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BEYOND EINSTEIN'S GENERAL RELATIVITY: HYBRID METRIC-PALATINI GRAVITY [1]

It has been established that both metric and Palatini versions of $f(R)$ gravity have interesting features, but also manifest several downsides. A hybrid combination of theories, containing elements from both formalisms, turns out to be very successful in accounting for the observed phenomenology and it is able to avoid some drawbacks of the original approaches. Here, we explore the formulation in a dynamically equivalent scalar-tensor form of this hybrid metric-Palatini approach. We present several of its main achievements, such as, passing the Solar System observational tests even if the scalar field is very light or massless, and outline several applications to astrophysical and cosmological scenarios. Furthermore, we also explore the viability of generalized hybrid metric-Palatini gravitational theories.

Keywords: general relativity, modified gravity, hybrid metric-Palatini gravity.

1. Introduction

The achievements of General Relativity (GR) on small scales, such as, within the Solar System, stellar models, or compact binary systems, have spurred investigations into the possibility of modified dynamics on larger scales. Over recent years, considerable attention has been devoted to exploring the theories' being broader combinations of curvature invariants beyond the conventional Einstein–Hilbert term [1–9]. These explorations have revealed a fundamental discrepancy between the standard metric formulation and its Palatini counterpart [10]. While the metric approach typically yields equations with higher-order derivatives, the Palatini formulation consistently produces second-order field equations. Although the simplicity of the second-order Palatini equations is attractive, they entail algebraic relationships between matter fields and the affine connection, determined by equations coupled to both matter fields and the metric.

The $f(R)$ theories serve as illustrative examples of these differing approaches. In the metric formulation, the quantity $\phi \equiv df/dR$ acts as a dynamic scalar field, governed by a second-order equation with self-interactions dependent on the Lagrangian's form, $f(R)$. For this scalar field to impact astrophysical and cosmological scales, it must possess an extremely low mass, suggesting a long interaction range. However, the presence of light scalars on smaller scales faces stringent constraints from laboratory and Solar System observations, unless screened by some mechanism

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[11–14]. In the Palatini framework, a scalar-tensor representation is also feasible, albeit with the scalar field satisfying an algebraic rather than differential equation. Consequently, the scalar ϕ becomes an algebraic function of the matter's stress-energy tensor, $\phi = \phi(T)$, potentially leading to undesired gradient instabilities in various contexts, as evidenced by studies on cosmological perturbations [15, 16] and atomic physics [17, 18].

This review focuses on the hybrid variation of these theories, where the purely metric Einstein–Hilbert action is supplemented with metric-affine correction terms akin to the Palatini approach [19, 20]. Recognizing that the metric and Palatini $f(R)$ theories offer simple extensions of GR with intriguing properties, but suffer from distinct drawbacks, a program was initiated to bridge these approaches [21–23]. Through a hybrid combination of metric and Palatini elements in constructing the gravity Lagrangian, viable models sharing attributes of both formalisms were discovered. Notably, these hybrid theories permit the generation of long-range forces without conflicting with local gravity tests or necessitating screening mechanisms. Expressing these hybrid $f(R)$ metric-Palatini theories with the use of a scalar-tensor representation facilitates, the analysis of field equations and solution construction.

In essence, adopting a theory like $R + f(\mathcal{R})$ retains GR's successes embodied in the Einstein–Hilbert action (R), while further gravitational effects are encapsulated in the metric-affine $f(\mathcal{R})$ component, where $\mathcal{R}$ denotes the Palatini curvature scalar derived from an independent connection. While metric-affine and purely metric formalisms coincide in GR, they diverge, when considering more general functions $f(\mathcal{R})$ [10]. Extensions of the $f(R)$ framework modify the matter's coupling to gravity by linearly [24] or nonlinearly depending on the matter Lagrangian [25–32], or its trace [33–35]. These modifications often induce non-geodesic motion via an extra force orthogonal to the four-velocity [36], leading to instabilities within the matter sector due to new nonlinear interactions [37, 38]. However, in the hybrid metric-Palatini approach, instabilities in the matter sector are expected to be absent, as conventional conservation laws are preserved.

