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INVERSE SQUARE POTENTIAL IN A SPACE WITH SPIN NONCOMMUTATIVITY OF COORDINATES

An attractive inverse square potential has been considered in a space with the spin noncommutativity of coordinates. The corresponding effective potential energy, as well as the total energy, is shown to be bounded from below. Using the variational method, the upper limit of the ground-state energy, which turns out to be negative for a sufficiently large coupling constant, is found. As a result, it is proved that the inverse square potential creates stationary levels in the space concerned, unlike the case of commutative space, where a particle falls to the center.

Keywords: inverse square potential, noncommutativity.

1. Introduction

The concept of noncommutative coordinates has been actively developing since the corresponding researches in the string theory, where it was shown that the coordinates on a D-brane in a magnetic field do not commute [1]. Furthermore, the coordinate noncommutativity also appears at the compactifaction of some variants of the M-theory [2]. In those problems, there emerges a coordinate commutator that looks like

$$[X_i, X_j] = i\theta_{ij}, \tag{1}$$

where θ_{ij} is an antisymmetric constant matrix.

It is of interest that a concept of minimum length, which arises at the quantum consideration of a gravitational field [3], can be included into commutative theories with the help of noncommutativity. One of the largest problems with relation (1) is its noninvariant character with respect to rotations. Some ways to construct noncommutative rotation-invariant algebras are known [3–6]. Another approach includes the algebras with spin noncommutativity of coordinates, where spatial coordinates are mixed up with spin operators.

For instance, in work [7], noncommutative coordinates were proposed that are formed by adding the spin operators $X_i = x_i + \theta s_i$ to commutative coordinates. The corresponding algebra reads

$$\begin{aligned} &[X_i, X_j] = i\theta^2\varepsilon_{ijk}s^k, \quad [X_i, P_j] = i\hbar\delta_{ij}, \\ &[X_i, s_j] = i\theta\varepsilon_{ijk}s^k, \quad [s_i, s_j] = i\hbar\varepsilon_{ijk}s^k, \\ &[P_i, P_j] = 0, \quad [P_i, s_j] = 0. \end{aligned} \tag{2}$$

It can be generalized onto the relativistic case by shifting the coordinates by the Dirac matrices: $X^\mu = x^\mu + i\theta\gamma^\mu$ [8]. It is easy to see that the coordinates introduced in such a way are Lorentz-invariant.

Another type of spin noncommutativity can be obtained by adding the Pauli–Lubanski vector to the coordinates: $X^\mu = x^\mu + \theta W^\mu$, where $W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}S_{\nu\rho}p_\sigma$ and $S_{\nu\rho} = \frac{i}{4}[\gamma_\nu, \gamma_\rho]$ [9, 10]. This algebra is also invariant (Lorentz-invariant) with respect to rotations and possesses the minimum length.

The inverse square potential was studied in the literature from various viewpoints. The corresponding interest is caused, on the one hand, by the realization of this potential in various systems, in particular, the Efimov effect [11], neutral atoms in the field of a charged wire [12–14], a magnetic moment in the field of a thin solenoid [15], a substance near the black hole horizon [16–19], an electron in the field of a molecule-dipole [20–23], and so forth. On the other hand, the inverse square potential allows a particle to fall to the attraction center [24]. In the quantum-mechanical case, it can be demonstrated that the mean $\langle r^2\rangle$ evolves following the law

$$\langle r^2\rangle = \langle r^2\rangle_0 + \frac{\langle \mathbf{rp}+\mathbf{pr}\rangle}{m}t + \frac{2\langle H\rangle}{m}t^2, \tag{3}$$

so that if $\langle H\rangle < 0$, the particle will fall to the center, i.e. $\langle r^2\rangle_{t_f} = 0$, after a certain finite time interval [25].

In a space with the generalized uncertainty principle and noncommutative coordinates, the potential $-\gamma/R^2$ is regularized, and there arise stationary levels instead of the falling [26]. Bound states also arise in the space with the minimum length [27].

In this work, we consider the influence of noncommutativity (2) on the behavior of a particle located in an inverse square potential. This noncommutativity effectively appears in systems with a strong dipole-dipole interaction, e.g., the Bose condensate of ^{52}Cr [7]. In addition, algebra (2) makes it possible to explain the triplet Cooper pairing mechanism [28]. Noncommutativity (2) is responsible for the anisotropy of the Aharonov–Bohm effect [29] and eliminates the degeneration of hydrogen atom levels in the orbital quantum number [30].

