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CALCULATION OF JOSEPHSON CURRENT IN A TWO-BARRIER TUNNEL JUNCTION

The Josephson current through a two-barrier SISIS tunnel junction has been calculated analytically in the quasiclassical approximation for the microscopic theory of superconductivity. Green's functions for a SISIS tunnel junction and an expression for the Josephson current through a point contact are obtained. The dependence of the tunnel current on the order parameter phase is determined, and the current dependence on the distance between the barriers is analyzed. The presence of resonance peaks in the Josephson current is demonstrated.

Keywords: Josephson effect, tunnel junction, critical current.

1. Introduction

Among a lot of important and interesting phenomena that were discovered in the last century, superconductivity occupies a special place. Superconductors attract considerable interest of scientists. There are plenty of reasons for that. One of them consists in the unordinary, from the classical standpoint, character of effects that arise in such systems. These are the dissipativeless behavior of the electric current, the expulsion of the magnetic field from the superconductor bulk, the magnetic flux quantization, and so forth [1, 2]. Another important circumstance that makes the researches of superconductors challenging is connected with the fact that such systems are a "window to the quantum world", because quantum-mechanical effects manifest themselves on the macroscopic level in this case.

Two Josephson effects are inherent in the physics of superconductivity. They are observed in a system composed of two superconductors (S) separated by the thin layer of an insulator (I) (the so-called SIS junction). One of those effects (stationary) consists in that if a current, whose magnitude is lower than a certain critical value, is passed through the junction, the voltage drop across the junction equals zero despite the presence of the insulator layer. The Josephson effects (stationary and non-stationary ones) are classed to the so-called weak-superconductivity ones [3].

In the recent years, there emerged technologies for the creation of multilayered tunnel junctions with an arbitrary geometry and a set of components: SINIS, SISIS, SIS′IS, and others [4]. The theoretical calculation of the effects of phase-coherent charge transfer in layered systems of the SISIS and SIS′IS types, whose research is challenging because of the creation of superconducting quantum interference devices (SQUIDs) [5] and superconducting qubits for a quantum computer [6, 7], has a large field of applications.

Multilayered superconductor junctions were studied in a number of works. The experimental regularities of a Josephson current in such structures are described in works [4, 8]. In the works by Brinkman and Kupriyanov *et al.* [9, 10], a theoretical analysis of the Josephson effect in the SISIS junction was carried out. Using the method of Green's temperature functions, the cited authors obtained an expression for the Josephson current in an integral form and analyzed it by numerical methods.

In this work, we propose a microscopic theory for the Josephson current in a two-barrier tunnel junction with the SISIS structure. Unlike work [10], where Green's functions for the Gor'kov equation were applied, we deal with the equations of superconductivity theory that have a lower order as differential equations, i.e. quasiclassical ones. This circumstance allows us to obtain an expression for the Josephson current through a junction in the analytical form. Note that, with the use of the method of quasiclassical equations, one of the authors (A.V.S.) theoretically described the Josephson effect in superconductor junctions with various structures: SIS, SNS, and SINS [1].

The researches performed in this work revealed some features in the tunnel current through a two-barrier junction, which are absent for the current through a SIS junction. First of all, this is a non-monotonic dependence of the critical current on the distance between the barriers and the presence of resonance maxima. A possibility of the resonance tunneling in a two-barrier structure was considered in work [11].

Another peculiarity of the Josephson effect in the SISIS tunnel junction is a non-sinusoidal dependence of the current on the phase difference. A similar dependence was obtained in works [12, 13] on the basis of the classical Ohta model [14]. The experimental observation of a non-sinusoidal dependence of the current through a two-barrier superconductor junction was described in works [15, 16].