This review comprehensively examines the formulation and applications of hybrid gravity models, with a focus on the weak-field limit, and the exploration of more general hybrid metric-Palatini theories. The structure of the article is outlined in the following manner: Section 2 outlines the basic features of hybrid metric-Palatini gravity, including the action, field equations, the equivalent scalar-tensor representation, and delves into the weak-field limit. Section 3 explores arbitrary hybrid gravity theories constructed from metric and independent connections, showcasing specific models that circumvent typical pathologies. Finally, Section 4 offers concluding remarks.

2. Hybrid Metric-Palatini Gravity: General Framework

2.1. Action and gravitational field equations

The action governing hybrid metric-Palatini gravity is given as [20, 39]:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R + f(\mathcal{R})\right] + \mathcal{L}_m(g^{\mu\nu}, \psi)\right\}, \quad (1)$$

where $\kappa^2 \equiv 8\pi G$, $\mathcal{L}_m$ denotes the standard minimally coupled matter Lagrangian, ψ collectively represents matter fields, R stands for the metric Einstein–Hilbert term, and $\mathcal{R} \equiv g^{\mu\nu}\mathcal{R}_{\mu\nu}$ is the Palatini curvature. The latter is defined in terms of an independent connection $\hat{\Gamma}^{\alpha}{}_{\mu\nu}$ as:

$$\mathcal{R} \equiv g^{\mu\nu}\left(\hat{\Gamma}^{\alpha}_{\mu\nu,\alpha} - \hat{\Gamma}^{\alpha}_{\mu\alpha,\nu} + \hat{\Gamma}^{\alpha}_{\alpha\lambda}\hat{\Gamma}^{\lambda}_{\mu\nu} - \hat{\Gamma}^{\alpha}_{\mu\lambda}\hat{\Gamma}^{\lambda}_{\alpha\nu}\right), \quad (2)$$

which generates the Ricci curvature tensor $\mathcal{R}_{\mu\nu}$ through:

$$\mathcal{R}_{\mu\nu} \equiv \hat{\Gamma}^{\alpha}_{\mu\nu,\alpha} - \hat{\Gamma}^{\alpha}_{\mu\alpha,\nu} + \hat{\Gamma}^{\alpha}_{\alpha\lambda}\hat{\Gamma}^{\lambda}_{\mu\nu} - \hat{\Gamma}^{\alpha}_{\mu\lambda}\hat{\Gamma}^{\lambda}_{\alpha\nu}. \quad (3)$$

Varying the action (1) with respect to the metric yields the gravitational field equation:

$$G_{\mu\nu} + F(\mathcal{R})\mathcal{R}_{\mu\nu} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \quad (4)$$

where $F(\mathcal{R}) = df(\mathcal{R})/d\mathcal{R}$, and the stress-energy tensor is defined as usual,

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta(g^{\mu\nu})}. \quad (5)$$

Varying the action with respect to the independent connection $\hat{\Gamma}^{\alpha}_{\mu\nu}$ leads to the equation of motion:

$$\hat{\nabla}_{\alpha}\left(\sqrt{-g}F(\mathcal{R})g_{\mu\nu}\right) = 0, \quad (6)$$

implying that $\hat{\Gamma}^{\alpha}_{\mu\nu}$ is the Levi-Civita connection of a metric $h_{\mu\nu} = F(\mathcal{R})g_{\mu\nu}$. Consequently, $h_{\mu\nu}$ is conformally related to the physical metric $g_{\mu\nu}$, with the conformal factor given by $F(\mathcal{R}) \equiv df(\mathcal{R})/d\mathcal{R}$.

2.2. Scalar-tensor representation

Similar to the pure metric and Palatini cases [40, 41], the action (1) for hybrid metric-Palatini gravity can be reformulated as a scalar-tensor theory by introducing an auxiliary field A such that:

$$S = \frac{1}{2\kappa^2} \int d^4x\sqrt{-g}\,[R + f(A) + f_A(\mathcal{R} - A)] + S_m, \tag{7}$$

where $f_A \equiv df/dA$, and S_m is the matter action.