The structure of the present work is as follows. In Section 2, the examined algebra is shown to really possess the minimum length, and an expression for the Hamiltonian of a particle in an inverse square potential in a space with spin noncommutativity is found. In Section 3, the effective potential energy and the particle total energy in this potential are demonstrated to be bounded from below. In Section 4, the variational method is used to determine the upper limit for the ground state 4. Conclusions can be found at the end of the paper.

2. Inverse Square Potential in Noncommutative Space

Let us consider an inverse square potential in a noncommutative space with algebra (2). We intend to demonstrate that there is a minimum length in this space. Really, let us determine the eigenvalues of the operator

$$\widehat{R}^2 = r^2 + \hbar\theta(\mathbf{r},\boldsymbol{\sigma}) + \frac{3}{4}(\hbar\theta)^2. \tag{4}$$

Depending on the spin direction, they equal

$$R^2_\pm = \left(r \pm \frac{\hbar\theta}{2}\right)^2 + \frac{(\hbar\theta)^2}{2}. \tag{5}$$

The minimum eigenvalue is obtained at $r = \hbar\theta/2$ and the sign "$-$", so that $\lambda^2_{\min} = (\hbar\theta)^2/2$. Since the mean value of any operator in any state cannot be less than its minimum eigenvalue, in particular,

$$\langle R^2\rangle \geq \lambda^2_{\min} = \frac{(\hbar\theta)^2}{2}, \tag{6}$$

we obtain that it is impossible to create a state with the localization in a region with linear sizes smaller than $\lambda_{\min} = \hbar\theta/\sqrt{2}$; therefore, the quantity $\lambda_{\min}$ is a minimum length in the space with algebra (2).

It is intuitively clear that a fall to a point-like center in a space with a nonzero minimum length is impossible. Let us prove this rigorously. Let us postulate that the problem is formulated in a noncommutative space by substituting noncommutative coordinates X instead of commutative ones x in the commutative-problem Hamiltonian $H(x,p)$. For instance, for the inverse square potential in a space with algebra (2), we have

$$H = \frac{\mathbf{p}^2}{2m} - \frac{\gamma}{R^2} = \frac{p_r^2}{2m} + \frac{\widehat{L}^2}{2mr^2} - \frac{\gamma}{r^2+\theta r(\mathbf{n},\boldsymbol{\sigma})+3\theta^2/4}, \tag{7}$$

where $p_r = -\frac{i\hbar}{r}\partial_r r$ is the radial part of the momentum operator, $\widehat{L}$ the angular momentum, and $\mathbf{n} = \mathbf{r}/r$.

For further calculations, it is convenient to introduce the dimensionless coordinates, $r = \theta x/2$, and Hamiltonian,

$$h = \frac{H}{H_0} = -\frac{1}{x}\frac{\partial^2}{\partial x^2}x + \frac{\widehat{l}^2}{x^2} - \frac{\widetilde{\gamma}}{x^2+2x(\mathbf{n},\boldsymbol{\sigma})+3}, \tag{8}$$

where $\widehat{l}^2 = \widehat{L}^2/\hbar^2$, $\widetilde{\gamma} = \gamma/\frac{\hbar^2}{2m}$, and $H_0 = \frac{2\hbar^2}{m\theta^2}$. Hence, the noncommutativity regularizes the potential. Therefore, the energy should be bounded from below. Really, since algebra (2) possesses the minimum length (6) and the kinetic energy is positive define, we obtain

$$\langle h\rangle \geq -\left\langle\frac{\widetilde{\gamma}}{X^2}\right\rangle \geq -\frac{\widetilde{\gamma}}{\lambda_{\min}^2} = -\frac{\widetilde{\gamma}}{2\theta^2}. \quad (9)$$

In the next section, we will improve the lower energy estimation by making allowance for the centrifugal term in the kinetic energy. In addition, we will demonstrate that Hamiltonian (7) really possesses bound states. For this purpose, it is enough to show that the ground state of this Hamiltonian has a finite negative energy.

