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DEFORMED DIRAC AND SHRÖDINGER EQUATIONS WITH IMPROVED MIE-TYPE POTENTIAL FOR DIATOMIC MOLECULES AND FERMIONIC PARTICLES IN THE FRAMEWORK OF EXTENDED QUANTUM MECHANICS SYMMETRIES

In this study, the bound-state solutions of the deformed Dirac equation (DDE) have been determined with the improved Mie-type potential including an improved Coulomb-like tensor potential (IMTPICLP) under the condition of the spin or pseudospin symmetry in the extended relativistic quantum mechanics (ERQM) symmetries. The IMTPICLP model includes a combination of the terms $1/r^3$ and $1/r^4$ which coupled with the couplings ($\mathbf{L\Theta}$ and $\widetilde{\mathbf{L}}\Theta$) between the physical properties of the system with the topological deformations of space-space. In the framework of the parametric Bopp's shift method and standard perturbation theory, the new relativistic and nonrelativistic energy eigenvalues for the improved Mie-type potential have been found. The new obtained values appeared sensitive to the quantum numbers $(j, k, l, \widetilde{l}, s, \widetilde{s}, m, \widetilde{m})$, the mixed potential depths (A, B, C, α), and noncommutativity parameters (Θ, σ, χ). The new energy spectra of the improved Kratzer–Fues potential within an improved Coulomb-like tensor interaction and the improved modified Kratzer potential within the Coulomb-like tensor interaction have been derived as particular cases of IMTPICLP. We recovered the usual relativistic and nonrelativistic results from the literature by applying the three simultaneous limits $(\Theta, \sigma, \chi) \to (0, 0, 0)$. One can notice that our results are in close agreement with the recent studies.

Keywords: Dirac equation, Schrödinger equation, Mie-type potential, noncommutative quantum mechanics, star product.

1. Introduction

In relativistic quantum physics, at the level of high energies, the Duffin–Kemmer–Petiau, Klein–Gordon, and Dirac equations are regularly utilized to describe the particle dynamics, according to the spin values of one, zero, or half, respectively. It should be noted that, in the case of low energies, the Schrödinger equation (SE) is the appropriate alternative, regardless of the spin values. In quantum mechanics, the exact and approximate bound-state solutions of the Schrödinger equation, as well as the other three relativistic equations with physically relevant potentials, are significant. Over the decades, exact solutions of the Dirac equation (DE) with various potentials have attracted a significant interest. The Mie-type potential which is considered as combined of $\frac{A}{r^2}$, $-\frac{B}{r}$, and C and is a typical diatomic potential that can be reduced to the Kratzer–Fues potential and the modified Kratzer potential when $A = D_e r_e^2$, $B = 2D_e r_e$, $C = 0$, and $A = D_e r_e^2$, $B = 2D_e r_e$,

$C = D_e$. Here, D_e is the energy of interaction between two atoms separated by r_e in a molecular system. Many researchers have solved the problems involving the Mie-type potential in DE. Aydoğdu and Sever (2010) found exact solutions of DE for the Mie-type potentials under the conditions of pseudospin (p-spin) and spin symmetries and obtained the bound-state energy equations, as well as the corresponding two-component spinor wave functions for Dirac particles using the asymptotic iteration method [1]. Hamzavi *et al.* treated the same physical problem, but with the Nikiforov–Uvarov (NU) method [2]. Hamzavi *et al.* studied this potential with a Coulomb-like tensor potential under a spin or p-spin symmetry with arbitrary spin-orbit coupling quantum number k [3] using the NU method. The spin symmetry is relevant to mesons (quark-antiquark particles), and the p-spin symmetry is utilized to feature deformed nuclei, superdeformation, and to build an effective shell-model within the context of DE. The Spin and p-spin symmetries are investigated as functions of the combining or differentiation of scalar and vector potentials $S(r)$ and $V(r)$. Eshghi and Ikhdair studied a relativistic Mie-type potential including the Coulomb potential for spin-1/2 particles under DE that contains both scalar and vector potentials and found an analytic solution for exact bound states using the Laplace transformation approach [4]. Recently, Onyenegecha *et al.* used the formula technique to present the analytic solutions of DE for the modified Kratzer potential under the p-spin and spin symmetries and found the energies analytically and numerically. They also investigated the thermodynamic properties of the $X^1\Sigma_g^+$ state of Cl_2 and N_2 diatomic molecules in the nonrelativistic spin symmetry limit [5]. Under the p-spin symmetry, Aydoğdu and Sever [6] found an explicit solution of DE for a pseudoharmonic potential in the presence of a linear tensor potential and demonstrated that tensor interactions erase all degeneracies between the members of pseudospin doublets. Moreover, this potential has been studied in the context of SE and deformed SE within the framework of ordinary nonrelativistic quantum mechanics [7] and in extended nonrelativistic quantum mechanics (ENRQM) symmetries [8]. Moreover, we used Bopp's shift (BS) method and standard perturbation theory to investigate the exact solvability of nonrelativistic quantum systems for ab isotropic harmonic oscillator plus an inverse quadratic potential in both noncommutativity two-dimensional real space and phase (NC: 2D-RSP), and we obtained the exact corrections for the corresponding spectrum. We also discovered the noncommutative anisotropic Hamiltonian (NAH) that corresponds to it [9]. In 2016, by using the BS method, we investigated new exact bound-state solutions of the deformed radial upper and lower components of DE, as well as the corresponding NAH operator for the modified Kratzer–Fues potential (NMKP). We also obtained the corrections of energy eigenvalues by using standard perturbation theory for interactions in one-electron atoms [10]. We used the BS method in the NC: 2D-RSP with NMKP to obtain an analytic expression for the nonrelativistic energy spectrum of some diatomic molecules. We also showed that the new energies of Cl_2, N_2, CO, NO, and CH diatomic molecule are the sums of the ordinary energies for a modified Kratzer-type potential in the commutative space and new additive terms due to the contribution of the additive part of the NMKP. We further demonstrated that, under NMKP interactions, the group symmetry of NC: 2D-RSP is reduced to a new symmetry subgroup (NC: 2D-RS) [11]. In 2018, we presented additional results in relativistic quantum systems with a modified pseudoharmonic potential for spin-1/2 particles using the BS method for solving the deformed DE in the framework of (NC: 3D-RS) symmetries, as well as exact corrections for excited states for interactions in one-electron atoms using standard perturbation theory [12]. Based on Ref. [4], in the framework of the (NC: 3D-RS) symmetries, we carrird out a novel theoretical analytic investigation for the modified Mie-type potential under DDE and found exact corrections for excited states for interactions in one-electron atoms [13]. Noncommutative quantum mechanics is another area of research that has recently gotten a lot of attention (NCQM), which studies physical and chemical processes in a new phase space that is more generalized than the Hilbert space. Through our current studies, we aim to re-examine the Mie-type potential in NCQM in the case of deformation Dirac theory (DDT) for a more profound investigation of new energy values and for the search for a possibility of new applications. We aimed to shed more light on this combined system within the framework of the DDT in an extended space with large symmetries based on the new postulates

$$\left[\widehat{x}_\mu^{(s,h,i)} \overset{*}{,} \widehat{x}_\nu^{(s,h,i)}\right] \neq 0 \text{ and } \left[\widehat{p}_\mu^{(s,h,i)} \overset{*}{,} \widehat{p}_\nu^{(s,h,i)}\right] \neq 0,$$

which create a noncommutative space-space (NCSS) and noncumulative phase-phase (NCPP), where they were absent within the framework of quantum mechanics. This alternative approach was suggested as a solution to many of the problems in relativistic and nonrelativistic quantum mechanics that it did not solve in the literature, such as quantum gravity, string theory, and the standard model's divergence problem [14–22]. NCSS and NCPP are important tools for altering the physical features of various quantum systems, and they have been a hot topic in academia for a long time. The concept of quantum mechanics with extensions (EQM) is not new, it has been proposed by Snyder for decades [23] in 1947, and its geometric analysis was introduced by Connes in 1991 and 1994 [24,25]. Seiberg and Witten extended earlier the ideas about the appearance of a NC geometry in string theory with a nonzero B-field and obtained a new version of gauge fields in noncommutative gauge theory [26]. The generation of new quantum fluctuations capable of erasing the observed unwanted divergences or infinities that appear to produce short-range in field theories such as gravitational theory is one of the potential goals of a NC deformation of space-space and phase-phase [27]. I hope for that this research will contribute to a more subatomic scale examinations and scientific knowledge of elementary particles in the field of nanoscales. The study of the improved Mie-type potential, including an improved Coulomb-like tensor potential (IMTPICLP, in short) in the DDT symmetries was motivated by the fact that it had not been reported in the available literature for Cl_2, N_2,CO, NO, and CH diatomic molecules. We also aspire to develop our previous study in [13], which is limited to the state of single-electron atoms in their outer orbit to include extended nonrelativistic quantum mechanics (ENRQM) symmetries and previously mentioned diatomic molecules. The vector and scalar IMTPICLP model will be employed in this study ($V_{mt}^{s}(\widehat{r})/V_{mt}^{ps}(\widehat{r})$, $S_{mt}^{s}(\widehat{r})/S_{mt}^{ps}(\widehat{r})$) are as follows:

$$\begin{cases} V_{mt}^{s}(\widehat{r}) = V_{mt}(r) - \dfrac{\partial}{\partial r}(V_{mt}(r))\dfrac{\mathbf{L}\Theta}{2r} + O(\Theta^2), \\ S_{mt}^{s}(\widehat{r}) = S_{mt}(r) - \dfrac{\partial}{\partial r}(S_{mt}(r))\dfrac{\mathbf{L}\Theta}{2r} + O(\Theta^2), \end{cases} \tag{1}$$

and

$$\begin{cases} V_{mt}^{ps}(\widehat{r}) = V_{mt}(r) - \dfrac{\partial}{\partial r}(V_{mt}(r))\dfrac{\widetilde{\mathbf{L}}\Theta}{2r} + O(\Theta^2), \\ S_{mt}^{ps}(\widehat{r}) = S_{mt}(r) - \dfrac{\partial}{\partial r}(S_{mt}(r))\dfrac{\widetilde{\mathbf{L}}\Theta}{2r} + O(\Theta^2), \end{cases} \tag{2}$$

where $(V_{mt}(r), S_{mt}(r))$ are the vector and scalar potentials according to the view of RQM known in the literature [3, 4]:

$$\begin{cases} V_{mt}(r) = \dfrac{A}{r^2} - \dfrac{B}{r} + C, \\ S_{mt}(r) = \dfrac{A_s}{r^2} - \dfrac{B_s}{r} + C_s, \end{cases} \tag{3}$$

where (A, A_s), (B, B_s) and (C, C_s) are some parameters representing the molecular properties, ($\widehat{r}$ and r) are the distances between the two particles in the deformation of Dirac theory symmetries and QM symmetries, respectively. The couplings $\mathbf{L\Theta}$ and $\widetilde{\mathbf{L}}\Theta$ are the scalar product of the usual components of the angular momentum operators $\mathbf{L}(L_x, L_y, L_z)/$ $/\widetilde{\mathbf{L}}(\widetilde{L}_x, \widetilde{L}_y, \widetilde{L}_z)$ and the modified noncommutativity vector $\mathbf{\Theta}\ (\theta_{12}, \theta_{23}, \theta_{13})/2$ which present the noncommutativity elements. In the case of G_{NC}, the noncentral generators can be suitably realized as self-adjoint differential operators ($\widehat{x}_\mu^{(s,h,i)}$, $\widehat{p}_\nu^{(s,h,i)}$) appear in three varieties. The first one is the canonical structure (CS), the second is the Lie structure (LS), while the latter corresponds to the quantum plane (QP) in the representations of Schrödinger, Heisenberg, and interaction pictures, satisfying a deformed algebra of the form (we have used the natural units $\hbar = c = 1$): [28–38]:

$$\left[x_\mu^{(s,h,i)}, p_\nu^{(s,h,i)}\right] = i\hbar\delta_{\mu\nu} \Longrightarrow$$
$$\Longrightarrow \left[\widehat{x}_\mu^{(s,h,i)} \overset{*}{,} \widehat{p}_\nu^{(s,h,i)}\right] = i\hbar_{\text{eff}}\delta_{\mu\nu} \tag{4a}$$

and

$$\left[x_\mu^{(s,h,i)}, x_\nu^{(s,h,i)}\right] = 0 \Longrightarrow \left[\widehat{x}_\mu^{(s,h,i)} \overset{*}{,} \widehat{x}_\nu^{(s,h,i)}\right] =$$
$$= \begin{cases} i\theta_{\mu\nu}:\ \theta_{\mu\nu} \in IC & \text{for CS}, \\ if_{\mu\nu}^{\alpha}\widehat{x}_\alpha^{(s,h,i)}:\ f_{\mu\nu}^{\alpha} \in IC & \text{for LS}, \\ iC_{\mu\nu}^{\alpha\beta}\widehat{x}_\alpha^{(s,h,i)}\widehat{x}_\beta^{(s,h,i)}:\ C_{\mu\nu}^{\alpha\beta} \in IC & \text{for QP}, \end{cases} \tag{4b}$$

with $\widehat{x}_\mu^{(s,h,i)} = \left(\widehat{x}_\mu^{s}, \widehat{x}_\mu^{h}, \widehat{x}_{nc\mu}^{i}\right)$ and $\widehat{p}_\mu^{(s,h,i)} = \left(\widehat{p}_\mu^{s}, \widehat{p}_\mu^{h}, \widehat{p}_\mu^{i}\right)$ are the generalized coordinates and the corresponding generalizing coordinates in the DDT symmetries, IC denoting the complex number field, while $x_\mu^{(s,h,i)} = \left(x_\mu^s, x_\mu^h, x_\mu^i\right)$ and $p_\mu^{(s,h,i)} = (p_\mu^s,\ p_\mu^h,\ p_\mu^i)$ are corresponding coordinates in the RQM symmetries. Furthermore, the usual uncertainty relation corresponds to the LHS of Eq. (4a) and will be extended to become two uncertainties. The formula for the new-form symmetries is as follows:

$$\left|\Delta x_\mu^{(s,h,i)}\Delta p_\nu^{(s,h,i)}\right| \geqslant \hbar\delta_{\mu\nu}/2 \Longrightarrow$$
$$\Longrightarrow \left|\Delta \widehat{x}_\mu^{(s,h,i)}\Delta \widehat{p}_\nu^{(s,h,i)}\right| \geqslant \hbar_{\text{eff}}\delta_{\mu\nu}/2 \tag{5a}$$

and

$$\left|\Delta \widehat{x}_\mu^{(s,h,i)}\Delta \widehat{x}_\nu^{(s,h,i)}\right| \geqslant \begin{cases} \left|\theta_{\mu\nu}\right|/2 & \text{for CS variety,} \\ f_{\mu\nu}/2 & \text{for LS variety,} \\ C_{\mu\nu}/2 & \text{for QP variety} \end{cases} \tag{5b}$$

with $f_{\mu\nu}$ and $C_{\mu\nu}$ are equal to the two average values $\left|\left\langle f_{\mu\nu}^{\alpha}\widehat{x}_\alpha^{(s,h,i)}\right\rangle\right|$ (summing according to the indice $\alpha = 1,2,3$) and $\left|\left\langle C_{\mu\nu}^{\alpha\beta}\widehat{x}_\alpha^{(s,h,i)}\widehat{x}_\beta^{(s,h,i)}\right\rangle\right|$ (summing according to the indices α, $\beta = 1,2,3$), respectively. The uncertainty relation in Eq. (5a) is obtained as a result of the generalization of LHS in Eq. (4a) to the RHS form, while the second uncertainty relation in Eq. (5b) is a result of the deformation of space-space that appears from RHS of Eq. (4b) that is divided into three varieties. The new uncertainty relation in Eq. (5b) has no equivalent in the framework of quantum mechanics known in the literature. It is worth to note that Eqs. (4) are covariant equations (the same behavior of $\widehat{x}_\mu^{(s,h,i)}$) under a Lorentz transformation, which includes boosts and/or rotations of the observer's inertial frame. In *DDT,* we extended the modified equal-time noncommutative canonical commutation relations (METNCCCRs) to include the Heisenberg and interaction pictures. Here, $\hbar_{\text{eff}} \cong \hbar$ is the effective Planck constant, $\theta_{\mu\nu} = \epsilon_{\mu\nu}\theta$ (θ is the noncommutative parameter, and $\epsilon_{\mu\nu}$ is just an antisymmetric number ($\epsilon_{\mu\nu} = -\epsilon_{\nu\mu} = 1$ for $\mu \neq \nu$ and $\epsilon_{\epsilon\epsilon} = 0$) which is an infinitesimal parameter, if compared to the energy values and elements of antisymmetric (3×3) real matrices, and $\delta_{\mu\nu}$ is the Kronecker symbol. The symbol $*$ denotes the Weyl–Moyal star product, which is generalized between two ordinary functions $h(x)g(x)$ to the new deformed form $h(x) * g(x)$ [39–49]

$$h(x) * g(x) = \begin{cases} \exp\left(i\epsilon^{\mu\nu}\theta\partial_\mu^x\partial_\nu^x\right)(hg)(x) & \text{for CS,} \\ \exp\left(\dfrac{i}{2}\widehat{x}_\mu^{(s,h,i)}g_k\left(i\partial_\mu^x, i\partial_\nu^x\right)\right)(hg)(x) & \text{for LS,} \\ iq^{G\left(u,v,\partial_\mu^u,\partial_\nu^v\right)}h(u,v)\,g(u',v')\rfloor_{\substack{v'\to v\\ u'\to u}} & \text{for QP,} \end{cases} \tag{6}$$

with

$$g_\alpha(k,p) = -k_\mu p_\nu f_k^{\nu\nu} + \frac{1}{6}k_\mu p_\nu\left(p_\alpha - k_\alpha\right) f_l^{\nu\nu} f_m^{l\alpha} + \ldots .$$

In the current paper, we apply the first variety, which allows us to rewrite $(h * g)(x)$ in the first order of the noncommutativity parameter $\epsilon^{\mu\nu}\theta$ as [50–57]:

$$(h * g)(x) = \exp\left(i\epsilon^{\mu\nu}\theta\partial_\mu^x\partial_\nu^x\right)(hg)(x) \approx$$
$$\approx (hg)(x) - \frac{i\epsilon^{\mu\nu}\theta}{2}\partial_\mu^x h\partial_\nu^x g\rfloor_{x^\mu = x^\nu} + O(\theta^2). \tag{7}$$

The indices (μ,ν) can take quantitative values $(1,2,3)$. Physically, the second term in Eq. (9) presents the effects of space-space noncommutativity. The present paper is organized as follows. The first section includes the scope and purpose of our investigation, while the remaining parts of the paper are structured as follows: A review of the DE with the Mie-type potential including a Coulomb-like tensor interaction is presented in Sect. 2. Section 3 is devoted to studying the DDE by applying the usual well-known Bopp's shift method to obtain the effective potentials of the IMTPICLP model. Furthermore, via standard perturbation theory, we will find the expectation values of the radial terms ($\frac{1}{r^3}$ and $\frac{1}{r^4}$) to calculate the corrected relativistic energy generated by the effect of the perturbed effective potentials $\Sigma_{\text{pert}}^{mt}(r)$ and $\Delta_{\text{pert}}^{mt}(r)$ of the IMTPICLP model, we derive the global corrected energies $E_{nc}^{sp}(n, A, B, C, \alpha, \Theta, \tau, \chi, j, l, s, m)$ and $E_{nc}^{ps}(n, A, B, C, \alpha, \Theta, \tau, \chi, j, \widetilde{l}, \widetilde{s}, \widetilde{m})$ for diatomic molecules and spin-1/2 fermions with the IMTPICLP model. We will also treat some important special cases, including the study of relativistic cases as a nonrelativistic limit of the spin symmetry, and we apply our results for Cl_2, N_2, CO, NO, and CH diatomic molecules. Section 5 is devoted to the conclusions.

2. Overview of DE under MTP within the CLP

In order to construct a physical model describing a physical system that interacts by means of a Mie-type potential (MTP) and a Coulomb-like potential (CLP) on the basis of DDE, we recall the eigenvalues and the corresponding eigenfunctions for such system within the framework of relativistic quantum mechanics are known from the literature. In this case, the system is governed by the following Dirac equation:

$$\begin{cases} \widehat{H}_D^{mt}\Psi_{nk}(r,\theta,\varphi) = E_{nk}\Psi_{nk}(r,\theta,\varphi), \\ \widehat{H}_D^{mt} = \widehat{\alpha}\mathbf{p} + \widehat{\beta}(M + S_{mt}(r)) - \\ - i\widehat{\beta}\widehat{\mathbf{r}}U(r) + V_{mt}(r). \end{cases} \tag{8}$$

Here, $\widehat{H}_D^{mt}$ is the Dirac Hamiltonian operator M is the reduced rest mass of a diatomic molecule or the studied fermionic particle/anti-particle, $\mathbf{p} = -\mathbf{i}\hbar\nabla$ is the momentum. The vector potential $V_{mt}(r)$ and space-time scalar potential $S_{mt}(r)$ are derived from the four-vector linear momentum operator A^μ ($V_{mt}(r)$, $\mathbf{A} = \mathbf{0}$) and the mass M, respectively, while E_{nk} is the relativistic eigenvalue, (n, k) represent the principal and spin-orbit coupling terms. The tensor interaction $U^{ctp}(r)$ equals $\left(-\frac{\alpha}{r}, r \geq R_c\right)$, $\alpha = \frac{Z_a Z_b e^2}{4\pi\epsilon_0}$ here $R_c = 7.78$ fm is the Coulomb radius, Z_a and Z_b denote the charges of the projectile a and the target nucleus b, respectively [58], $\widehat{\alpha}_i = \text{anti\_diag}(\sigma,\sigma)$, $\widehat{\beta} = \text{diag}(I_{2\times2}, -I_{2\times2})$ and σ are three-vector spin matrices. Since the Mie-type potential has spherical symmetry, allowing the spinor solutions $\Psi_{nk}(r,\theta,\varphi)$ known form

$$\begin{pmatrix} \dfrac{F_{nk}^s(r)}{r} Y_{jm}^l(\theta,\varphi) \\ i\dfrac{G_{nk}^s(r)}{r} Y_{jm}^l(\theta,\varphi) \end{pmatrix}$$

for the spin symmetry and

$$\begin{pmatrix} \dfrac{F_{nk}^{ps}(r)}{r} Y_{j\widetilde{m}}^{\widetilde{l}}(\theta,\varphi) \\ i\dfrac{G_{nk}^{ps}(r)}{r} Y_{j\widetilde{m}}^{\widetilde{l}}(\theta,\varphi) \end{pmatrix}$$

for the p-spin symmetry, $F_{nk}(r)$ and $G_{nk}(r)$ represent the upper and lower components of the Dirac spinors, while $Y_{jm}^l(\theta,\varphi)$ and $Y_{j\widetilde{m}}^{\widetilde{l}}$ are the spin and p-spin spherical harmonics, and $(m, \widetilde{m})$ are the projections on the z-axis. The upper and lower components $F_{nk}^s(r)$ and $G_{nk}^{ps}(r)$ for the spin and p-spin symmetries satisfy the two uncoupled differential equations as follows:

$$\left(\frac{d^2}{dr^2} - k(k+1)r^{-2} + U_{\text{eff}}^{clp-s}(r) - \right.$$
$$- (M + E_{nk} - \Delta_{mt}(r)) \times$$
$$\times (M - E_{nk} + \Sigma_{mt}(r)) +$$
$$\left. + \frac{\frac{d\Delta_{mt}(r)}{dr}\left(\frac{d}{dr} + \frac{k}{r} - U(r)\right)}{M + E_{nk} - \Delta_{mt}(r)}\right) F_{nk}^s(r) = 0, \tag{9}$$

$$\left(\frac{d^2}{dr^2} - k(k-1)r^{-2} + U_{\text{eff}}^{clp-p}(r) - \right.$$
$$- (M + E_{nk}^{ps} - \Delta_{mt}(r)) \times$$
$$\times (M - E_{nk} + \Sigma_{mt}(r)) +$$
$$\left. + \frac{\frac{d\Sigma_{mt}(r)}{dr}\left(\frac{d}{dr} - \frac{k}{r} + U(r)\right)}{M + E_{nk} + \Sigma_{mt}(r)}\right) G_{nk}^{ps}(r) = 0. \tag{10}$$