The field A is dynamically equivalent to the Palatini scalar curvature $\mathcal{R}$, if $f''(\mathcal{R}) \neq 0$. Defining:

$$\phi \equiv f_A, \quad V(\phi) = Af_A - f(A), \tag{8}$$

action (7) becomes:

$$S = \frac{1}{2\kappa^2} \int d^4x\sqrt{-g}\,[R + \phi\mathcal{R} - V(\phi)] + S_m. \tag{9}$$

Action (9) is equivalent to our original starting point Eq. (1). Variation of the above action with respect to the metric, the scalar ϕ, and the connection results in the field equations:

$$R_{\mu\nu} + \phi\mathcal{R}_{\mu\nu} - \frac{1}{2}\left(R + \phi\mathcal{R} - V\right) g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{10}$$

$$\mathcal{R} - V_\phi = 0, \tag{11}$$

$$\hat{\nabla}_\alpha \left(\sqrt{-g}\phi g^{\mu\nu}\right) = 0, \tag{12}$$

respectively, where $V_\phi = V'(\phi)$.

The solution of Eq. (12) implies that the independent connection is the Levi-Civita connection of a metric $h_{\mu\nu} = \phi g_{\mu\nu}$. Thus, the relation between the tensors $\mathcal{R}_{\mu\nu}$ and $R_{\mu\nu}$ can be written as (see [20] for more details):

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2\phi^2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi + \frac{1}{2}g_{\mu\nu}\Box\phi\right), \tag{13}$$

which can be used in the action (9) to remove the independent connection, resulting in the scalar-tensor representation of hybrid metric-Palatini gravity [42]. Consequently, we obtain the following action:

$$S = \frac{1}{2\kappa^2} \int d^4x\sqrt{-g}\,\Big[(1 + \phi)R + \\ + \frac{3}{2\phi}\partial_\mu\phi\partial^\mu\phi - V(\phi)\Big] + S_m. \tag{14}$$

It's worth noting that, accounting for the substitution $\phi \to -(\kappa\phi)^2/6$, action (14) simplifies to the familiar scenario of a conformally coupled scalar field featuring a self-interaction potential. This substitution transforms the kinetic term in action (14) into the standard form, rendering the action equivalent to that of a massive scalar field conformally coupled to the Einstein gravity. However, it deviates from Brans–Dicke gravity, where the scalar field remains massless. Now, we introduce a nonzero $V(\phi)$ as specified in Eq. (9).

Indeed, we have arrived at Brans–Dicke-type of theories specified by the non-trivial coupling function:

$$\omega_{\rm BD} = \frac{3\phi}{2\phi - 2}. \tag{15}$$

This formulation extends beyond the specific cases of $\omega_{\rm BD} = 0$ and $\omega_{\rm BD} = -3/2$, which correspond to the scalar-tensor representations of the metric $f(R)$ and Palatini-$f(\mathcal{R})$ gravities, respectively, as discussed in the literature [2].

By employing the derived equations (11) and (13) within the context of Eq. (10), the metric's field equation can be expressed as follows:

$$(1 + \phi)R_{\mu\nu} = \kappa^2\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + \frac{1}{2}g_{\mu\nu}\left(V + \Box\phi\right) + \\ + \nabla_\mu\nabla_\nu\phi - \frac{3}{2\phi}\partial_\mu\phi\partial_\nu\phi, \tag{16}$$

or equivalently as:

$$(1 + \phi)G_{\mu\nu} = \kappa^2 T_{\mu\nu} + \nabla_\mu\nabla_\nu\phi - \Box\phi\, g_{\mu\nu} - \\ - \frac{3}{2\phi}\nabla_\mu\phi\nabla_\nu\phi + \frac{3}{4\phi}\nabla_\lambda\phi\nabla^\lambda\phi\, g_{\mu\nu} - \frac{1}{2}V g_{\mu\nu}, \tag{17}$$

from which it is seen that the spacetime curvature is generated by both the matter and the scalar field.