3. Lower Estimate of the Ground-State Energy

Let us improve the lower estimate (9) of the ground-state energy by considering the centrifugal term in the kinetic energy. For this purpose, it is convenient to multiply the potential energy operator by the conjugate denominator and rewrite Hamiltonian (8) in the form

$$h = -\frac{1}{x}\frac{\partial^2}{\partial x^2}x + \frac{\widehat{l}^2}{x^2} - \widetilde{\gamma}\frac{x^2+3-2x(\mathbf{n},\boldsymbol{\sigma})}{x^4+2x^2+9}. \quad (10)$$

In this Hamiltonian, the spin and angular variables are separated, and the spin-angular equation is solved exactly in the form of a linear combination of spherical spinors $\Omega_{j,l,m}(\theta,\varphi)$ [31]:

$$\psi = R_+\Omega_{j,j+1/2,m} + R_-\Omega_{j,j+1/2,m} = \begin{pmatrix} R_+ \\ R_- \end{pmatrix}. \quad (11)$$

Taking into consideration that spherical spinors are eigenfunctions of the squared angular momentum operator, we have

$$\widehat{l}^2\Omega_{j,l,m} = l(l+1)\Omega_{j,l,m}, \quad (12)$$

and the action of the operator $(\mathbf{n},\boldsymbol{\sigma})$ on spherical spinors looks like

$$(\mathbf{n},\boldsymbol{\sigma})\Omega_{j,j\pm1/2,m} = \mp i\Omega_{j,j\mp1/2,m}. \quad (13)$$

Let us seek the effective potential energy $\widetilde{U}(x)$ for the radial Schrödinger equation in the spherical spinor representation in the matrix form:

$$\widetilde{U}(x) = \begin{pmatrix} I_{j+1/2}-V_1 & -iV_2 \\ iV_2 & I_{j-1/2}-V_1 \end{pmatrix}, \quad (14)$$

where $V_1 = -\widetilde{\gamma}\frac{x^2+3}{x^4+2x^2+9}$, $V_2 = -\widetilde{\gamma}\frac{2x}{x^4+2x^2+9}$, and $I_l = l(l+1)/x^2$.

Since the kinetic energy operator is positively defined, it is evident that

$$\varepsilon \geq \langle\psi|\widetilde{U}|\psi\rangle \geq U_0, \quad (15)$$

where U_0 is the minimum eigenvalue of the effective potential energy operator (14). Let us find the minimum eigenvalue $U(x)$ for matrix (14). It is reached at $j = 1/2$ and is equal to

$$U(x) = V_1 + \frac{1}{x^2} - \sqrt{\frac{1}{x^4}+V_2^2}. \quad (16)$$

Minimizing expression (16) as a function of x, we obtain $U_0 = \min U(x)$, which, according to Eq. (15), is the lower estimate of the particle energy:

$$\varepsilon_{\text{low}}(\widetilde{\gamma}) = U_0. \quad (17)$$

The corresponding calculations give the following values for this quantity: $\varepsilon_{\text{low}}(\widetilde{\gamma}=1) \approx -0.3482$, $\varepsilon_{\text{low}}(\widetilde{\gamma}=10) \approx -4.3475$, and $\varepsilon_{\text{low}}(\widetilde{\gamma}=100) \approx -49.0618$. The dependence $\varepsilon_{\text{low}}(\widetilde{\gamma})$ is plotted in Figure. Note that, for real experiments with the potential $-\gamma/r^2$, the value of $\widetilde{\gamma}$ has an order of 100 [14].

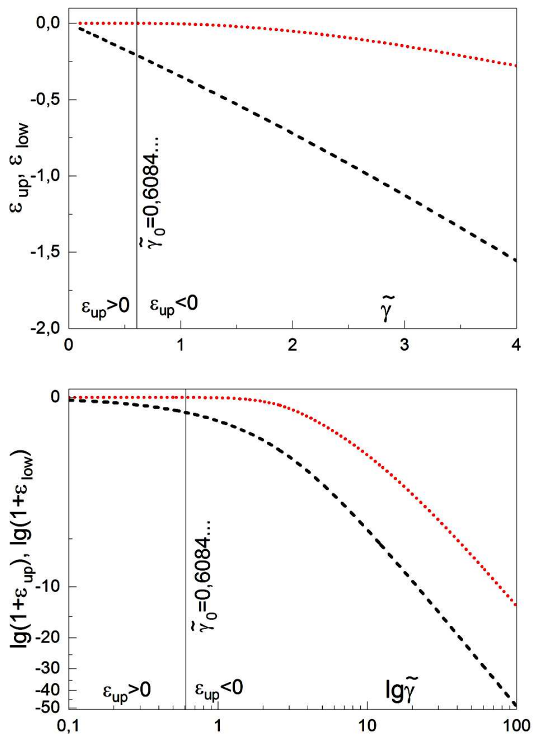
4. Upper Estimate of the Ground-State Energy

The boundedness of the potential energy from below does not yet guarantee the presence of bound states, because the potential well may turn out too shallow or narrow for the bound levels to be formed. In order to demonstrate that they do exist, let us apply the variational method to show that the ground-state energy is really negative.