2. Quasiclassical Equations of Superconductivity Theory for the Description of Current States

It is known that the theoretical description of superconductors in the mean-field approximation is based on the Bogolyubov equations. The latter are equations for the eigenvalues and eigenvectors of the coefficients in the canonical Bogolyubov transformations from particles to quasiparticles, $u_{\mathbf{p}}(\mathbf{r})$ and $v_{\mathbf{p}}(\mathbf{r})$:

$$\begin{cases}\hat{\xi}u_{\mathbf{p}}(\mathbf{r})-\Delta(\mathbf{r})v_{\mathbf{p}}(\mathbf{r})=\varepsilon_{\mathbf{p}}u_{\mathbf{p}}(\mathbf{r}),\\ \hat{\xi}v_{\mathbf{p}}(\mathbf{r})+\Delta^*(\mathbf{r})u_{\mathbf{p}}(\mathbf{r})=-\varepsilon_{\mathbf{p}}v_{\mathbf{p}}(\mathbf{r}).\end{cases} \tag{1}$$

Here, $\Delta(\mathbf{r})$ is the order parameter (mean field), $\hat{\xi} = \frac{\hat{\mathbf{p}}^2}{2m}+U(\mathbf{r})-\mu$, $U(\mathbf{r})$ is an external field, μ the chemical potential, and $\varepsilon_{\mathbf{p}}=\sqrt{\xi_{\mathbf{p}}^2+|\Delta|^2}$.

The Bogolyubov equations (1) have to be solved under the self-consistency condition

$$\Delta(\mathbf{r})=g\sum_{\mathbf{p}}u_{\mathbf{p}}(\mathbf{r})v^*_{\mathbf{p}}(\mathbf{r})\tanh\frac{\varepsilon_{\mathbf{p}}}{2T}.$$

From the system of Bogolyubov equations (1), we can change to a system of equations for Matsubara Green's functions, which are called the Gor'kov equations:

$$\begin{cases}(i\omega_n-\hat{\xi})G_{\omega_n}(\mathbf{r},\mathbf{r}')+\Delta(\mathbf{r})F_{\omega_n}(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'),\\ (i\omega_n+\hat{\xi})F_{\omega_n}(\mathbf{r},\mathbf{r}')+\Delta^*(\mathbf{r})G_{\omega_n}(\mathbf{r},\mathbf{r}')=0.\end{cases}$$

The latter have to be solved provided the condition

$$\Delta^*(\mathbf{r})=|g|T\sum_{\omega_n}F_{\omega_n}(\mathbf{r},\mathbf{r}).$$

In the Nambu matrix formalism, this system of equations is written as follows:

$$\left(i\omega_n-\sigma_z\hat{\xi}-\hat{\Delta}(\mathbf{r})\right)\hat{G}_{\omega_n}(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r},\mathbf{r}'), \tag{2}$$

where

$$\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \hat{\Delta}(\mathbf{r})=\begin{pmatrix}0&\Delta(\mathbf{r})\\ \Delta^*(\mathbf{r})&0\end{pmatrix}.$$

From the mathematical viewpoint, the system of equations of superconductivity theory in the form of Bogolyubov's equations – or, equivalently, Gor'kov's equations – is rather difficult, because the sought quantities – these are the coefficients $u_{\mathbf{p}}(\mathbf{r})$ and $v_{\mathbf{p}}(\mathbf{r})$ or the functions $G_{\omega_n}(\mathbf{r},\mathbf{r}')$ and $F_{\omega_n}(\mathbf{r},\mathbf{r}')$ – are nonlinear functionals of the function $|\Delta(\mathbf{r})|$, which is spatially inhomogeneous in the general case. The results of researches [1] testify that the velocity of the Cooper pair motion as an ensemble, v_s, is much lower than the characteristic velocity of the electrons that form this pair, i.e. the Fermi velocity v_{F}. This fact means that, although electrons in a superconductor are strongly degenerate, the motion of Cooper pairs is quasiclassical. The account for this circumstance allows the equations of superconductivity theory to be substantially simplified. Namely, the quasiclassical equations have a lower order than the initial differential equations of superconductivity theory (written in spatial coordinates). Note that a similar simplification takes place, when changing from the Dirac equation for a relativistic electron to the Pauli equations in the nonrelativistic limit. The simplification