Here, $U_{\text{eff}}^{clp-s/p}(r)$ equals $\left(\frac{2kU(r)}{r} \mp \frac{dU(r)}{dr} - U^2(r)\right)$ which can be expressed analytically as

$$U_{\text{eff}}^{clp-s/p}(r) = \frac{-2k\alpha \mp \alpha - \alpha^2}{r^2}, \tag{11}$$

while $\Sigma_{mt}(r) = V_{mt}(r) + S_{mt}(r)$ and $\Delta_{mt}(r) = V_{mt}(r) - S_{mt}(r)$ are determined by

$$\begin{cases} \Sigma_{mt}(r) = \dfrac{A}{r^2} - \dfrac{B}{r} + c \quad \text{and} \quad \dfrac{d\Delta_{mt}(r)}{dr} = 0 \Longrightarrow \\ \Longrightarrow \Delta_{mt} = C_s : \text{spin sy.} \\ \Delta_{mt}(r) = \dfrac{A}{r^2} - \dfrac{B}{r} + c \quad \text{and} \quad \dfrac{d\Sigma_{mt}(r)}{dr} = 0 \Longrightarrow \\ \Longrightarrow \Sigma_{mt} = C_p : \text{p-spin sy.} \end{cases} \tag{12}$$

We obtain the following second-order Schrödinger-like equation under RQM symmetries, respectively:

$$\left[\frac{d^2}{dr^2} - k(k+1)r^{-2} + U_{\text{eff}}^{clp-s}(r) - \right.$$
$$\left. - \gamma_s V_{mt}(r) - \beta_s^2\right] F_{nk}^s(r) = 0 \tag{13}$$

and

$$\left[\frac{d^2}{dr^2} - k(k-1)r^{-2} + U_{\text{eff}}^{clp-p}(r) - \right.$$
$$\left. - \gamma_p V_{mt}(r) - \beta_p^2\right] G_{nk}^{ps}(r) = 0, \tag{14}$$

where $k(k-1)$ and $k(k+1)$ are equal to $\widetilde{l}(\widetilde{l}-1)$ and $l(l+1)$, respectively, $\gamma_s = M + E^s_{nk} - C_s$, $\beta^2_s = \gamma_s (M - E^{sp}_{nk})$, $\gamma_p = E^{ps}_{nk} - M - C_p$, $\beta^2_p = (M+E^{ps}_{nk})(M-E^{ps}_{nk}+C_p)$. The authors of Refs. [3, 4] used the NU method to obtain the expressions for the upper and lower components $F^s_{nk}(r)$ and $G^{ps}_{nk}(r)$ as generalized Laguerre polynomials $L^q_n\left(2\sqrt{\epsilon^{s2}}r\right)$ and $L^q_n\left(2\sqrt{\epsilon^{p2}}r\right)$ in RQM symmetries. They showed that

$$F^s_{nk}(r) = \frac{\left(2\sqrt{\epsilon^{s2}}\right)^{\frac{1}{2}(1+q^s)}}{n!}\sqrt{\frac{n-q^s}{n!}}\, r^{\frac{1}{2}(1+q^s)} \times \\ \times \exp\left(\sqrt{\epsilon^{s2}}r\right) L^{q^s}_n\left(2\sqrt{\epsilon^{s2}}r\right) \tag{15}$$

and

$$G^{ps}_{nk}(r) = \frac{\left(2\sqrt{\epsilon^{p2}}\right)^{\frac{1}{2}(1+q^p)}}{n!}\sqrt{\frac{n-q^p}{n!}}\, r^{\frac{1}{2}(1+q^p)} \times \\ \times \exp\left(\sqrt{\epsilon^{p2}}r\right) L^{q^p}_n\left(2\sqrt{\epsilon^{p2}}r.\right) \tag{16}$$

Here,

$\epsilon^{s2} = 4(k+\alpha+1)(k+\alpha) + 4\gamma_s A,$

$\epsilon^{p2} = \gamma_p c + \beta^2_p$

and

$q^p = \sqrt{1+4(k+\alpha)(k+\alpha-1)+4\gamma_p A}.$

For the spin and p-spin symmetries, the equations of energy are given by [3, 4]:

$$(M - E^s_{nk} + C)\left[2n+1+ \\ + \sqrt{(2k+1)^2+4\alpha(\alpha+2k+1)+4(M+E^s_{nk}-C_s)A}\right]^2 = \\ = (M+E^s_{nk}-C_s)B^2 \tag{17}$$

and

$$(M + E^p_{nk} - C)\left[2n+1+ \\ +\sqrt{(2k-1)^2+4\alpha(\alpha+2k-1)+4(E^p_{nk}-M-C_p)A}\right]^2 = \\ = (E^p_{nk}-M-C_p)B^2. \tag{18}$$

The lower component $G^s_{nk}(r)$ of the spin symmetry and the upper component $F^{ps}_{nk}(r)$ of the pseudospin symmetry are obtained as:

$$G^s_{nk}(r) = \left(\frac{d}{dr} + \frac{k}{r} - U^{\text{ctp}}(r)\right)\frac{F^s_{nk}(r)}{M+E^s_{nk}-C_s} \tag{19}$$

and

$$F^{ps}_{nk}(r) = \left(\frac{d}{dr} - \frac{k}{r} - U^{\text{ctp}}(r)\right)\frac{G^{ps}_{nk}(r)}{M-E^{ps}_{nk}+C_p}. \tag{20}$$

3. The New Solutions of DDE in the IMTPICLP model under the DDT Symmetries

3.1. Review of the BS method

Let us begin in this subsection by finding the DDE in the symmetries of the deformation Dirac theory with IMTPICLP. Our objective is achieved by applying the new principles which were mentioned in Introduction, are presenred in Eqs. (4) and (7), and summarized in new relationships MASCCCRs and the notion of the Weyl–Moyal star product. These data allow us to rewrite the usual radial Dirac equations in Eq. (8) in the *DDT* symmetries as follows:

$$\left(\widehat{\alpha}\mathbf{p} + \widehat{\beta}(M + S_{mt}(r)) - i\widehat{\beta}\widehat{\mathbf{r}}U(r) - \\ - (E_{nk} - V_{mt}(r))\right) * \Psi_{nk}(r,\theta,\varphi) = 0. \tag{21}$$

Thus, the upper and lower components $F^s_{nk}(r)$ and $G^{ps}_{nk}(r)$ satisfy the following second-order differential equations under the *DDT* symmetries:

$$\left[\frac{d^2}{dr^2} - k(k+1)r^{-2} + U^{clp-s}_{\text{eff}}(r) - \\ - \gamma_s V_{mt}(r) - \beta^2_s\right] * F^s_{nk}(r) = 0 \tag{22}$$

and

$$\left[\frac{d^2}{dr^2} - k(k-1)r^{-2} + U^{clp-p}_{\text{eff}}(r) - \\ - \gamma_p V_{mt}(r) - \beta^2_p\right] * G^{ps}_{nk}(r) = 0. \tag{23}$$

Among the possible paths to finding solutions of Eqs. (22) and (23), we mention, with regard for applications of the Connes method [23, 24], that based on the Seiberg–Witten map [26]. It is known that the star product can be translated into the ordinary product known in the literature as the Bopp's shift method. F. Bopp was the first who considered pseudodifferential operators obtained by the quantization rules $(x, p) \to (\widehat{x} = x - \frac{i}{2}\partial_p, \widehat{p} = p + \frac{i}{2}\partial_x)$ instead of the ordinary correspondence $(x,p) \to (\widehat{x} = x, \widehat{p} = p + \frac{i}{2}\partial_x)$, respectively. This procedure is known as the Bopp's shift (BS) method. This quantization procedure is known as the Bopp quantization [59–62]. This method has achieved a considerable success in recent years. In the search for solutions of the

NR deformed Schrödinger equation, we can consider many different potentials (see Refs. [63–67]). This method is not limited by the DSE, but can be extended to the study of various relativistic physical problems, for example, of the deformed KGE (see the Refs. [68–75]), for the DDE (see Refs. [12, 13, 76–78]) for deformed Duffin–Kemmer–Petiau equation (DDKPE) [79, 80]. Thus, the Bopp's shift method is based on reducing second-order linear differential equations such as DSE, DKG, DDE, and DDKPE with Weyl–Moyal star product to second-order linear differential equations, namely, SE, KGE, DE, and DKPE without Weyl–Moyal star product with a simultaneous translation in the space-space. It is worth notiing that the BS method permutes to reduce the above equations to the simplest form:

$$\left(\frac{d^2}{dr^2} - k\,(k+1)\,\widehat{r}^{-2} + U_{\rm eff}^{clp-s}\,(\widehat{r}) - \right.$$
$$\left. - \gamma_s V_{mt}\,(\widehat{r}) - \beta_s^2\right) F_{nk}^{s}\,(r) = 0 \tag{24}$$

and

$$\left(\frac{d^2}{dr^2} - k\,(k-1)\,\widehat{r}^{-2} + U_{\rm eff}^{clp-p}\,(\widehat{r}) - \right.$$
$$\left. - \gamma_p V_{mt}\,(\widehat{r}) - \beta_p^2\right) G_{nk}^{ps}\,(r) = 0. \tag{25}$$

The modified algebraic structure of covariant canonical commutation relations with the notion of Weyl–Moyal star product in Eqs. (4) becomes new METNCCCRs with ordinary known products (see, e.g., [59–62]):

$$\begin{cases} \left[\widehat{x}_\mu^{(s,h,i)}, \widehat{p}_\nu^{(s,h,i)}\right] = i\hbar_{\rm eff}\delta_{\mu\nu}, \\ \left[\widehat{x}_\mu^{(s,h,i)}, \widehat{x}_\nu^{(s,h,i)}\right] = i\theta_{\mu\nu}. \end{cases} \tag{26}$$

The generalized positions and momentum coordinates $\widehat{x}_\mu^{(s,h,i)}$ and $\widehat{p}_\mu^{(s,h,i)}$in the symmetries of DDT are defined as [59–62]:

$$\begin{cases} \widehat{x}_\mu^{(s,h,i)} = x_\mu^{(s,h,i)} - \sum_{\nu=1}^{3} \frac{i\theta_{\mu\nu}}{2} p_\nu^{(s,h,i)}, \\ \widehat{p}_\mu^{(s,h,i)} = p_\mu^{(s,h,i)}. \end{cases} \tag{27}$$