Let us choose the trial wave function in the form

$$\psi(x,\theta,\varphi) = Ce^{-\alpha x/2}\left(x\Omega_{1/2,1,0} + i\frac{\beta}{\alpha}\Omega_{1/2,0,0}\right), \quad (18)$$

where α and β are dimensionless variational parameters, and C the normalization constant. The linear combination of spherical spinors in Eq. (18) is selected as in the exact solution of the angular Schrödinger equation with Hamiltonian (10). The exponential factor in Eq. (18) provides a correct asymptotics for the radial wave function at infinity, and the corresponding power exponent x provides a correct asymptotics at the center.



Dependences of the lower ($\varepsilon_{\rm low}$, dashed curve) and upper ($\varepsilon_{\rm up}$, dotted curve) estimates for the ground-state energy on the coupling constant $\widetilde{\gamma}$. The vertical line at $\widetilde{\gamma}_0 \approx 0.6084$ corresponds to the threshold value of the coupling constant: the upper estimate of the ground-state energy is positive at $\widetilde{\gamma} < \widetilde{\gamma}_0$ and negative at $\widetilde{\gamma} > \widetilde{\gamma}_0$. A true value of the ground-state energy is located between the upper and lower curves

From the normalization condition

$$\int\limits_0^\infty dx \int\limits_0^\pi d\theta \int\limits_0^{2\pi} d\varphi\, x^2 \sin\theta |\psi|^2 = 1$$

with regard for the orthogonality of spherical spinors,

$$\int\limits_0^\pi d\theta \int\limits_0^{2\pi} d\varphi \, \sin\theta \, \Omega^\dagger_{jlm} \Omega_{j'l'm'} = \delta_{jj'}\delta_{ll'}\delta_{mm'},$$

we obtain the normalization constant $|C|^2 = \alpha^5/ /(24 + 2\beta^2)$.

Taking Eq. (12) into consideration, the mean kinetic energy for the wave function (18) can be easily calculated:

$$\left\langle -\frac{1}{x}\frac{\partial^2}{\partial x^2}x + \frac{\widehat{l}^2}{x^2} \right\rangle = \frac{\alpha^2}{4}. \qquad (19)$$

At the same time, it is difficult to calculate the mean potential energy analytically. Therefore, let us make some transformations in the potential. For this purpose, let us multiply both the numerator and denominator of the fraction by the conjugate denominator:

$$-\frac{\widetilde{\gamma}}{x^2 + 2x(\mathbf{n},\boldsymbol{\sigma}) + 3} = -\widetilde{\gamma}\frac{x^2 + 3 - 2x(\mathbf{n},\boldsymbol{\sigma})}{(x^2+3)^2 - 4x^2}. \qquad (20)$$

Since we are interested in the upper estimate for the ground-state energy, the potential of the system can be substituted by a potential that is larger at every point x:

$$-\widetilde{\gamma}\frac{x^2 + 3 - 2x(\mathbf{n},\boldsymbol{\sigma})}{(x^2+3)^2 - 4x^2} \le -\widetilde{\gamma}\frac{x^2 + 3 - 2x(\mathbf{n},\boldsymbol{\sigma})}{(x^2+3)^2}. \qquad (21)$$

The integrals that arise when calculating $\langle U \rangle$ can be found analytically:

$$\int\limits_0^\infty dx \frac{x^n e^{-\alpha x}}{(x^2+b)^{m+1}} = \frac{(-1)^{m+n}}{m!}\frac{d^n}{d\alpha^n}\frac{d^m}{db^m}\int\limits_0^\infty dx \frac{e^{-\alpha x}}{x^2+b}.$$

The integral on the right-hand side equals [32]

$$\int\limits_0^\infty dx \frac{e^{-\alpha x}}{x^2+b} = \frac{1}{\sqrt{b}} f(\sqrt{b}\alpha), \qquad (22)$$

where $f(x) = \operatorname{ci} x \sin x - \operatorname{si} x \cos x$, and $\operatorname{ci} x$ and $\operatorname{si} x$ are the integral cosine and sine, respectively.