The scalar field equation can be approached in two distinct manners, each offering insights into how the hybrid models illustrate the physical characteristics of the $\omega_{\rm BD} = 0$ and $\omega_{\rm BD} = -3/2$ scalar-tensor models. First, tracing Eq. (10) with $g^{\mu\nu}$, we find $-R - \phi\mathcal{R} + 2V = \kappa^2 T$, and, using Eq. (11), it takes the following form:

$$2V - \phi V_\phi = \kappa^2 T + R. \tag{18}$$

Similar to the Palatini case ($\omega_{\rm BD} = -3/2$), this equation indicates that the field ϕ can be represented as an algebraic function of the scalar $X \equiv \kappa^2 T + R$, i.e., $\phi = \phi(X)$. However, in the pure Palatini scenario, ϕ solely depends on T. Notably, the right-hand side of Eq. (16) encompasses new matter terms linked to the trace T and its derivatives, alongside the curvature R and its derivatives. Consequently, this theory can be interpreted as a higher-derivative theory involving both matter and metric fields. Nonetheless, such an interpretation can be circumvented, if R is substituted in Eq. (18) with the relation:

$$R = \mathcal{R} + \frac{3}{\phi}\Box\phi - \frac{3}{2\phi^2}\,\partial_\mu\phi\partial^\mu\phi, \tag{19}$$

together with $\mathcal{R} = V_\phi$. This yields a second-order evolution equation for the scalar field, given by:

$$-\Box\phi + \frac{1}{2\phi}\partial_\mu\phi\partial^\mu\phi + \frac{\phi}{3}[2V - (1+\phi)V_\phi] = \frac{\phi\kappa^2}{3}T, \tag{20}$$

which is an effective Klein–Gordon equation. This last expression shows that, unlike in the Palatini ($\omega_{\rm BD} = -3/2$) case, the scalar field is dynamical. Consequently, the theory remains unaffected by the microscopic instabilities inherent in Palatini models with infrared corrections [10].

2.3. Linearized field equations

Determining the post-Newtonian parameters of the theory is crucial for assessing its weak field limit and compatibility with local precision gravity tests. For analyses of the post-Newtonian behavior in metric and Palatini $f(R)$ theories, we refer to works such as [40, 41]. The unified analyses can be found in references like [43]. Here, we particularly focus on the parameter γ, representing the fractional difference of the Newtonian potentials, especially when cosmological expansion effects can be neglected.

To embark this endeavor, we need to examine the perturbations of Eqs. (16) and (20) within a Minkowskian background. Typically, this involves assuming $\phi = \phi_0 + \varphi(x)$, where ϕ_0 represents the field's asymptotic value far from local systems, usually determined by the cosmological background solution. Concurrently, we adopt a quasi-Minkowskian coordinate system, where $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$. Instead of detailing the analysis here, we refer the reader to [20, 39] for further insights and present the relevant findings. Specifically, the effective Newton constant, the post-Newtonian parameter γ, and the scalar field mass are expressed as:

$$G_{\rm eff} \equiv \frac{\kappa^2}{8\pi(1+\phi_0)}\left(1 + \frac{\phi_0}{3}e^{-m_\varphi r}\right), \tag{21}$$

$$\gamma \equiv \frac{[1 + \phi_0 \exp\left(-m_\varphi r\right)/3]}{[1 - \phi_0 \exp\left(-m_\varphi r\right)/3]}, \tag{22}$$

$$m_\varphi^2 \equiv \frac{1}{3}\,[2V - V_\phi - \phi(1+\phi)V_{\phi\phi}]|_{\phi=\phi_0}\,, \tag{23}$$

respectively. These results represent the standard post-Newtonian metric up to second order for this class of theories.

To compare these results with $f(R)$ gravity, where we typically have:

$$G_{\rm eff} \equiv \frac{G}{\phi_0}\left(1 + \frac{1}{3}e^{-m_f r}\right), \tag{24}$$

$$\gamma \equiv \frac{(1 - e^{-m_f r}/3)}{(1 + e^{-m_f r}/3)}, \tag{25}$$

which requires a large mass $m_f^2 \equiv (\phi V_{\phi\phi} - V_\phi)/3$ to render the Yukawa-type corrections negligible in local experiments. Thus, achieving $\gamma \approx 1$ hinges on one possibility [40, 41]: $m_\varphi r \gg 1$ across millimeter to astronomical scales, signifying that the scalar interaction range $1/m_\varphi$ should be smaller than a few millimeters.

In the current hybrid scenario, however, two avenues lead to $\gamma \approx 1$. The first mirrors $f(R)$ theories, needing a highly massive scalar field. The second option entails a very small value of ϕ_0. If $\phi_0 \ll 1$, the Yukawa-type corrections become negligible irrespective of the magnitude of m_φ. This could permit the existence of a long-range scalar field capable of altering cosmological and galactic dynamics without affecting the Solar System. Under optimistic circumstances, subtle modifications might be detectable as anomalies in the local gravitational field [44].

