hat{\partial}(\gamma_\mu\psi_\mu)\right] + + \hat{\partial}\ [\lambda_7(\gamma_\mu\psi_\mu) - i\lambda_8\psi_0]\right\}.$$

After some additional rather long combining of two last equations, we derive, for a new vector-bispinor

$$\Phi_\lambda = \left[\psi_\lambda - \frac{1}{4}\gamma_\lambda\gamma_\mu\psi_\mu\right] + \frac{1}{4}\gamma_\lambda[\lambda_7\gamma_\mu\psi_\mu - i\lambda_8\psi_0], \tag{16}$$

the equivalent system

$$-\frac{1}{2}\hat{\partial}\gamma_\mu\Phi_\mu - \frac{1}{3}\partial_\lambda\left[\Phi_\lambda - \frac{1}{4}\gamma_\lambda\gamma_\mu\Phi_\mu\right] + M\gamma_\mu\Phi_\mu = 0,$$
$$\left[\partial_\lambda - \frac{1}{4}\gamma_\lambda\hat{\partial}\right]\gamma_\mu\Phi_\mu + M\left[\Phi_\lambda - \frac{1}{4}\gamma_\lambda\gamma_\mu\Phi_\mu\right] +$$
$$+\frac{1}{8}[8\hat{\partial}\Phi_\lambda + 3\gamma_\lambda\hat{\partial}\gamma_\mu\Phi_\mu - 4\partial_\lambda\gamma_\mu\Phi_\mu - 4\gamma_\lambda\partial_\mu\Phi_\mu] = 0.$$

These two equations can be joined (equivalently) into one equation

$$\frac{1}{8}\left[8\hat{\partial}\Phi_\lambda + \frac{1}{6}\gamma_\lambda\hat{\partial}\gamma_\mu\Phi_\mu + 4\partial_\lambda\gamma_\mu\Phi_\mu - \right.$$
$$\left. -\frac{14}{3}\gamma_\lambda\partial_\mu\Phi_\mu\right] + M\Phi_\lambda = 0, \qquad (17)$$

which coincides with the minimal equation by Pauli and Fierz for a spin-3/2 particle. So, these two models for a spin-3/2 particle are equivalent in the absence of external fields.

5. External Electromagnetic Fields

Now, we start with the vector-bispinor equations (14), but with generalized derivative $D_\mu = \partial_\mu - ieA_\mu$:

$$\lambda_1\hat{D}\gamma_\mu\psi_\mu - \lambda_2\hat{D}\psi_0 -$$
$$-2\lambda_3\left[D_\mu\psi_\mu - \frac{1}{4}\hat{D}\gamma_\mu\psi_\mu\right] + M\gamma_\mu\psi_\mu = 0,$$
$$i\lambda_4\hat{D}\gamma_\mu\psi_\mu + \lambda_5\hat{D}\psi_0 - 2i\lambda_6\left[D_\mu\psi_\mu - \right.$$
$$\left.-\frac{1}{4}\hat{D}\gamma_\mu\psi_\mu\right] + M\psi_0 = 0,$$
$$\left[D_\lambda - \frac{1}{4}\gamma_\lambda\hat{D}\right][\lambda_7\gamma_\mu\psi_\mu - i\lambda_8\psi_0] +$$
$$+\frac{1}{8}[8\hat{D}\psi_\lambda + 3\gamma_\lambda\hat{D}\gamma_\mu\psi_\mu - 4D_\lambda(\gamma_\mu\psi_\mu - 4\gamma_\lambda D_\mu\psi_\mu] +$$
$$+M\left(\psi_\lambda - \frac{1}{4}\gamma_\lambda\gamma_\mu\psi_\mu\right) = 0.$$

We will use operator identities

$$D_\mu\gamma_\mu\, D_\nu\gamma_\nu = D_\mu D_\mu - \frac{ie}{4}F_{[\mu\nu]}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu),$$
$$\hat{D} = D_\mu\gamma_\mu, \quad D_\mu\hat{D} = \hat{D}D_\mu + \gamma_\nu(-ieF_{[\mu\nu]}).$$