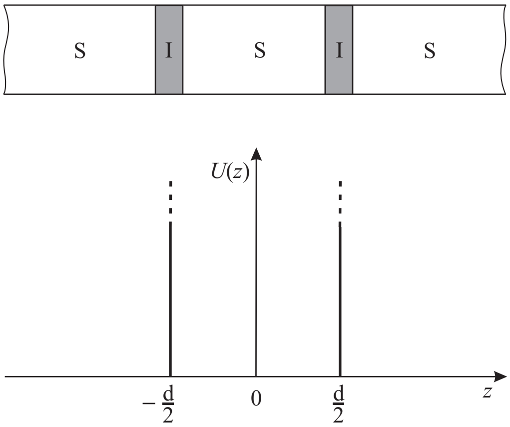

Fig. 1. Model of a symmetric SISIS tunnel junction and the corresponding potential

of the equations associated with the quasiclassical character of the Cooper pair motion has a spatial aspect, because $(\mathrm{v}_s)_{\mathrm{crit}}/\mathrm{v}_{\mathrm{F}} \sim T_{\mathrm{c}}/T_{\mathrm{F}} \sim a/\xi_0 \ll 1$, where T_{c} is the critical temperature, T_{F} the Fermi temperature, a the interatomic distance, and ξ_0 the coherence length. Hence, in a ceratin sense, the quasiclassical equations are smoothed out over the atomic lengths and contain only large-scale spatial variations of the order parameter. The principles used, while constructing the quasiclassical equations of superconductivity theory for tunnel junctions, are described in [1].

In this work, we will construct quasiclassical equations for a tunnel junction of the SISIS type. The junction geometry is shown in Fig. 1. Each insulator is simulated by a δ-like potential barrier

$$U(z) = \alpha \left[\delta\left(z - \frac{d}{2}\right) + \delta\left(z + \frac{d}{2}\right)\right]. \tag{3}$$

Let us expand the Matsubara Green's function (2) in a series of eigenstates of the one-particle Hamiltonian with potential (3):

$$\hat{G}_{\omega_n}(\mathbf{r},\mathbf{r}') = \sum_{i,k}\int d\mathbf{p}\int d\mathbf{p}'\,\hat{G}^{ik}_{\omega_n}(\mathbf{p},\mathbf{p}')\chi^{(i)}_{\mathbf{p}}(\mathbf{r})\chi^{(k)}_{\mathbf{p}'}(\mathbf{r}'), \tag{4}$$

where

$$\chi^{(1)}_{\mathbf{p}}(\mathbf{r}) = \frac{1}{2\pi}e^{i\mathbf{p}_\perp\mathbf{r}}\Psi^{(1)}_{p_z},\ \chi^{(2)}_{\mathbf{p}}(\mathbf{r}) = \frac{1}{2\pi}e^{i\mathbf{p}_\perp\mathbf{r}}\Psi^{(2)}_{p_z}.$$

Here, $p_\perp$ and p_z are the perpendicular and longitudinal, respectively, momentum components; and the functions $\Psi^{(1)}_{p_z}$ and $\Psi^{(2)}_{p_z}$ are solutions of the one-dimensional Schrödinger equation with potential (3). We have

$$\Psi^{(1)}_{p_z}(z) = \frac{1}{\sqrt{2\pi}}\left[\left\{e^{ip_z z} + C_1 e^{-ip_z z}\right\}\theta\left(-z - \frac{d}{2}\right) + \right.$$
$$+ \left\{C_2 e^{ip_z z} + C_3 e^{-ip_z z}\right\}\theta\left(z + \frac{d}{2}\right)\theta\left(-z + \frac{d}{2}\right) +$$
$$\left. + C_4 e^{ip_z z}\theta\left(z - \frac{d}{2}\right)\right] \tag{5}$$