This allows us to find the operator $\widehat{r}^2$ equal to $r^2 - \mathbf{L\Theta}$ [12, 13, 76–78], while the new operators $V_{mt}^{s}\,(\widehat{r})$, $V_{mt}^{p}\,(\widehat{r})$, $U_{\rm eff}^{clp-s}\,(\widehat{r})$, $U_{\rm eff}^{clp-p}\,(\widehat{r})$, $k\,(k+1)\,\widehat{r}^{-2}$ and $k\,(k-1)\,\widehat{r}^{-2}$ in the DDT symmetries, are expressed as

$$\begin{cases} V_{mt}^{s}\,(\widehat{r}) = V_{mt}\,(r) - \dfrac{\partial V_{mt}\,(r)}{\partial r}\dfrac{\mathbf{L\Theta}}{2r} + O\left(\Theta^2\right), \\ V_{mt}^{p}\,(\widehat{r}) = V_{mt}\,(r) - \dfrac{\partial V_{mt}\,(r)}{\partial r}\dfrac{\widetilde{\mathbf{L}}\Theta}{2r} + O\left(\Theta^2\right), \\ U_{\rm eff}^{clp-s}\,(\widehat{r}) = U_{\rm eff}^{clp-s}\,(r) - \dfrac{\partial U_{\rm eff}^{clp-s}\,(r)}{\partial r}\dfrac{\mathbf{L\Theta}}{2r} + O\left(\Theta^2\right), \\ U_{\rm eff}^{clp-p}\,(\widehat{r}) = U_{\rm eff}^{clp-p}\,(r) - \dfrac{\partial U_{\rm eff}^{clp-p}\,(r)}{\partial r}\dfrac{\widetilde{\mathbf{L}}\Theta}{2r} + O\left(\Theta^2\right), \\ k\,(k+1)\,\widehat{r}^{-2} = k\,(k+1)\,r^{-2} + \\ + k\,(k+1)\,r^{-4}\mathbf{L}\Theta + O\left(\Theta^2\right), \\ k\,(k-1)\,\widehat{r}^{-2} = k\,(k-1)\,r^{-2} + \\ + k\,(k-1)\,r^{-4}\widetilde{\mathbf{L}}\Theta + O\left(\Theta^2\right). \end{cases} \tag{28}$$

Substituting Eqs. (28) into Eqs. (24) and (25), we obtain the following two Schrödinger-like equations:

$$\left(\frac{d^2}{dr^2} - k\,(k+1)\,r^{-2} + U_{\rm eff}^{clp-s}\,(r) - \right.$$
$$\left. - \gamma_s V_{mt}\,(r) - \beta_s^2 - \Sigma_{mt}^{\rm pert}\,(r)\right) F_{nk}^{s}\,(r) = 0 \tag{29}$$

and

$$\left(\frac{d^2}{dr^2} - k\,(k-1)\,r^{-2} + U_{\rm eff}^{clp-p}\,(r) - \right.$$
$$\left. - \gamma_p V_{mt}\,(r) - \beta_p^2 - \Delta_{mt}^{\rm pert}\,(r)\right) G_{nk}^{ps}\,(r) = 0 \tag{30}$$

with

$$\Sigma_{mt}^{\rm pert}\,(r) = \left(-\frac{1}{2r}\frac{\partial U_{\rm eff}^{clp-s} - s\,(r)}{\partial r} + \frac{k\,(k+1)}{r^4} - \right.$$
$$\left. - \frac{\gamma_s}{2r}\frac{\partial V_{mt}\,(r)}{\partial r}\right)\mathbf{L}\Theta + O\left(\Theta^2\right) \tag{31}$$

and

$$\Delta_{mt}^{\rm pert}\,(r) = \left(-\frac{1}{2r}\frac{\partial U_{\rm eff}^{clp-p}\,(r)}{\partial r} + \frac{k\,(k-1)}{r^4} - \right.$$
$$\left. - \frac{\gamma_p}{2r}\frac{\partial V_{mt}\,(r)}{\partial r}\right)\widetilde{\mathbf{L}}\Theta + O\left(\Theta^2\right). \tag{32}$$

By comparing Eqs. (13) and (14) and Eqs. (29) and (30), we observe two additive potentials $\Sigma_{mt}^{\rm pert}\,(r)$

and $\Delta_{mt}^{\text{pert}}(r)$. Moreover, these terms are proportional to the infinitesimal noncommutativity parameter Θ. From a physical point of view, this means that these two spontaneously generated terms $\Sigma_{mt}^{\text{pert}}(r)$ and $\Delta_{mt}^{\text{pert}}(r)$ as a result the topological properties of the deformed space-space can be considered very small compared to the fundamental terms $\Sigma_{mt}(r)$ and $\Delta_{mt}(r)$, respectively. A direct calculation gives $\frac{\partial V_{mt}(r)}{\partial r}$ and $\frac{\partial U_{\text{eff}}^{clps/p}(r)}{\partial r}$ as follows:

$$\begin{cases} \dfrac{\partial V_{mt}(r)}{\partial r} = -\dfrac{2A}{r^3} + \dfrac{B}{r^2}, \\ \dfrac{\partial U_{\text{eff}}^{clps/p}(r)}{\partial r} = -2\dfrac{-2k\alpha \mp \alpha - \alpha^2}{r^3}. \end{cases} \tag{33}$$

Substituting Eq. (33) into Eqs. (31) and (32), we obtain the spontaneously generated terms $\Sigma_{mt}^{\text{pert}}(r)$ and $\Delta_{mt}^{\text{pert}}(r)$ as follows:

$$\Sigma_{mt}^{\text{pert}}(r) = \left(\frac{k(k+1) + \gamma_s A - 2k\alpha - \alpha - \alpha^2}{r^4} - \right.$$
$$\left. - \frac{\gamma_s}{2}\frac{B}{r^3}\right)\mathbf{L}\Theta + O\left(\Theta^2\right) \tag{34}$$

and

$$\Delta_{mt}^{\text{pert}}(r) = \left(\frac{k(k-1) + \gamma_p A - 2k\alpha + \alpha - \alpha^2}{r^4} - \right.$$
$$\left. - \frac{\gamma_p}{2}\frac{B}{r^3}\right)\widetilde{\mathbf{L}}\Theta + O\left(\Theta^2\right). \tag{35}$$

Furthermore, we use the unit step function (also known as the Heaviside step function $\theta(x)$ or simply the theta function) and rewrite the global induced two potentials $\Sigma_{t-mt}^{\text{pert}}(r)$ and $\Delta_{t-mt}^{\text{pert}}(r)$ for the spin and pseudospin symmetries corresponding to the upper and lower components ($F_{nk}^{s}(s)$ and $G_{nk}^{s}(s)$) and ($F_{nk}^{ps}(s)$ and $G_{nk}^{ps}(s)$), respectively, as:

$$\Sigma_{t\_mt}^{\text{pert}}(r) = \Sigma_{mt}^{\text{pert}}(r)\,\theta\left(E_{nc}^{mt-s}\right) -$$
$$- \Sigma_{mt}^{\text{pert}}(r)\,\theta\left(-E_{nc}^{mt-s}\right) =$$
$$= \begin{cases} \Sigma_{mt}^{\text{pert}}(r) & \text{for } F_{nk}^{s}(r), \\ -\Sigma_{mt}^{\text{pert}}(r) & \text{for } G_{nk}^{s}(r), \end{cases} \tag{36}$$

and

$$\Delta_{t-mt}^{\text{pert}}(r) = \Delta_{ts}^{\text{pert}}(r)\,\theta\left(E_{nc}^{mt-ps}\right) -$$
$$-\Delta_{mt}^{\text{pert}}(r)\,\theta\left(-E_{nc}^{mt-ps}\right) =$$
$$= \begin{cases} \Delta_{mt}^{\text{pert}}(r) & \text{for } F_{nk}^{ps}(r), \\ -\Delta_{mt}^{\text{pert}}(r) & \text{for } G_{nk}^{ps}(r), \end{cases} \tag{37}$$

Where the step function $\theta(x)$ is given by

$$\theta(x) = \begin{cases} 1 \text{ for } x \geq 0, \\ 0 \text{ for } x < 0. \end{cases} \tag{38}$$

The Mie-type potential including a Coulomb-like tensor interaction is extended by including new additive potentials $\Sigma_{mt}^{\text{pert}}(r)$ and $\Delta_{mt}^{\text{pert}}(r)$ expressed to the radial terms $\frac{1}{r^3}$ and $\frac{1}{r^4}$ to become the improved Mie-type potential including an improved Coulomb-like tensor potential under DDT symmetries. The global induced two potentials $\Sigma_{t-mt}^{\text{pert}}(r)$ and $\Delta_{t-mt}^{\text{pert}}(r)$ represent the physical interaction between the system's physical properties that correspond to the spin and p-spin symmetries ($\mathbf{L}$ and $\widetilde{\mathbf{L}}$) and the distance between diatomic molecules r with topological deformations of the space-space characterized by the noncommutativity vector $\boldsymbol{\Theta}$. The generated new two effective potentials $\Sigma_{mt}^{\text{pert}}(r)$ and $\Delta_{mt}^{\text{pert}}(r)$ are also proportional to the infinitesimal vector $\boldsymbol{\Theta}$. This allows us to consider the new additive parts of the effective potential $\Sigma_{mt}^{\text{pert}}(r)$ and $\Delta_{mt}^{\text{pert}}(r)$ as perturbation potentials as compared with the main potentials $\Sigma_{mt}(r)$ and $\Delta_{mt}(r)$ which are also known with the parent potential operator in the symmetries of DDT. That is, the two inequalities $\Sigma_{mt}^{\text{pert}}(r) \ll \Sigma_{mt}(r)$ and $\Delta_{mt}^{\text{pert}}(r) \ll \Delta_{mt}(r)$ have become achieved. All physical justifications for applying the time-independent perturbation theory become satisfied to calculate the expectation values of the previous radial terms. This allows us to give a complete prescription for determining the energy levels of the generalized $(n, l, \widetilde{l}, m, \widetilde{m}, s, \widetilde{s})^{\text{th}}$ excited states.

3.2. The expectation values in the IMTPICLP model in the DDT for the spin symmetry

Here, we want to apply the perturbative theory in the case of deformation Dirac theory symmetries. We will find the expectation values:

$$M_{1(nlms)}^{sp-mt} \equiv \left\langle \frac{1}{r^3} \right\rangle_{(nlms)}^{sp-mt} \text{ and } M_{2(nlms)}^{sp-mt} \equiv \left\langle \frac{1}{r^4} \right\rangle_{(nlms)}^{sp-mt}$$

for the spin symmetry accounting for the unperturbed upper component $F_{nk}^{s}(r)$ which we have seen previously in Eq. (15). After straightforward calculations, we obtain the following results:

$$M_{1(nlms)}^{sp-mt} = \frac{\left(2\sqrt{\epsilon^{s2}}\right)^{(1+q^s)}(n-q^s)}{n!^3} \times$$
$$\times \int_0^{+\infty} r^{(1+q^s)-3}\exp\left(2\sqrt{\epsilon^{s2}}r\right)\left[L_n^{q^s}\left(2\sqrt{\epsilon^{s2}}r\right)\right]^2 dr \quad (39a)$$

and

$$M_{2(nlms)}^{sp-mt} = \frac{\left(2\sqrt{\epsilon^{s2}}\right)^{(1+q^s)}(n-q^s)}{n!^3} \times$$
$$\int_0^{+\infty} r^{(1+q^s)-4}\exp\left(2\sqrt{\epsilon^{s2}}r\right)\left[L_n^{q^s}\left(2\sqrt{\epsilon^{s2}}r\right)\right]^2 dr. \quad (39b)$$

We have used the useful abbreviations $\langle R\rangle_{(nlms)}^{sp-mt}$ which equal the average value $\langle n,l,m|R|n,l,m\rangle$ to avoid the extra burden of writing, with $R = \left\{\frac{1}{r^3} \text{ or } \frac{1}{r^4}\right\}$. Furthermore, we have applied the property of the spherical harmonics, which has the form

$$\int_0^{+\infty} Y_l^m(\theta',\varphi')\, Y_{l'}^{m'}(\theta,\varphi)\sin(\theta)\, d\theta d\varphi = \delta_{ll'}\delta_{mm'}.$$