3. General Hybrid Metric-Palatini Theories

The space of "hybrid" theories is quite vast. Alongside the metric and its Levi-Civita connection, there is an additional independent connection that is utilized to construct curvature invariants [22]. This expansion allows the consideration of various new terms such as:

$$\mathcal{R}R,\quad \mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu},\quad R^{\mu\nu}\mathcal{R}_{\mu\nu},$$
$$\mathcal{R}^{\mu\nu\alpha\beta}\mathcal{R}_{\mu\nu\alpha\beta},\quad R^{\mu\nu\alpha\beta}\mathcal{R}_{\mu\nu\alpha\beta},\quad \text{etc.}$$

While a comprehensive analysis of these hybrid theories has not been conducted, there is evidence suggesting that the hybrid metric-Palatini class of theories we are focusing on is a unique class of viable higher-order hybrid gravity theories. In the more restricted context of purely metric theories, the $f(R)$ class of theories stands out by avoiding Ostrogradski instabilities, due to a separation of additional degrees of freedom into a harmless scalar degree of freedom [45]. Similarly, as we have seen, such a separation is feasible for hybrid metric-Palatini theories. This exceptional feature extends to the larger realm of metric-affine theories, where generic theories often harbor ghosts, superluminalities, or other unphysical degrees of freedom.

As a representative class of more general theories, consider the following action:

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^4x\sqrt{-g}, f(R, \mathcal{R}, \hat{Q}_H), \tag{26}$$

where

$$\hat{Q}_H = R^{\mu\nu}\mathcal{R}_{\mu\nu}, \tag{27}$$

which was studied in [46]. Here, the precise field content of this action in the weak-field limit was determined. The variation of action (26) with respect to the metric produces the following field equations:

$$\begin{aligned} & f_{,R}R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f + g_{\mu\nu}\Box f_{,R} - \nabla_\mu\nabla_\nu f_{,R} + \\ & + f_{,\mathcal{R}}\mathcal{R}_{\mu\nu} + 2f_{,\hat{Q}}R^\lambda_\mu\mathcal{R}_{\nu\lambda} + \frac{1}{2}\Box\left(f_{,\hat{Q}}\mathcal{R}_{\mu\nu}\right) + \\ & + \frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta\left(f_{,\hat{Q}}\mathcal{R}^{\alpha\beta}\right) - \nabla_\lambda\nabla_{(\nu}\left(f_{,\hat{Q}}\mathcal{R}^\lambda_{\mu)}\right) = \kappa^2T_{\mu\nu}, \end{aligned} \tag{28}$$

where $f_{,R}$, $f_{,\mathcal{R}}$, and $f_{,\hat{Q}}$ are derivatives of f with respect to R, $\mathcal{R}$, and $\hat{Q}_H$, respectively. The solution to the equation of motion for the connection dictates that it is the Levi-Civita connection of the metric $\hat{g}_{\mu\nu}$ given by

$$\hat{g}^{\mu\nu} = \frac{\sqrt{-r}}{\sqrt{-g}}\, r^{\mu\nu}, \tag{29}$$

where

$$r^{\mu\nu} = f_{,\mathcal{R}}g^{\mu\nu} + f_{,\hat{Q}}R^{\mu\nu}. \tag{30}$$

This allows us to eliminate the auxiliary metric $\hat{g}_{\mu\nu}$ in terms of the physical metric $g_{\mu\nu}$.

Considering perturbations $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ around the Minkowski space, where $g_{\mu\nu} = \eta_{\mu\nu}$, and subsequently inverting the linearized field equations for the physical metric, we obtain the propagators for both the graviton and any additional degrees of freedom that may be present in $h_{\mu\nu}$. The propagator $\Pi^{\alpha\beta\gamma\delta}$ is defined as:

$$\Pi^{-1\gamma\delta}_{\alpha\beta}h_{\gamma\delta} = \kappa^2\tau_{\alpha\beta}, \tag{31}$$