Adhering the same way as in a free particle case, for a new vector-bispinor

$$\Phi_\lambda = \left[\psi_\lambda - \frac{1}{4}\gamma_\lambda(\gamma_\mu\psi_\mu)\right] + \frac{1}{4}\gamma_\lambda[\lambda_7(\gamma_\mu\psi_\mu) - i\lambda_8\psi_0],$$

we constructed the equation

$$\frac{1}{8}\left[8\hat{D}\Phi_\lambda + \frac{1}{6}\gamma_\lambda\hat{D}\gamma_\rho\Phi_\rho + 4D_\lambda\gamma_\rho\Phi_\rho - \frac{14}{3}\gamma_\lambda D_\rho\Phi_\rho\right] +$$
$$+M\Phi_\lambda - \frac{(\lambda_1\lambda_5 - \lambda_2\lambda_4)}{3M}\gamma_\lambda\gamma_\nu ieF_{[\nu\mu]} \times$$
$$\times\left(\Phi_\mu + \frac{1}{4}\gamma_\mu\gamma_\rho\Phi_\rho\right) = 0. \qquad (18)$$

In comparison with the Pauli–Fierz equation, it contains the additional interaction term and refers to a particle with additional electromagnetic characteristic governed by the factor $(\lambda_1\lambda_5 - \lambda_2\lambda_4)/3M$.

6. Wave Equation in the Riemannian Space

Now, we extend the Fradkin model to the Riemannian space-time. Instead of the *ict*-metric in the Minkowski space, we will use the metrical tensor $g_{\alpha\beta}(x)$ for a curved space-time geometry with the signature $(+,-,-,-)$. Correspondingly, we employ other Dirac matrices

$$\gamma^0 = \begin{vmatrix} 0 & I \\ I & 0 \end{vmatrix}, \quad \gamma^i = \begin{vmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{vmatrix}, \quad M \to iM. \qquad (19)$$

The derivatives are changed to a more general form $D_\alpha(x) = \nabla_\alpha + \Gamma_\alpha(x) + ieA_\alpha(x), \hat{D} = \gamma^\alpha(x)D_\alpha(x)$,

where $\Gamma_\alpha(x)$ stands for the bispinor connection, and local Dirac matrices are determined through tetrads: $\gamma^\alpha(x) = \gamma^a e^\alpha_{(a)}(x)$.

We should recall the following important commutation rules:

$$\hat{D}(x) = \gamma^\rho(x)D_\beta = D_\beta\gamma^\rho(x), D^2 = D^\alpha D_\alpha,$$
$$D_\alpha D_\beta - D_\beta D_\alpha = ieF_{\alpha\beta} + \frac{1}{2}\sigma^{\nu\rho}(x)R_{\nu\rho\alpha\beta}(x),$$
$$D_\mu\hat{D} - \hat{D}D_\mu = (ieF_{\mu\nu} + \Sigma_{\mu\nu}),$$
$$\Sigma_{\mu\nu} = \frac{1}{2}\sigma^{\rho\sigma}(x)R_{\rho\sigma\mu\nu}(x), \hat{D}\hat{D} = D^2 + \Sigma(x),$$
$$\Sigma(x) = \left[ieF_{\alpha\beta} + \frac{1}{2}\sigma^{\nu\rho}(x)R_{\nu\rho\alpha\beta}(x)\right]\sigma^{\alpha\beta}(x),$$
$$\sigma^{\nu\rho}(x) = \frac{\gamma^\nu(x)\gamma^\rho(x) - \gamma^\rho(x)\gamma^\nu(x)}{4},$$

where the Riemann curvature tensor is used. The definition of a general covariant matrix $\gamma^5(x)$ (note that $\epsilon_{0123} = -1$) is as follows:

$$\gamma^5(x) = \frac{i}{4!}\epsilon_{\alpha\beta\rho\sigma}(x)\ \gamma^\alpha(x)\gamma^\beta(x)\gamma^\rho(x)\gamma^\sigma(x),$$
$$\epsilon^{\alpha\beta\rho\sigma}(x) = \epsilon^{abcd}e^\alpha_{(a)}(x)e^\beta_{(b)}(x)e^\rho_{(c)}(x)e^\sigma_{(d)}(x). \qquad (20)$$

The covariant Levi-Civita symbol $\epsilon^{\alpha\beta\rho\sigma}(x)$ is transformed by the rule $\epsilon'^{\alpha\beta\rho\sigma}(x) = -\det[L_{ai}(x)]\epsilon^{\alpha\beta\rho\sigma}(x)$. In particular, under a tetrad P-reflection, it is transformed as

$$\epsilon^{(p)\alpha\beta\rho\sigma}(x) = (-1)\,\epsilon^{\alpha\beta\rho\sigma}(x). \tag{21}$$

We readily derive the identity $\gamma^5(x) = \gamma^5$.