Let us compare Eqs. (39a) and (39b) with the integral of the form [81]

$$\int_0^{+\infty} t^{\eta-1}\exp(-\omega t)\, L_m^{\lambda}(\omega t)\, L_n^{\beta}(\omega t)\, dt =$$
$$= \frac{\omega^{-\eta}\Gamma(n-\eta+\beta+1)\,\Gamma(m+\lambda+1)}{m!n!\Gamma(1-\eta+\beta)\,\Gamma(\lambda+1)} \times$$
$$\times\ {}_3F_2(-m,\eta,\eta-\beta;-n+\eta,\lambda+1,1) \quad (40)$$

with $\mathrm{Rel}(\varepsilon)\rangle 0$. We note that ${}_3F_2(-m,\varepsilon,\varepsilon-\beta;-n+\varepsilon,\lambda+1,1)$ is obtained from the generalized hypergeometric function ${}_pF_q(\alpha_1,...,\alpha_p;\beta^1,...,\beta^q,1)$ for $p=3$ and $q=2$, while Γ denotes the usual Gamma function. After straightforward calculations, we find

$$M_{1(nlms)}^{sp-mt} = \frac{\left(2\sqrt{\epsilon^{s2}}\right)^{(1+q^s)}(n-q^s)}{n!^4} \times$$
$$\times \frac{\left(2\sqrt{\epsilon^{s2}}\right)^{1-q^s}\Gamma(n+q^s+1)}{\Gamma(q^s+1)} \times$$
$$\times\ {}_3F_2(-n,q^s-1,-1;-n+q^s-1,q^s+1,1) \quad (41a)$$

and

$$M_{2(nlms)}^{sp-mt} = \frac{\left(2\sqrt{\epsilon^{s2}}\right)^{1+q^s}(n-q^s)}{n!^4} \times$$
$$\times \frac{\left(2\sqrt{\epsilon^{s2}}\right)^{2-q^s}(n+2)(n+1)\Gamma(n+q^s+1)}{2\Gamma(q^s+1)} \times$$
$$\times\ {}_3F_2(-n,q^s-2,-2;-n+q^s-2,q^s+1,1), \quad (41b)$$

where we have used the property $\Gamma(n+1) = n!$.

3.3. The expectation values in the IMTPICLP model in the DDT for the p-spin symmetry

In this subsection, we want to apply the perturbative theory in the case of deformation Dirac theory symmetries. We will find the expectation values:

$$M_{1(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-mt} \equiv \left\langle \frac{1}{r^3}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-mt}$$

and

$$M_{2(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-mt} \equiv \left\langle \frac{1}{r^4}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-mt}$$

for the p-spin symmetry with tensor interaction accounting for the wave function which we have seen previously in Eq. (16). By examining the two expressions of the upper and lower components ($F_{nk}^s(r)$ and $G_{nk}^{ps}(r)$) shown in Eqs. (15) and (16), we note that there is a possibility to move from the unperturbed upper component $F_{nk}^s(r)$ to the other lower component $G_{nk}^{ps}(r)$ by making the following substitutions:

$$q^s \Longleftrightarrow q^p \quad \text{and} \quad \epsilon^s \Leftrightarrow \epsilon^p. \quad (42)$$

This allows us to obtain the expectation values for the p-spin symmetry from Eqs. (41a) and (41b) without re-calculation, as follows:

$$M_{1(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-mt} = \frac{\left(2\sqrt{\epsilon^{s2}}\right)^{(1+q^p)}(n-q^p)}{n!^4} \times$$
$$\times \frac{\left(2\sqrt{\epsilon^{p2}}\right)^{1-q^s}\Gamma(n+q^p+1)}{\Gamma(q^p+1)} \times$$
$$\times\ {}_3F_2(-n,q^p-1,-1;-n+q^p-1,q^p+1,1) \quad (43a)$$

and

$$M_{2(n\widetilde{l}\widetilde{m}\widetilde{s})}^{sp-mt} = \frac{\left(2\sqrt{\epsilon^{p2}}\right)^{1+q^p}(n-q^{sp})}{n!^4} \times$$
$$\times \frac{\left(2\sqrt{\epsilon^{p2}}\right)^{2-q^{sp}}(n+2)(n+1)\Gamma(n+q^p+1)}{2\Gamma(q^p+1)} \times$$
$$\times\ {}_3F_2(-n,q^p-2,-2;-n+q^p-2,q^p+1,1). \quad (43b)$$

3.4. The corrected energy for the IMTPICLP model in DDT symmetries

The main objective underlined in this subsection is to find the contribution resulting from topological properties based on our strategy that we have successfully applied in previous works and which we try to develop in each new work. We can say that the global relativistic energy in the perspective of the deformation Dirac theory produced with the IMTPICLP model as a result of the major contribution to relativistic energy known in the literature under the MTPICLP model in the usual Dirac theory which we paved for through a quick look for the spin(p-spin)-symmetry in Eqs. (17) and (18), while the new contribution is produced from the topological properties under a space-space deformation. It can be evaluated through several contributions, we will address three of them. The first one is generated from the effect of the perturbed spin-orbit effective potentials $\Sigma_{mt}^{\text{pert}}(r)$ and $\Delta_{mt}^{\text{pert}}(r)$ corresponds to the spin symmetry and pseudospin symmetry. These perturbed effective potentials are obtained by replacing the coupling of the angular momentum ($\mathbf{L}$ and $\widetilde{\mathbf{L}}$) operators and the NC vector Θ with the new equivalent couplings ($\Theta\mathbf{L}S$ and $\Theta\widetilde{\mathbf{L}}\widetilde{\mathbf{S}}$) for the spin-symmetry and p-spin-symmetry, respectively (with $\Theta^2 = \Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2$). This degree of freedom comes considering that the infinitesimal NC vector $\boldsymbol{\Theta}$ is arbitrary. We have oriented the two spin-$\mathbf{s}$ and spin-$\widetilde{\mathbf{s}}$ of the fermionic particles to become parallels to the vector Θ which interactes in the IMTPICLP model. Moreover, we replace the new spin-orbit couplings $\Theta\mathbf{L}S$ and $\Theta\widetilde{\mathbf{L}}\widetilde{\mathbf{S}}$ with the corresponding new physical forms $(\Theta/2)\,\mathbf{G}^2$ and $(\Theta/2)\,\widetilde{\mathbf{G}}^2$: with

$$\begin{cases}\mathbf{G}^2 = \mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2,\\ \widetilde{\mathbf{G}}^2 = \mathbf{J}^2 - \widetilde{\mathbf{L}}^2 - \widetilde{\mathbf{S}}^2\end{cases}$$

for the spin/(p-spin)-symmetry, respectively. Furthermore, in RQM, the operators ($\widehat{\mathbf{H}}_{rnc}^{mt}$, $\mathbf{J}^2$, $\mathbf{L}^2$, $\mathbf{S}^2$ and $\mathbf{J}_z$) form a complete set of conserved physical quantities, the eigenvalues of the operators $\mathbf{G}^2$ and $\widetilde{\mathbf{G}}^2$ are equal to the values:

$$\begin{cases}2F(j,l,s) = [j(j+1) - l(l+1) - s(s+1)],\\ 2F\left(j,\widetilde{l},\widetilde{s}\right) = [j(j+1) - \widetilde{l}(\widetilde{l}-1) - \widetilde{s}(\widetilde{s}+1)]\end{cases}$$

with $|l-s| \le j \le |l+s|$ and $|\widetilde{l}-\widetilde{s}| \le j \le |\widetilde{l}+\widetilde{s}|$ for the spin-symmetry and p-spin-symmetry, respectively. As a direct consequence, the partially corrected energies $\Delta E_{mt}^{\text{so-sp}}(n,A,B,C,\alpha,\Theta,j,l,s) \equiv \Delta E_{mt}^{so-sp}$ and $\Delta E_{mt}^{so-ps}(n,A,B,C,\alpha,\ \Theta,\ j,\ \widetilde{l},\ \widetilde{s}) \equiv\ \Delta E_{mt}^{so-ps}$ due to the perturbed effective potentials $\Sigma_{mt}^{\text{pert}}(r)$ and $\Delta_{mt}^{\text{pert}}(r)$ produced for the n^{th} excited state, in the deformation Dirac theory symmetries as follows:

$$\Delta E_{mt}^{so-sp} = \Theta\,(j(j+1) - k(k+1) - s(s+1))\ \langle Z\rangle_{(nlms)}^{mt}(n,A,B,C,\alpha)\ \text{for-uppe-component}\ F^s(r)\text{-of-spin symmetry} \tag{44a}$$

and

$$\Delta E_{mt}^{so-ps} = \Theta\,(j(j+1) - k(k-1) - \widetilde{s}(\widetilde{s}+1))\ \langle \widetilde{Z}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{mt}(n,A,B,C,\alpha)\ \text{for-lower-component}\ G^{ps}(r)\text{-of-p-spin symmetry.} \tag{44b}$$

The global two expectation values $\langle Z\rangle_{(nlms)}^{mt}(n,A,B,C,\alpha)$ and $\langle \widetilde{Z}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{mt}(n,A,B,C,\alpha)$ for the spin/(p-spin)-symmetry, respectively, are determined from the following expressions:

$$\langle Z\rangle_{(nlms)}^{mt}(n,A,B,C,\alpha) = \left(k\,(k+1) + \gamma_s A - 2k\alpha - \alpha - \alpha^2\right)\times \left\langle \frac{1}{r^4}\right\rangle_{(nlms)}^{s-mt} - \frac{\gamma_s B}{2}\left\langle \frac{1}{r^3}\right\rangle_{(nlms)}^{s-mt} \tag{45a}$$

and

$$\langle \widetilde{Z}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{mt}(n,A,B,C,\alpha) = \left(k\,(k-1) + \gamma_p A - 2k\alpha + \alpha - \alpha^2\right)\times \left\langle \frac{1}{r^4}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-mt} - \frac{\gamma_p B}{2}\left\langle \frac{1}{r^3}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-mt}. \tag{45b}$$

The second main part is obtained from the magnetic effect of the perturbative effective potentials $\Sigma_{mt}^{\text{pert}}(r)$ and $\Delta_{mt}^{\text{pert}}(r)$ in the IMTPICLP model in the deformation Dirac theory symmetries. These effective potentials are achieved, when we replace both ($\mathbf{L}\Theta$ and $\widetilde{\mathbf{L}}\Theta$) by ($\tau\aleph L_z$ and $\tau\aleph\widetilde{L}_z$), respectively, and Θ_{13} by $\tau\aleph$. Here, ($\aleph$ and τ) present the intensity of the magnetic field induced by the effect of a deformation of the space-space geometry and a new infinitesimal noncommutativity parameter, so that the physical unit of the original noncommutativity parameter

$\Theta_{12}(\text{length})^2$ is the same unit of $\tau\aleph$. We also need to apply:

$$\begin{cases}\langle n', l', m'|L_z|n, l, m\rangle = m\delta_{m'm}\delta_{l'l}\delta_{n'n},\\ \langle n', \widetilde{l'}, \widetilde{m'}|\widetilde{L}_z|n, \widetilde{l}, \widetilde{m}\rangle = \widetilde{m}\delta_{\widetilde{m'}\widetilde{m}}\delta_{\widetilde{l'}\widetilde{l}}\delta_{n'n}\end{cases}$$

with $((|l|, -|\widetilde{l}|) \leq (m, \widetilde{m}) \leq (|l|, |\widetilde{l}|))$ for the spin/(p-spin)-symmetry, respectively. All of these data allow the discovery of a new energy shift $\Delta E_{mt}^{mg-s}(n, A, B, C, \alpha, \tau, m)$, and $\Delta E_{mt}^{mg-ps}(n, A, B, C, \alpha, \tau, \widetilde{m})$ due to the perturbed Zeeman effect created by the influence of the IMTPICLP model for the $(n, l, \widetilde{l}, m, \widetilde{m}, s, \widetilde{s})^{\text{th}}$ excited state in deformation Dirac theory symmetries as follows:

$$\Delta E_{mt}^{mg-sp}(n, A, B, C, \alpha, \tau, m) = \tau\aleph\,\langle Z\rangle_{(nlms)}^{mt}\, m \quad \text{for-uppe-component } F^s(r) \tag{46a}$$

and

$$\Delta E_{mt}^{mg-ps}(n, A, B, C, \alpha, \tau, \widetilde{m}) = \tau\aleph\langle\widetilde{Z}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{mt}\widetilde{m} \quad \text{for-lower-component } G^{ps}(r). \tag{46b}$$