for the wave incident from the left side and

$$\Psi^{(2)}_{p_z}(z) = \frac{1}{\sqrt{2\pi}}\left[C_4 e^{-ip_z z}\theta\left(-z - \frac{d}{2}\right) + \right.$$
$$+ \left\{C_2 e^{-ip_z z} + C_3 e^{ip_z z}\right\}\theta\left(-z + \frac{d}{2}\right)\theta\left(z + \frac{d}{2}\right) +$$
$$\left. + \left\{C_1 e^{ip_z z} + e^{-ip_z z}\right\}\theta\left(z - \frac{d}{2}\right)\right] \tag{6}$$

for the wave incident from the right-hand side. The constants C_1, C_2, C_3, and C_4 are determined from the conditions of the wave function continuity across the barriers and the given jumps of its first derivative.

The coefficient of electron transmission through the barrier is determined by the relation

$$D = \frac{8\kappa^4}{(8\kappa^4 + 4\kappa^2 + 1) + (4\kappa^2 - 1)\cos(2p_z d) + 4\kappa\sin(2p_z d)},$$

where

$$\kappa = \frac{p_z}{2m\alpha} = \frac{\sqrt{E}}{\sqrt{2m}\alpha}.$$

On the basis of the Gor'kov equation (2), we obtain the following integral equation for the coefficients of Green's function expansion (4):

$$(i\omega_n - \sigma_z\xi)\hat{G}^{ik}_{\omega_n}(\mathbf{p},\mathbf{p}') - \sum_j\int d\mathbf{p}''\langle i,\mathbf{p}|\hat{\Delta}(z)|j,\mathbf{p}''\rangle \times$$
$$\times\,\hat{G}^{jk}_{\omega_n}(\mathbf{p}'',\mathbf{p}') = \delta_{ik}\delta(\mathbf{p} - \mathbf{p}').$$

Let us take into consideration that the spatial homogeneity is broken only in the z axis direction, and the order parameter Δ depends only on the coordinate z. Therefore, the matrix elements $\langle i,\mathbf{p}|\hat{\Delta}(z)|i'\mathbf{p}''\rangle$ are diagonal in the transverse momenta, i.e. they contain the δ-function $\delta(\mathbf{p}_\perp - \mathbf{p}'_\perp)$. This is also valid for Green's function, which looks like

$$\hat{G}^{ik}_{\omega_n}(\mathbf{p},\mathbf{p}') = \hat{G}^{ik}_{\omega_n}(\mathbf{p}_\perp, p_z, p'_z)\delta(\mathbf{p}_\perp - \mathbf{p}'_\perp).$$

The indicated symmetry of the problem makes it possible to transform the equations for the coefficients of the Green's function expansion to the following form:

$$(i\omega_n-\sigma_z\xi)\hat{G}^{ik}_{\omega_n}(\mathbf{p}_\perp,p_z,p'_z)-\sum_j\int dp''_z\langle i,p_z|\hat{\Delta}(z)|j,p''_z\rangle\times$$
$$\times\hat{G}^{jk}_{\omega_n}(\mathbf{p}_\perp,p''_z,p'_z)=\delta_{ik}\delta(p-p'_z). \qquad (7)$$

The obtained form (7) of the equation for the Gor'kov Green's functions is convenient, when constructing the quasiclassical approximation. Let us take into account that the characteristic momenta that give a contribution to the values of physical quantities are close to the Fermi momentum p_F. Therefore, we may put $p=p_\mathrm{F}+\xi/\mathrm{v}_\mathrm{F}$, where ξ has the order of T_c. For the difference between the momentum projections, we obtain

$$p_z-p'_z\cong\frac{\xi-\xi'}{\mathrm{v}_\mathrm{F}x},\quad x\equiv\cos\theta,$$

where θ is the electron incidence angle at the barrier.