where $\tau_{\alpha\beta}$ denotes the linearized stress-energy source. Using the formalism of the spin-projector operators as outlined in Ref. [47] and further elucidated in Ref. [48], the outcome can be expressed in Fourier space (where essentially $\Box \to -k^2$) in the form of two functions, a and c, as:

$$k^2\Pi_{\alpha\beta\gamma\delta} = \frac{\mathcal{P}^2_{\alpha\beta\gamma\delta}}{a(-k^2)} - \frac{\mathcal{P}^0_{\alpha\beta\gamma\delta}}{a(-k^2) - 3c(-k^2)}, \tag{32}$$

where $\mathcal{P}^2_{\alpha\beta\gamma\delta}$ selects the spin-2 modes, while $\mathcal{P}^0_{\alpha\beta\gamma\delta}$ corresponds to the scalar modes of the fluctuations (for detailed insights, we refer the reader to Refs [46–48]). The functions a and c are readily determinable once a theory of the form Eq. (26) is provided, and they rely on the combinations:

$$A = \frac{6f^{(0)}\mathcal{R}\mathcal{R} + f^{(0)},\hat{Q}}{2f^{(0)},\mathcal{R}}, \quad B = \frac{f^{(0)},\hat{Q}}{f^{(0)}\mathcal{R}}, \tag{33}$$

in the following way:

$$a(\Box) = f^{(0)}_{,R} + f^{(0)},\mathcal{R} - f^{(0)}_{,\hat{Q}}\frac{B}{4}\Box^2, \tag{34}$$

$$\begin{aligned} c(\Box) = {} & f^{(0)}_{,R} + f^{(0)}_{,\mathcal{R}} - 2\left(f^{(0)}_{,RR} + 4f^{(0)}_{,R\mathcal{R}} + f^{(0)}_{,\hat{Q}}\right)\Box + \\ & + \left[f^{(0)}_{,R\mathcal{R}}(6A + B) + f^{(0)}_{,\hat{Q}}\left(2A + \frac{B}{4}\right)\right]\Box^2. \end{aligned} \tag{35}$$

To gain some insight into the method, we consider several special cases and, to simplify the analysis, we assume $f^{(0)}_{\mathcal{R}\mathcal{R}} = 0$.

3.1. Metric $f(R)$ models

In the pure metric $f(R)$ case, $f^{(0)}_{,R\mathcal{R}} = A = 0$, and we have

$$\Pi^{\alpha\beta\gamma\delta}_{f(R)} = \Pi^{\alpha\beta\gamma\delta}_{GR} + \frac{1}{2\left(k^2 + (3f^{(0)}_{,RR})^{-1}\right)}\mathcal{P}^{0\alpha\beta\gamma\delta}. \tag{36}$$

Hence, as anticipated, we encounter an additional scalar degree of freedom, consistent with the fact that $f(R)$ models are equivalent to Brans–Dicke theories with a parameter $\omega_{\rm BD} = 0$. The mass of the "scalaron" is determined by $m^2 = (3f^{(0)}_{,RR})^{-1}$, and, as long as $f''(R) > 0$, the theory is stable; otherwise, a tachyonic mass undermines stability in a vicinity of the Minkowski space.

3.2. Palatini $f(\mathcal{R})$ models

As mentioned above, the Palatini-type $f(\mathcal{R})$ models are equivalent to Brans–Dicke theories with $\omega_{\rm BD} = -3/2$. This particular value renders the kinetic term of the field null, implying its non-dynamical nature. Consequently, we anticipate the absence of any extra scalar degree of freedom. To ensure proper normalization, let us assume $f^{(0)},\mathcal{R} = 1$, leading to $f^{(0)}_{,RR} = f^{(0)}_{,R\mathcal{R}} = f^{(0)}, \hat{Q} = 0$. Thus, we have:

$$\Pi^{\alpha\beta\gamma\delta}_{f(\mathcal{R})} = \Pi^{\alpha\beta\gamma\delta}_{GR}, \tag{37}$$

confirming our expectation.