After we have completed the first and second stages of the self-production of energy, we are heading to another very important case in the IMTPICLP model in DDT symmetries. This physical phenomenon is produced automatically from the influence of perturbed effective potentials $\Sigma_{mt}^{\text{pert}}(r)$ and $\Delta_{mt}^{\text{pert}}(r)$ which we have seen in Eqs. (37a) and (37b). We consider the fermionic particles undergoing a rotation with angular velocity ω. The features of this subjective phenomenon are determined through the replacement of an arbitrary vector $\mathbf{\Theta}$ with $\chi\omega$. Now, we replace the two couplings ($\mathbf{L}\Theta$ and $\widetilde{\mathbf{L}}\Theta$) with ($\chi\mathbf{L}\omega$ and $\chi\widetilde{\mathbf{L}}\omega$), respectively, as follows:

$$\begin{pmatrix}\mathbf{L}\Theta\\ \widetilde{\mathbf{L}}\Theta\end{pmatrix} \to \chi\begin{pmatrix}\mathbf{L}\omega\\ \widetilde{\mathbf{L}}\omega\end{pmatrix}. \tag{47}$$

Here, χ is just an infinitesimal real proportional constant. The effective potentials $\Sigma_{\text{pert}}^{t-\text{rot}}(s)$ and $\Delta_{\text{pert}}^{mt-\text{rot}}(s)$, which induces the rotational movements of the fermionic particles, can be expressed as follows:

$$\begin{pmatrix}\Sigma_{mt}^{\text{pert}}(r)\\ \Delta_{mt}^{\text{pert}}(r)\end{pmatrix} \longrightarrow \begin{pmatrix}\Sigma_{\text{pert}}^{mt-\text{rot}}(r)\\ \Delta_{\text{pert}}^{mt-\text{rot}}(r)\end{pmatrix} = = \chi\begin{pmatrix}\langle Z\rangle_{(nlms)}^{mt}(n, A, B, C, \alpha)\,\mathbf{L}\omega\\ \langle Z\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{mt}(n, A, B, C, \alpha)\,\mathbf{L}\omega\end{pmatrix}. \tag{48}$$

We chose a rotational velocity ω parallel to the (Oz) axis ($\omega = \omega\mathbf{e}_z$) to simplify the calculations; this, of course, does not change the physical characteristics of the examined problem as much as it simplifies the calculations. The spin-orbit couplings are then transformed into new physical phenomena as follows:

$$\begin{pmatrix}\Sigma_{\text{pert}}^{mt-\text{rot}}(r)\,\mathbf{L}\omega\\ \Delta_{\text{pert}}^{mt-\text{rot}}(r)\,\widetilde{\mathbf{L}}\omega\end{pmatrix} = \chi\omega\begin{pmatrix}\Sigma_{\text{pert}}^{mt-\text{rot}}(r)\,L_z\\ \Delta_{\text{pert}}^{mt-\text{rot}}(r)\,\widetilde{L}_z\end{pmatrix}. \tag{49}$$

All of this data allow the discovery of the new corrected energy $\Delta E_{mt}^{\text{rot}-sp}(n, A, B, C, \alpha, \chi, m)$ and $\Delta E_{mt}^{\text{rot}-ps}(n, A, B, C, \alpha, \chi, \widetilde{m})$ due to the perturbed effective potentials $\Sigma_{\text{pert}}^{mt-\text{rot}}(r)$ and $\Delta_{\text{pert}}^{mt-\text{rot}}(r)$ which are generated automatically by the influence of the improved Mie-type potential including an improved Coulomb-like tensor interaction for the $(n, l, \widetilde{l}, m, \widetilde{m}, s, \widetilde{s})^{\text{th}}$ excited state in DDT symmetries as follows:

$$\begin{pmatrix}\Delta E_{mt}^{\text{rot}-sp}\\ \Delta E_{mt}^{\text{rot}-ps}\end{pmatrix} = \chi\omega\begin{pmatrix}\langle Z\rangle_{(nlms)}^{mt}(n, A, B, C, \alpha)m\\ \langle\widetilde{Z}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{mt}(n, A, B, C, \alpha)\widetilde{m}\end{pmatrix}. \tag{50}$$

It is worth noting that the authors of Ref. [82] investigated rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gases in two- and three-dimensional spaces at zero temperature, but, in this case, the rotational term was added to the Hamiltonian operator, whereas, in our case, the two rotation operators $\Sigma_{\text{pert}}^{mt-\text{rot}}(r)\,\mathbf{L}\omega$ and $\Delta_{\text{pert}}^{mt-\text{rot}}(r)\,\widetilde{\mathbf{L}}\omega$ appear automatically due to the deformation of the space-space for the improved Mie-type potential including an improved Coulomb-like tensor interaction. The eigenvalues of the operations $\mathbf{G}^2$ and $\widetilde{\mathbf{G}}^2$ for a fermionic particle and antiparticle (negative energy) with spin $s = \widetilde{s} = \frac{1}{2}$ are equal to the following values:

$$\begin{cases}F(j, l, s) = [j(j+1) - l(l+1) - 3/4]/2,\\ F(j, \widetilde{l}, \widetilde{s}) = [j(j+1) - \widetilde{l}(\widetilde{l}-1) - 3/4]/2\end{cases}$$

respectively. In the case of spin-1/2 fields, the possible values of j are ($l \pm 1/2$ and $\widetilde{l} \pm 1/2$) for the spin symmetry $F(j, l, s)$ and the pseudospin symmetry $F(j, \widetilde{l}, \widetilde{s})$, as follows:

$$F^f(j = l \pm 1/2, s = 1/2) = = \frac{1}{2}\begin{cases}l & \text{Up polarity: for } j = l + 1/2,\\ -(l+1) & \text{Down polarity: for } j = l - 1/2\end{cases} \tag{51}$$

and

$$\digamma^{f}(j=\widetilde{l}\pm 1/2,\widetilde{s}=1/2)=$$

$$=\frac{1}{2}\begin{cases}\widetilde{l} \text{ Up polarity: for } j=\widetilde{l}+1/2,\\ -(\widetilde{l}+1) \text{ Down polarity: for } j=\widetilde{l}-1/2.\end{cases} \quad (52)$$

In the symmetries of the DDT symmetries, the total relativistic energy $E_{nc}^{sp}(n,A,B,C,\alpha,\Theta,\tau,\chi,j,l,s,m)$ and $E_{nc}^{ps}(n,A,B,C,\alpha,\Theta,\tau,\chi,j,\widetilde{l},\widetilde{s},\widetilde{m})$ in the case of spin 1/2 with improved Mie-type potential including an improved Coulomb-like tensor interaction, corresponding to the generalized $(n,l,\widetilde{l},m,\widetilde{m},s,\widetilde{s})^{\text{th}}$ excited states are expressed as:

$$E_{nc}^{sp}(n,A,B,C,\alpha,\Theta,\tau,\chi,j,l,s,m)=$$

$$=E_{nk}^{s}+\langle Z\rangle_{(nlms)}^{mt}(n,A,B,C,\alpha)\left((\tau\aleph+\chi\omega)\,m+\right.$$

$$\left.+\frac{\Theta}{2}\begin{cases}l \text{ Up polarity: } j=l+1/2\\ -(l+1) \text{ Down polarity: } j=l-1/2\end{cases}\right) \quad (53a)$$

and

$$E_{nc}^{ps}(n,A,B,C,\alpha,\Theta,\tau,\chi,j,\widetilde{l},\widetilde{s},\widetilde{m})=$$

$$=E_{nk}^{ps}+\langle\widetilde{Z}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{mt}(n,A,B,C,\alpha)\left((\tau\aleph+\chi\omega)\,\widetilde{m}+\right.$$

$$\left.+\frac{\Theta}{2}\begin{cases}\widetilde{l} \text{ Up polarity: for } j=\widetilde{l}+1/2\\ -(\widetilde{l}+1) \text{ Down polarity: for } j=\widetilde{l}-1/2\end{cases}\right), \quad (53b)$$

where E_{nk}^{sp} and E_{nk}^{ps} are usual relativistic energies for Mie-type potential including a Coulomb-like tensor interaction obtained from the equations of energy in Eqs. (17) and (18). These results describe the spin and p-spin new energies in DDE for atoms with one electron. This is consistent with the results we found previously in Ref. [13]. For Cl_2, N_2, CO, NO, and CH diatomic molecules, we replace $\digamma^{f}(j,l,s)$ and $\digamma^{f}(j,\widetilde{l},\widetilde{s})$ by the generalized two previous values $\digamma(j,l,s)$ and $\digamma(j,\widetilde{l},\widetilde{s})$. Now, we obtain the total relativistic energy $E_{nc}^{sp}(n,A,B,C,\alpha,\Theta,\tau,\chi,j,l,s,m)$ and $E_{nc}^{ps}(n,A,B,C,\alpha,\Theta,\tau,\chi,j,\widetilde{l},\widetilde{s},\widetilde{m})$ for Cl_2, N_2, CO, NO, and CH diatomic molecules with improved Mie-type potential including an improved Coulomb-like tensor interaction, corresponding to the generalized $(n,l,\widetilde{l},m,\widetilde{m},s,\widetilde{s})^{\text{th}}$ excited states are expressed as:

$$E_{nc}^{sp}(n,A,B,C,\alpha,\Theta,\tau,\chi,j,l,s,m)=$$

$$=E_{nk}^{s}+\langle Z\rangle_{(nlms)}^{mt}(n,A,B,C,\alpha)\times$$

$$\times\left[(\tau\aleph+\chi\omega)\,m+\digamma(j,l,s)\right] \quad (54a)$$

and

$$E_{nc}^{ps}(n,A,B,C,\alpha,\Theta,\tau,\chi,j,\widetilde{l},\widetilde{s},\widetilde{m})=$$

$$=E_{nk}^{ps}+\langle\widetilde{Z}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{mt}(n,A,B,C,\alpha)\times$$

$$\times\left[(\tau\aleph+\chi\omega)\,\widetilde{m}+\frac{\Theta}{2}\digamma^{f}(j,\widetilde{l},\widetilde{s})\right]. \quad (54b)$$

We can now generalize our obtained energies E_{g-nc}^{mt-s} and E_{g-nc}^{mt-p} for the improved Mie-type potential which are produced with the global induced two potentials $\Sigma_{t-mt}^{\text{pert}}(r)$ and $\Delta_{t-mt}^{\text{pert}}(r)$ for the spin and pseudospin symmetries corresponding to the upper and lower components ($F_{nk}^{s}(s)$ and $G_{nk}^{s}(s)$) and ($F_{nk}^{ps}(s)$ and $G_{nk}^{ps}(s)$), respectively, as:

$$E_{g-nc}^{mt-s}=E_{nc}^{mt-s}\theta\left(\left|E_{nc}^{mt-s}\right|\right)-E_{nc}^{mt-s}\theta\left(-\left|E_{nc}^{mt-s}\right|\right)=$$

$$=\begin{cases}E_{nc}^{mt-s} \text{ for upper component}\\ \text{of spin symmetry,}\\ -E_{nc}^{mt-s} \text{ for lower component}\\ \text{of spin symmetry}\end{cases} \quad (55)$$

and

$$E_{g-nc}^{mt-ps}=E_{nc}^{mt-ps}\theta\left(\left|E_{nc}^{mt-ps}\right|\right)-$$

$$-E_{nc}^{mt-ps}\theta\left(-\left|E_{nc}^{mt-ps}\right|\right)=$$

$$=\begin{cases}E_{nc}^{mt-ps} \text{ for upper component}\\ \text{of p-symmetry,}\\ -E_{nc}^{mt-ps} \text{ for lower component}\\ \text{of p-symmetry.}\end{cases} \quad (56)$$

4. Study of Important Relativistic Particular Cases in DDT

We will look at some specific examples involving the new bound-state energy eigenvalues in Eqs. (53a) and (53b). By adjusting relevant parameters of the IMT-PICLP model in a deformation of Dirac theory symmetries, we will derive some specific potentials useful for other physical systems such as ones with the improved Kratzer–Fues potential within an improved Coulomb-like tensor interaction, the improved modified Kratzer potential within the Coulomb-like tensor interaction, and the improved Mie-type potential in the symmetries of extended nonrelativistic quantum mechanics.