All previous calculations are repeated with only minor changes. For the covariant vector-bispinor

$$\Phi_\lambda = \left[\psi_\lambda - \frac{1}{4}\gamma_\lambda\gamma^\mu\psi_\mu\right] + \frac{1}{4}\gamma_\lambda[\lambda_7\gamma^\mu\psi_\mu - i\lambda_8\psi_0],$$

we get

$$\frac{i}{8}\left(8\hat{D}\Phi^\alpha + \frac{1}{6}\gamma^\alpha\hat{D}\gamma^\rho\Phi_\rho + 4D^\alpha\gamma^\rho\Phi_\rho - \frac{14}{3}\gamma^\alpha D^\rho\Phi_\rho\right) - M\Phi^\alpha - \frac{(\lambda_1\lambda_5 - \lambda_2\lambda_4)}{3M}\gamma^\alpha\gamma^\nu[ieF_{\nu\mu}(x) + \Sigma_{\nu\mu}(x)] \times \left[\Phi^\mu + \frac{1}{4}\gamma^\mu\gamma^\rho\Phi_\rho\right] = 0. \tag{22}$$

In view of the symmetry properties of the Riemann tensor, and the multiplication rule for three Dirac matrices, we can transform the last term in Eq. (22) as follows: $\gamma^\nu(ieF_{\nu\mu} + \Sigma_{\nu\mu}) = \gamma^\nu(ieF_{\nu\mu} - \frac{1}{2}R_{\nu\mu})$, where $R_{\nu\mu}$ is the Ricci tensor. Therefore, Eq. (22) can be presented as

$$i\left(\hat{D}\Phi^\alpha + \frac{1}{48}\gamma^\alpha\hat{D}\gamma^\rho\Phi_\rho + \frac{1}{2}D^\alpha\gamma^\rho\Phi_\rho - \frac{7}{12}\gamma^\alpha D^\rho\Phi_\rho\right) - M\Phi^\alpha - \frac{(\lambda_1\lambda_5 - \lambda_2\lambda_4)}{3M}\gamma^\alpha(ieF_{\nu\mu} - \frac{1}{2}R_{\nu\mu}) \times \gamma^\nu\left(\Phi^\mu + \frac{1}{4}\gamma^\mu\gamma^\rho\Phi_\rho\right) = 0. \tag{23}$$

This is the Fradkin equation for a spin-3/2 particle in the presence of the curved space-time background. If the electric charge of the particle is zero, the last equation becomes simpler: the term with the tensor $F_{\mu\nu}$ vanishes, and $D_\rho = \nabla_\rho + \Gamma_\rho(x)$.

It should be noted that, in any of Majorana bases [2], the following properties of the Dirac matrices with respect to the complex conjugation hold: $(\gamma^\alpha)* = -\gamma^\alpha$, $(\Gamma_\alpha)^* = +\Gamma_\alpha$. For this cause, the wave equation does not mix the real and imaginary parts of the wave function. This means that the theory is valid for a neutral Fradkin spin-3/2 particle as well.

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Received 27.08.19

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РІВНЯННЯ ФРАДКІНА
ДЛЯ ЧАСТИНКИ ЗІ СПІНОМ 3/2
У ЗОВНІШНЬОМУ ЕЛЕКТРОМАГНІТНОМУ
ТА ҐРАВІТАЦІЙНОМУ ПОЛЯХ

Р е з ю м е

Досліджено модель Фрадкіна для частинки зі спіном 3/2 у присутності зовнішніх полів. Застосовуючи загальний формалізм Гельфанда–Яглома, ми розвинули цю модель на основі набору із шести незвідних представлень власної групи Лоренца, що породжує 20-компонентну хвильову функцію. Накладаючи стандартні вимоги, такі як релятивістська інваріантність, ненульова маса, спін $S = 3/2$, P-симетрія та існування лагранжіану для цієї моделі, ми спочатку отримуємо систему спінорних рівнянь за відсутності зовнішніх полів. Хвильова 20-компонентна функція складається з біспінора і вектор-спінора. Якщо зовнішні поля відсутні, модель Фрадкіна зводиться до мінімальної теорії Паулі–Фірца (або Раріти–Швінгера). Детально розглянута ця еквівалентність. Далі ми враховуємо зовнішнє електромагнітне поле. Виявляється, що рівняння Фрадкіна у мінімальній формі містить додатковий член взаємодії, який визначається електромагнітним тензором $F_{\alpha\beta}$. Крім того, ми враховуємо викривленість простору-часу. У загальноковаріантному випадку рівняння Фрадкіна містить додатковий член гравітаційної взаємодії, який визначається тензором Річчі $R_{\alpha\beta}$. Якщо електричний заряд частинки рівний нулеві, модель Фрадкіна залишається правильною і описує нейтральну частинку майоранівського типу зі спіном 3/2, яка додатково взаємодіє з викривленням простору через тензор Річчі.