3.3. Hybrid metric-Palatini models

As noted in [42], in Ricci-flat spacetimes, the hybrid metric-Palatini theories exhibit characteristics akin to Palatini-$f(\mathcal{R})$ theories, which reduce in vacuum to GR with a cosmological constant. Hence, it is unsurprising that we do not identify any new propagating degrees of freedom in the Minkowski vacuum:

$$\Pi^{\alpha\beta\gamma\delta}_{f(X)} = \Pi^{\alpha\beta\gamma\delta}_{GR}. \tag{38}$$

Remarkably, this class of theories is not equivalent to either of the preceding scenarios. When examining curved spacetimes, a new scalar degree of freedom emerges. In this context, hybrid metric-Palatini gravity stands as a minimalistic scalar-tensor expansion of GR, with the scalar propagating solely in the presence of a background curvature.

3.4. The hybrid $f(R,\mathcal{R})$ models

The generalized hybrid Ricci scalar theories were first introduced in [49, 50], revealing qualitatively distinct properties compared to the more restricted hybrid models discussed earlier. Specifically, the $f(R,\mathcal{R})$ theories were demonstrated to be equivalent to a class of biscalar-tensor theories. These theories entail an additional $\mathcal{P}^{0\alpha\beta\gamma\delta}$ spin-0 propagator with a double pole, representing two propagating scalar degrees of freedom. The masses of these scalar fields can be readily inferred and are given by:

$$m^2_{\pm} = \frac{f^{(0)}_{,\mathcal{R}}}{18\left(f^{(0)}_{,R\mathcal{R}}\right)^2}\left(f^{(0)}_{,RR} + 4f^{(0)}_{,R\mathcal{R}} \pm S\right), \tag{39}$$

where, for convenience, the quantity S is defined by

$$S \equiv \sqrt{\left(f^{(0)}_{,RR} + 4f^{(0)}_{,R\mathcal{R}}\right)^2 - 12\frac{\left(f^{(0)}_{,R\mathcal{R}}\right)^2}{f^{(0)}_{,\mathcal{R}}}}\,. \tag{40}$$

It is worth noting that the scalar particle with mass squared m^2_- corresponds to the scalaron mentioned in Eq. (36) when dealing with pure metric $f(R)$ gravity. However, in general, its mass is now shifted. The other scalar represents a new particle arising from the nontrivial dependence on $\mathcal{R}$, and, unlike in the scenario of hybrid metric-Palatini gravity, it propagates even in Ricci-flat spaces. The criterion ensuring the absence of tachyonic instabilities for either scalar is expressed as:

$$f^{(0)}_{,\mathcal{R}} > 0, \quad \text{and} \quad f^{(0)}_{,RR} + 4f^{(0)}_{,R\mathcal{R}} - S > 0. \tag{41}$$

In order' for neither of these scalars, to be a ghost, we should have both $r_+ > 0$ and $r_- > 0$. The second condition would require that

$$f^{(0)}_{,RR} + 4f^{(0)}_{,R\mathcal{R}} - S < 0, \tag{42}$$

in contradiction with condition (41). It seems then that we cannot avoid both tachyons and ghosts in this theory.

3.5. The hybrid Ricci-squared $f(\mathcal{R},\hat{Q})$ theories

Finally, let us consider the $\hat{Q}_H$-invariant. For simplicity, we restrict to models here without a nonlinear dependence on the metric Ricci scalar. Basically, the graviton propagator acquires its structure from the function $a(\Box)$ in Eq. (34). Now, only the higher-derivative term $\hat{Q}_H$ modifies it. We can arrange the result for the propagator in the form

$$\Pi^{\alpha\beta\gamma\delta}_{f(\mathcal{R},\hat{Q})} = \frac{\Pi^{\alpha\beta\gamma\delta}_{GR}}{\left(1 - \frac{1}{4}\left(f^{(0)}_{,\hat{Q}}\right)^2 k^4\right)} + \frac{3f^{(0)}_{,\hat{Q}}\left(1 + \frac{3}{4}f^{(0)}_{,\hat{Q}}k^2\right)}{2\left(1 - \frac{1}{4}\left(f^{(0)}_{,\hat{Q}}\right)^2 k^4\right)\left(1 + 3f^{(0)}_{,\hat{Q}}k^2 + 2\left(f^{(0)}_{,\hat{Q}}\right)^2 k^4\right)}\mathcal{P}^{0\alpha\beta\gamma\delta}. \tag{43}$$

The sixth-order theory we are dealing with exhibits a modulated graviton propagator that introduces two extra poles. Moreover, there is a scalar propagator featuring five poles. This stands in stark contrast to the metric Q-theory, which involves just one additional spin-2 particle and entails fourth-order field equations. We won't delve into the detailed properties of these new degrees of freedom here, as it is evident that the theory is plagued by ghosts, rendering it non-physical. It's easy to see that this is a generic occurrence when constructing the action from any hybrid curvature invariant, except for $\mathcal{R}$ in the specific scenario of separable functional dependence $R+f(\mathcal{R})$.