4.1. Deformed Dirac equation with improved Kratzer–Fues potential within ICLP

Using Eq. (3), i.e., $A = D_e r_e^2$, $B = 2D_e r_e$, $C = 0$, and Eqs. (3) and (33) wirh improved Kratzer–Fues potential $V_{kf}(\widehat{r})$ in DDT, we have

$$V_{kf}(\widehat{r}) = \frac{D_e r_e^2}{r^2} - \frac{2D_e r_e}{r} - \frac{1}{2r}\left(-\frac{2D_e r_e^2}{r^3} + \frac{2D_e r_e}{r^2}\right) \times \\ \times \begin{cases} \mathbf{L}\Theta \text{ for spin-symmetry} \\ \widetilde{\mathbf{L}}\Theta \text{ for p-spin-symmetry} \end{cases} + O(\Theta^2). \quad (57)$$

The energy eigenvalue corresponding to the upper and lower components $F_{nk}^{s}(r)$ and $G_{nk}^{ps}(r)$ under spin and p-spin symmetries for the improved Kratzer–Fues potential within an improved Coulomb-like tensor potential (ICLP) are determined from Eqs. (53) and (54) as follows:

$$E_{nc}^{sp}(n, D_e, r_e, \alpha, \Theta, \tau, \chi, j, l, s, m) = E_{nk}^{kf-s} + \\ + \langle Z\rangle_{(nlms)}^{kf}(n, D_e, r_e, \alpha)\left((\tau\aleph + \chi\omega)\, m + \\ + \frac{\Theta}{2}\begin{cases} l \quad \text{Up polarity: } j = l + 1/2 \\ -(l+1) \quad \text{Down polarity: } j = l - 1/2 \end{cases}\right) \quad (58)$$

and

$$E_{nc}^{ps}(n, D_e, r_e, \alpha, \Theta, \tau, \chi, j, \widetilde{l}, \widetilde{s}, \widetilde{m}) = E_{nk}^{kf-ps} + \\ + \langle \widetilde{Z}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{kf}(n, D_e, r_e, \alpha)\left((\tau\aleph + \chi\omega)\, \widetilde{m} + \\ + \frac{\Theta}{2}\begin{cases} \widetilde{l} \quad \text{Up polarity: } j = \widetilde{l} + 1/2 \\ -(\widetilde{l}+1) \quad \text{Down polarity: } j = \widetilde{l} - 1/2 \end{cases}\right). \quad (59)$$

Here, E_{nk}^{kf-s} and E_{nk}^{kf-ps} are determined from the energy equations for the Mie-type potential within Coulomb-like tensor interaction under spin and p-spin symmetries of the Dirac theory as follows:

$$\left(M + E_{nk}^{kf-s}\right)\Big[2n + 1 + \\ + \sqrt{(2k+1)^2 + 4\alpha(\alpha + 2k + 1) + 4\gamma_s D_e r_e^2}\,\Big]^2 = \\ = 4\gamma_s D_e^2 r_e^2 \quad (60)$$

and

$$\left(M + E_{nk}^{kf-ps}\right)\Big[2n + 1 + \\ + \sqrt{(2k-1)^2 + 4\alpha(\alpha + 2k - 1) + 4\gamma_p D_e r_e^2}\,\Big]^2 = \\ = 4\gamma_p D_e^2 r_e^2, \quad (61)$$

while the new expectation values $\langle Z\rangle_{(nlms)}^{kf}(n, D_e, r_e, \alpha)$ and $\langle \widetilde{Z}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{kf}(n, D_e, r_e, \alpha)$ which re determined from Eq. (45) by applying the compensation referred to the beginning of the current subsection as follows:

$$\langle Z\rangle_{(nlms)}^{kf}(n, D_e, r_e, \alpha) = (k(k+1) + \gamma_s D_e r_e^2 - \\ - 2k\alpha - \alpha - \alpha^2)\left\langle \frac{1}{r^4}\right\rangle_{(nlms)}^{s-kf} - \gamma_s D_e r_e \left\langle \frac{1}{r^3}\right\rangle_{(nlms)}^{s-kf} \quad (62a)$$

and

$$\langle \widetilde{Z}\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{kf}(n, D_e, r_e, \alpha) = (k(k-1) + \gamma_p D_e r_e^2 - \\ - 2k\alpha + \alpha - \alpha^2)\left\langle \frac{1}{r^4}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-kf} - \gamma_p D_e r_e \left\langle \frac{1}{r^3}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-kf} \quad (62b)$$

with

$$\left(\left\langle \frac{1}{r^4}\right\rangle_{(nlms)}^{s-kf}, \left\langle \frac{1}{r^3}\right\rangle_{(nlms)}^{s-kf}\right) = \\ = \operatorname*{Im}_{A\to D_e r_e^2,\ B\to 2D_e r_e}\left(\left\langle \frac{1}{r^4}\right\rangle_{(nlms)}^{s-mt}, \left\langle \frac{1}{r^3}\right\rangle_{(nlms)}^{s-mt}\right) \quad (63)$$

and

$$\left(\left\langle \frac{1}{r^4}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-kf}, \left\langle \frac{1}{r^3}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-kf}\right) = \\ = \operatorname*{Im}_{A\to D_e r_e^2,\ B\to 2D_e r_e}\left(\left\langle \frac{1}{r^4}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-mt}, \left\langle \frac{1}{r^3}\right\rangle_{(n\widetilde{l}\widetilde{m}\widetilde{s})}^{ps-mt}\right). \quad (64)$$

4.2. Deformed Dirac equation with improved modified Kratzer potential within ICLP

Using Eq. (3), i.e. $A = D_e r_e^2$, $B = 2D_e r_e$, $C = D_e$ and Eqs. (3) and (33), we get the improved Kratzer–Fues potential $V_{\text{kf}}(\widehat{r})$ in DDT as

$$V_{\text{kf}}(\widehat{r}) = \frac{D_e r_e^2}{r^2} - \frac{2D_e r_e}{r} + D_e - \\ - \frac{1}{2r}\left(-\frac{2D_e r_e^2}{r^3} + \frac{2D_e r_e}{r^2}\right) \times \\ \times \begin{cases} \mathbf{L}\Theta \quad \text{for spin-symmetry} \\ \widetilde{\mathbf{L}}\Theta \quad \text{for p-spin-symmetry} \end{cases} + O(\Theta^2), \quad (65)$$

The energy eigenvalue corresponding to the upper and lower components $F^{s}_{nk}(r)$ and $G^{ps}_{nk}(r)$ under spin and p-spin symmetries for the improved Kratzer–Fues potential within an improved Coulomb-like tensor interaction are determined from Eqs. (53) and (54) as follows:

$$E^{sp}_{nc}(n, D_e, r_e, \alpha, \Theta, \tau, \chi, j, l, s, m) = E^{mk-s}_{nk} + \\ + \langle Z\rangle^{kf}_{(nlms)}(n, D_e, r_e, \alpha)\left((\tau\aleph + \chi\omega)\, m + \\ + \frac{\Theta}{2}\begin{cases} l & \text{for up polarity } j = l + 1/2 \\ -(l+1) & \text{for down polarity } j = l - 1/2 \end{cases}\right) \quad (66)$$

and

$$E^{ps}_{nc}(n, D_e, r_e, \alpha, \Theta, \tau, \chi, j, \widetilde{l}, \widetilde{s}, \widetilde{m}) = E^{mk-ps}_{nk} + \\ + \langle \widetilde{Z}\rangle^{kf}_{(n\widetilde{l}\widetilde{m}\widetilde{s}t)}(n, D_e, r_e, \alpha)\left((\tau\aleph + \chi\omega)\, \widetilde{m} + \frac{\Theta}{2} \times \\ \times \begin{cases} \widetilde{l} & \text{for up polarity } j = \widetilde{l} + 1/2 \\ -(\widetilde{l}+1) & \text{for down polarity } j = \widetilde{l} - 1/2 \end{cases}\right), \quad (67)$$

where E^{mk-s}_{nk} and E^{mk-ps}_{nk} are determined from the energy equations for the Mie-type potential within Coulomb-like tensor interaction in the spin and p-spin symmetries in the Dirac theory obtained as follows:

$$\left(M + E^{mk-s}_{nk} - D_e\right)\Big[2n + 1 + \\ + \sqrt{(2k+1)^2 + 4\alpha(\alpha + 2k + 1) + 4\gamma_s D_e r_e^2}\,\Big]^2 \times \\ \times 4\gamma_s D_e^2 r_e^2 \quad (68)$$

and

$$(M + E^{mk-ps}_{nk} - D_e)\Big[2n + 1 + \\ + \sqrt{(2k-1)^2 + 4\alpha(\alpha + 2k - 1) + 4\gamma_p D_e r_e^2}\,\Big]^2 \times \\ \times 4\gamma_p D_e^2 r_e^2. \quad (69)$$

4.2.1. The improved Mie-type potential problems in NREQM symmetries

To realize a study of the nonrelativistic limit in extended nonrelativistic quantum mechanics symmetries with the improved Mie-type potential, two steps must be applied. The first step corresponds to the nonrelativistic limit for a usual nonrelativistic quantum energy. This is done by applying the following steps:

$$(\alpha, C_s) \to (0,0),\ E^{s}_{nk} + M \to 2\mu,\ E^{s}_{nk} - M \to E^{nr}_{nl}, \\ k \longrightarrow l \Rightarrow \gamma_s = M + E^{s}_{nk} - C_s \to \gamma^{nr}_{s} = 2\mu. \quad (70)$$

This allows us to obtain the nonrelativistic energy levels for the Mie-type potential in NRQM symmetries as:

$$E^{nr}_{nl} = C - \frac{2\mu B^2}{\left[2n + 1 + \sqrt{(2l+1)^2 + 8\mu A}\right]^2}. \quad (71)$$

Now, the second step corresponds to the reexport of relativistic expectation values $\langle Z\rangle^{mt}_{(nlms)}(n, A, B, C, \alpha)$ of the spin symmetry in Eq. (45) from the corresponding nonrelativistic expectation values $\langle Z\rangle^{nr-mt}_{(nlms)}(n, A, B, C, \alpha)$ as:

$$\langle Z\rangle^{mt}_{(nlms)}(n, A, B, C, \alpha) \to \langle Z\rangle^{nr-mt}_{(nlms)}(n, A, B, C, \alpha) = \\ = \left((l(l+1) + 2\mu A)\left\langle \frac{1}{r^4}\right\rangle^{s-mt}_{(nlms)} - \mu B\left\langle \frac{1}{r^3}\right\rangle^{s-mt}_{(nlms)}\right). \quad (72)$$

This allows us to express the nonrelativistic energy corrections $\Delta E^{mt}_{nc-nr}(n, A, B, C, \alpha, \Theta, \tau, \chi, j, l, s, m)$ produced for the improved Mie-type potential problems as

$$\Delta E^{mt}_{nc-nr}(n, A, B, C, \alpha, \Theta, \tau, \chi, j, l, s, m) = \\ \langle Z\rangle^{nr-mt}_{(nlms)}(n, A, B, C, \alpha)\left((\tau\aleph + \chi\omega)\, m + \\ + \frac{\Theta}{2}\begin{cases} l & \text{for up polarity } j = l + 1/2 \\ -(l+1) & \text{for down polarity } j = l - 1/2 \end{cases}\right). \quad (73)$$