Hence, in view of Eq. (13) and the equality

$$\frac{d^2}{d\alpha^2} f(\sqrt{b}\alpha) = \frac{\sqrt{b}}{\alpha} - b f(\sqrt{b}\alpha), \qquad (23)$$

we obtain the upper estimate for the mean potential energy,

$$\langle U \rangle \le -\frac{\widetilde{\gamma}}{2}\frac{a\beta^2 + b\beta + c}{12 + \beta^2}, \qquad (24)$$

where $a = 1 - \sqrt{3}\alpha f$, $b = 4a - 4\sqrt{3}\alpha^2 f - 6a^3 f'$, $c = 2 - 3\alpha + 3\sqrt{3}\alpha^3 f$, $f = f(\sqrt{3}\alpha)$, and $f' = = d/d\alpha f(\sqrt{3}\alpha)$. For the mean value for Hamiltonian (8), we obtain

$$\langle h \rangle \le E(\alpha,\beta) = \frac{\alpha^2}{4} - \frac{\widetilde{\gamma}}{2}\frac{a\beta^2 + b\beta + c}{12 + \beta^2}. \qquad (25)$$

Its minimization with respect to the parameter β gives

$$\beta = \frac{12a - c}{b} \pm \sqrt{\left(\frac{12a - c}{b}\right)^2 + 12}. \qquad (26)$$

In order to minimize Eq. (25) with respect to the parameter α, we have to solve a complicated transcendental equation, which cannot be done analytically. Numerical calculations give the following estimates for the ground-state energy: $\varepsilon_{\text{up}}(\tilde{\gamma} = 1) \approx \approx -0.0037$, $\varepsilon_{\text{up}}(\tilde{\gamma} = 10) \approx -1.0753$, and $\varepsilon_{\text{up}}(\tilde{\gamma} = = 100) \approx -13.0336$. The dependence $\varepsilon_{\text{up}}(\tilde{\gamma})$ is exhibited in Figure. A true value of the ground-state energy is located between the upper and lower plots.

There is a threshold value, $\tilde{\gamma}_0 \approx 0.6084$, below which the variational method gives positive values for the ground-state energy, so one cannot talk about the presence of bound states. However, if $\tilde{\gamma} > \tilde{\gamma}_0$, the upper estimate of the ground-state energy is negative, which proves the existence of bound states for the inverse square potential in the space with spin noncommutativity.

5. Conclusions

In this work, the attractive inverse square potential $-\gamma/R^2$ in a space with the spin noncommutativity of coordinates [Eq. (2)] is considered. The minimum length for this algebra is calculated: $\lambda_{\min} = = \hbar\theta/\sqrt{2}$. The effective potential energy (14) for the radial motion of a particle in the ground state in the inverse square potential is determined. The corresponding value is found to be bounded from below, which testifies that the ground-state energy is also bounded from below. Using the variational method, it is shown that, if the coupling constant $\gamma/\frac{\hbar^2}{2m} \gtrapprox \gtrapprox 0.6084$, the corresponding estimated value for the energy is negative. Hence, it is proved that if the coupling constant in a noncommutative space is large enough, the minimum energy of a particle in the inverse square potential has a finite negative value. As a result, bound states can be formed, unlike the commutative case, when the particle falls to the center.

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ОБЕРНЕНО КВАДРАТИЧНИЙ
ПОТЕНЦІАЛ У ПРОСТОРІ ЗІ СПІНОВОЮ
НЕКОМУТАТИВНІСТЮ КООРДИНАТ

Р е з ю м е

Розглянуто притягальний обернено квадратичний потенціал у просторі зі спіновою некомутативністю координат. Показано, що для такого потенціалу ефективна потенціальна, а отже і повна, енергія обмежена знизу. За допомогою варіаційного методу знайдено верхню границю енергії основного стану, яка для достатньо великих констант зв'язку є від'ємною. Таким чином доведено, що замість падіння на притягальний центр в комутативному просторі, в обернено квадратичному потенціалі в просторі зі спіновою некомутативністю утворюються стаціонарні рівні.