These observations support our assertion that hybrid metric-Palatini theories hold special theoretical significance.

4. Conclusion

This article introduces a hybrid metric-Palatini framework for gravity theories and examines the implications of these new theories through various theoretical consistency checks. From a field theory standpoint, the hybrid metric-Palatini or $f(X)$ class of theories, where $X = R+\kappa^2 T$, holds a special status similar to that of $f(R)$ theories in purely metric gravity [45]. These theories stand out because, among gravity theories devoid of ghost-like or otherwise problematic degrees of freedom, $f(X)$ actions are the only viable constructs utilizing both the metric and an independent Palatini connection. The uniqueness of $f(X)$ actions lies in their ability to separate higher derivatives in the gravity sector into a scalar mode, avoiding Ostrogradskian instability.

Having established the theoretical consistency and significance of hybrid metric-Palatini theories, as evidenced by our post-Newtonian analysis, they show promise in modifying cosmological dynamics at large scales while circumventing local gravity constraints [51]. This is because, as a scalar-tensor theory, the hybrid theory features an evolving the Brans–Dicke coupling, allowing potentially significant deviations from General Relativity (GR) in the past and future. In contrast, in metric $f(R)$ models, the Brans–Dicke coupling remains a finite constant, necessitating the use of various "screening mechanisms".

Cosmological perturbations have also been analyzed in these models up to linear order [20, 39, 52], suggesting that the formation of a large-scale structure is viable, albeit with subtle features that may be detectable in future experiments. Notably, numerical studies imply that differences in gravitational potentials may exhibit oscillations at higher redshifts, potentially observable in cross-correlations of matter and lensing power spectra.

At an effective level, $f(X)$ modifications involve both the trace of matter stress-energy and the Ricci scalar of the metric curvature, suggesting relevance to both dark energy and dark matter problems [19]. Various aspects of dark matter phenomenology have been discussed, from astronomical to galactic and extragalactic scales [19, 53–55]. Additionally, wormhole [56–59] and stellar [60, 61] geometries, cosmic strings solutions [62–65], and thick brane structures [66] have been explored in these theories, but the nature of possible black hole solutions requires further investigation [67]. Constraints on these theories from astrophysical data, such as measurements of binary pulsars [68], also warrant exploration.

In conclusion, while the physics of metric and Palatini versions of $f(R)$ gravity have been extensively studied in various contexts, studies on the hybrid $f(X)$ version of the theory remain largely unexplored. The results presented in this work provide compelling motivation for the further exploration of these specific theories.

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ЗА МЕЖАМИ ЗАГАЛЬНОЇ ТЕОРІЇ ВІДНОСНОСТІ ЕЙНШТЕЙНА: ГІБРИДНА ҐРАВІТАЦІЯ ІЗ МЕТРИКОЮ ПАЛАТІНІ

Встановлено, що як метричні, так і варіанти Палатіні ґравітації $f(R)$, мають цікаві особливості і, водночас, також виявляють декілька недоліків. Гібридна комбінація теорій, що містить елементи обох формалізмів, виявляється дуже успішною у поясненні спостережуваної феноменології і здатна уникнути деяких недоліків первісних підходів. В цій статті досліджується формулювання цього гібридного підходу із метрикою Палатіні у динамічно еквівалентній скалярно-тензорній формі. Ми наводимо кілька основних досягнень цього підходу, таких як перевірка спостережуваних даних для Сонячної системи, навіть якщо скалярне поле є дуже легким або безмасовим, і окреслюємо кілька застосувань до астрофізичних і космологічних сценаріїв. Крім того, ми також досліджуємо життєздатність узагальнених гібридних теорій ґравітації із метрикою Палатіні.

Ключові слова: загальна теорія відносності, модифікована ґравітація, гібридна ґравітація із метрикою Палатіні.