The global NR energy

$$E^{mt}_{nc-nr}(n, A, B, C, \alpha, \Theta, \tau, \chi, j, l, s, m)$$

produced with the improved Mie-type potential in ENRQM symmetries as a result of the topological properties of a deformation space-space is the sum of usual energy E^{mt}_{nl} in Eq. (71) under Mie-type potential in NRQM symmetries and the obtained correction $\Delta E^{mt}_{nc-nr}(n, A, B, C, \alpha, \Theta, \tau, \chi, j, l, s, m)$ in Eq. (73) as follows:

$$E^{mt}_{nc-nr} = C - \frac{2\mu B^2}{\left[2n + 1 + \sqrt{(2l+1)^2 + 8\mu A}\right]^2} +$$

$$+\langle Z\rangle_{(nlms)}^{nr-mt}(n,A,B,C,\alpha)\left((\tau\aleph+\chi\omega)m+\right.$$
$$\left.+\frac{\Theta}{2}\begin{cases} l & \text{for up polarity } j=l+1/2\\ -(l+1) & \text{for down polarity } j=l-1/2\end{cases}\right). \quad (74)$$

It should be noted that the corrected energy ΔE_{nc-nr}^{mt} expressed in Eq. (74) is due to the effect of the perturbed potential $V_{nr-\mathrm{pert}}^{mt}(r)$:

$$V_{nr-\mathrm{pert}}^{mt}(r)=$$
$$=\left(l(l+1)r^{-4}-\frac{1}{2r}\frac{\partial V_{mt}(r)}{\partial r}\right)\mathbf{L}\Theta+O(\Theta^2). \quad (75)$$

The first term in Eq. (75) is due to the centrifuge term $l(l+1)\widehat{r}^{-2}$ in ENRQM symmetries which equals the usual centrifuge term $l(l+1)r^{-2}$ plus the perturbative centrifuge term $l(l+1)r^{-4}\mathbf{L}\Theta$, while the second term is produced due to the effect of the improved Mie-type potential. Using Eq. (3), i.e., $A=D_er_e^2$, $B=2D_er_e$, $C=0$, and Eq. (74), the new nonrelativistic energy eigenvalue with the improved Mie-type potential reduce to the new nonrelativistic energy eigenvalue E_{nc-nr}^{kf} for the improved Kratzer–Fues potential in ENRQM symmetries:

$$E_{nc-nr}^{kf}=-\frac{8\mu D_e^2r_e^2}{\left[2n+1+\sqrt{(2l+1)^2+8\mu D_er_e^2}\right]^2}+$$
$$+\langle Z\rangle_{(nlms)}^{nr-kf}(n,D_e,r_e,\alpha)\left((\tau\aleph+\chi\omega)m+\right.$$
$$\left.+\frac{\Theta}{2}\begin{cases} l & \text{Up polarity: } j=l+1/2\\ -(l+1) & \text{Down polarity: } j=l-1/2\end{cases}\right). \quad (76)$$

The first part of Eq. (76) is the nonrelativistic energy eigenvalue E_{nr}^{kf} for the Kratzer–Fues potential in NRQM symmetries, while the second part is due to the effect of deformation of the space-space for the Kratzer–Fues potential. Now, using Eq. (3), i.e., $A=D_er_e^2$, $B=2D_er_e$, $C=D_e$, and Eq. (74), we get the new nonrelativistic energy eigenvalue with the improved Mie-type potential which is reduced to a new nonrelativistic energy eigenvalue $E_{\mathrm{nc-nr}}^{kf}$ for the improved modified Kratzer potential in ENRQM symmetries:

$$E_{nc-nr}^{kf}=D_e-\frac{8\mu D_e^2r_e^2}{\left[2n+1+\sqrt{(2l+1)^2+8\mu D_er_e^2}\right]^2}+$$
$$+\langle Z\rangle_{(nlms)}^{nr-mk}(n,D_e,r_e,\alpha)\left((\tau\aleph+\chi\omega)m+\right.$$
$$\left.+\frac{\Theta}{2}\begin{cases} l & \text{Up polarity: } j=l+1/2\\ -(l+1) & \text{Down polarity: } j=l-1/2\end{cases}\right). \quad (77)$$

The first part of Eq. (77) is the nonrelativistic energy eigenvalue E_{nr}^{mk} for the modified Kratzer potential consistent with the energy in Refs. [5,83,84] under NRQM symmetries, while the second part is due to the effect of deformation of the space-space for the modified Kratzer potential.

4.3. Study of composite systems

Now, considering composite systems such as molecules made of $N=2$ particles with masses m_n $(n=1,2)$ in the frame of a noncommutative algebra, it is worth to consider the descriptions of systems in NRQM symmetries. It was obtained that composite systems with different masses are described with different noncommutative parameters [31, 33, 44]:

$$\left[\overset{\wedge(s,h,i)}{x_\alpha}\overset{*}{,}\overset{\wedge(s,h,i)}{x_\beta}\right]=i\eta_{\alpha\beta}^c. \quad (78)$$

The noncommutativity parameters $\eta_{\alpha\beta}^c$ and α_n are equal to $\sum_{n=1}^2\alpha_n^2\eta_{\alpha\beta}^{(n)}$ and $\frac{m_n}{\sum_n}m_n$, respectively. The indices $n=1$, 2 label the particle, and $\eta_{\alpha\beta}^{(n)}$ is the parameter of noncommutativity, corresponding to the particle with mass m_n. Note that, in the case of a system of two particles with the same mass $m_1=m_2$ such as the homogeneous chlorine (Cl_2) and nitrogen (N_2) diatomic molecules, the parameter $\eta_{\alpha\beta}^{(n)}=\theta_{\alpha\beta}$. Thus, the three parameters (Θ,σ,χ) which appear in Eq. (78) are changed to the new form:

$$\Lambda^{c2}=\left(\sum_{n=1}^2\alpha_n^2\Lambda_{12}^{(n)}\right)^2+\left(\sum_{n=1}^2\alpha_n^2\Lambda_{23}^{(n)}\right)^2+$$
$$+\left(\sum_{n=1}^2\alpha_n^2\Lambda_{13}^{(n)}\right)^2, \quad (79)$$

with Λ^{c2} can take all three roles $(\Theta^{c2},\tau^{c2},\chi^{c2})$. As was mentioned above, in the case of a system of two particles with the same mass $m_1=m_2$ such as the homogeneous Cl_2 and N_2 diatomic molecules, $\Theta_{\alpha\beta}^{(n)}=\Theta_{\alpha\beta}$, $\sigma_{\alpha\beta}^{(n)}=\sigma_{\alpha\beta}$ and $\chi_{\alpha\beta}^{(n)}=\chi_{\alpha\beta}$. Finally, we can generalize the nonrelativistic global energy $E_{r-nc}^{mt}(n,A,B,C,\alpha,\Theta,\tau,\chi,j,l,s,m)$ for the modified Morse potential considering that composite systems with different masses are described with different noncommutative parameters for the diatomic (CO, NO, and CH) molecule as:

$$E_{\mathrm{r-nc}}^{mhp}(n,A,B,C,\alpha,\Theta,\tau,\chi,j,l,s,m) = \\ = C - \frac{2\mu B^2}{\left[2n+1+\sqrt{(2l+1)^2+8\mu A}\right]^2} + \\ + \langle Z\rangle_{(nlms)}^{nr-mt}(n,A,B,C,\alpha)\,\Theta^c[j(j+1) - \\ - l(l+1) - s(s+1)]/2 + \tau^c \aleph m + \chi^c \omega m). \quad (80)$$

The Schrödinger equation, as the most well-known nonrelativistic wave equation describing the state of a low-energy particle, describes the energy regardless of its spin value, but its extension in ENRQM symmetries for the improved Mie-type potential model has a physical behavior similar to the Dirac equation for fermionic particles with spin-1/2. It can describe a dynamic state of a particle with spin-1/2. This is one of the most important new results of this research. Worthwhile it is better to mention that, for the three simultaneous limits (Θ,σ,χ) and $(\Theta^c,\sigma^c,\chi^c) \to (0,0,0)$, we recover the equations of energy for the spin symmetry and the p-spin symmetry in Refs. [3, 4].

5. Summary and Conclusions

This work presents an approximate analytic solution of the 3-dimensional deformed Dirac equation with the improved Mie-type potential within the improved Coulomb-like tensor interaction under the pseudospin- and spin-symmetry limits with an arbitrary spin-orbit coupling quantum number k. We have obtained the new approximate bound-state energies that appeared sensitive to the quantum numbers $(j,k,l,\widetilde{l},s,\widetilde{s},m,\widetilde{m})$, the potential depths (A,B,C,α), and noncommutativity parameters (Θ,σ,χ) under the condition of spin and pseudospin symmetries. Finally, as we know, we derived some specific potentials useful for other physical systems such as the improved Kratzer–Fues potential within an improved Coulomb-like tensor interaction and the improved modified Kratzer potential within the Coulomb-like tensor interaction. We have ended our research with the treatment of the nonrelativistic limit of the improved Mie-type potential in ENRQM symmetries. It is worth mentioning that, in all cases, by making the three simultaneous limits $(\Theta,\sigma,\chi) \to (0,0,0)$, we recover the ordinary physical quantities as in Refs. [3, 4]. Finally, the feature of a noncommutative geometry on the 3-dimensional deformed Dirac equation with the improved Mie-type potential within the improved Coulomb-like tensor interaction would be present in many physical problems such as spin-orbit and pseudospin-orbit couplings, modified Zeeman effect, and other ones and would cause the behavior of topological properties of the deformed space-space. Our findings in this work could be used in condensed matter physics, atomic physics, and chemical physics.

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ДЕФОРМОВАНІ РІВНЯННЯ ДІРАКА І ШРЬОДІНГЕРА З МОДИФІКОВАНИМ ПОТЕНЦІАЛОМ МІ-ТИПУ ДЛЯ ДВОАТОМНИХ МОЛЕКУЛ ТА ФЕРМІ-ЧАСТИНОК З УРАХУВАННЯМ СИМЕТРІЙ УЗАГАЛЬНЕНОЇ КВАНТОВОЇ МЕХАНІКИ

Для зв'язаних станів знайдено розв'язки деформованого рівняння Дірака з модифікованим потенціалом Мі-типу, що містить модифіковану тензорну взаємодію кулонівського типу за умов спінової або псевдоспінової симетрії та симетрій узагальненої релятивістської квантової механіки. В цьому потенціалі є доданки, пропорційні $1/r^3$ та $1/r^4$, які пов'язані із взаємозв'язками ($\mathbf{L\Theta}$ та $\widetilde{\mathbf{L}}\Theta$) між фізичними властивостями системи з топологічними деформаціями простір-простору. Використовуючи параметричний метод зсуву Боппа та теорію збурень, ми знаходимо нові релятивістичні і нерелятивістичні власні значення енергії для модифікованого потенціалу Мі-типу. Виявилось, що нові власні значення є чутливими до квантових чисел $(j, k, l, \widetilde{l}, s, \widetilde{s}, m, \widetilde{m})$, глибин змішаного потенціалу (A, B, C, α) та параметрів некомутативності (Θ, σ, χ). В окремих випадках отримано нові спектри енергії з модифікованими потенціалами Кратцера–Фьюса і Кратцера для модифікованої кулонівського типу тензорної взаємодії. Ми відтворили відомі результати, використовуючи одночасно три границі $(\Theta, \sigma, \chi) \rightarrow (0, 0, 0)$. Відмітимо, що наші результати є близькими до результатів, отриманих іншими авторами.

Ключові слова: рівняння Дірака, рівняння Шрьодінгера, потенціал Мі-типу, некомутативна квантова механіка, зірковий добуток.