Hence, we have the following approximate relations:

$$\delta(p_z-p'_z)\cong\mathrm{v}_\mathrm{F}x\delta(\xi-\xi'),$$

$$\int\limits_0^\infty dp_z...\cong\frac{1}{\mathrm{v}_\mathrm{F}x}\int\limits_{-\infty}^\infty d\xi.$$

Taking them into account and changing from the variable ξ to the Fourier-conjugate variable t (whose dimensionality is the inverse temperature) with the use of the formulas

$$\langle\xi|t\rangle=\frac{1}{\sqrt{2\pi\mathrm{v}_\mathrm{F}x}}e^{-i\xi t},\ \langle t|\xi\rangle=\frac{1}{\sqrt{2\pi\mathrm{v}_\mathrm{F}x}}e^{i\xi t},$$

we arrive at a differential equation of a lower order for Green's functions (7):

$$\left(i\omega_n+i\sigma_z\frac{d}{dt}\right)\langle t,i|\hat{G}_{\omega_n}|k,t'\rangle-$$
$$-\sum_j\int dt''\langle t,i|\hat{\Delta}(z)|j,t''\rangle\langle t'',j|\hat{G}_{\omega_n}|k,t'\rangle=$$
$$=\delta_{ik}\delta(t-t'). \qquad (8)$$

The calculations show that the matrix elements of the order parameter are diagonal in the variables t and t', i.e. the following relation is obeyed:

$$\langle t,i|\hat{\Delta}(z)|j,t'\rangle=\delta(t-t')\langle t,i|\hat{\Delta}(z)|j,t\rangle.$$

The final form of the Gor'kov equations for a tunnel junction in the quasiclassical approximation looks like

$$\begin{aligned}
&\left(i\omega_n+i\sigma_z\frac{d}{dt}-\hat{\Delta}^{ik}\right)\hat{G}^{ik}_{\omega_n}(t,t')-\hat{\Delta}^{12}\hat{G}^{21}_{\omega_n}(t,t')=\\
&=\delta(t-t'),\\
&\left(i\omega_n+i\sigma_z\frac{d}{dt}-\hat{\Delta}^{22}\right)\hat{G}^{21}_{\omega_n}(t,t')-\hat{\Delta}^{21}\hat{G}^{11}_{\omega_n}(t,t')=0,\\
&\left(i\omega_n+i\sigma_z\frac{d}{dt}-\hat{\Delta}^{22}\right)\hat{G}^{22}_{\omega_n}(t,t')-\hat{\Delta}^{21}\hat{G}^{12}_{\omega_n}(t,t')=\\
&=\delta(t-t'),\\
&\left(i\omega_n+i\sigma_z\frac{d}{dt}-\hat{\Delta}^{11}\right)\hat{G}^{12}_{\omega_n}(t,t')-\hat{\Delta}^{12}\hat{G}^{22}_{\omega_n}(t,t')=0.
\end{aligned} \qquad (9)$$

This system of equations has to be completed by the expression for the current density through the junction,

$$\mathbf{j}(\mathbf{r})=\frac{ie}{m}T\sum_{\omega_n}\lim_{\mathbf{r}'\to\mathbf{r}}(\nabla_{\mathbf{r}'}-\nabla_\mathbf{r})G_{\omega_n}(\mathbf{r},\mathbf{r}').$$

After changing to the t-representation, we obtain

$$j(z)=\frac{1}{2}\frac{e}{m}N(0)T\sum_{\omega_n}\sum_{i,k}\int\limits_0^1dx\int dtdt'\times$$
$$\times\langle t,i|\hat{G}_{\omega_n}|k,t'\rangle J^{ki}(x,t,t',z), \qquad (10)$$

Here, in the expression

$$J^{ki}(x,t,t',z)=\int d\xi d\xi' e^{-i\xi t+i\xi' t'}\times$$
$$\times\left(\Psi^{*(k)}_{p'_z}(z)\hat{p}_z\Psi^{(i)}_{p_z}(z)-\Psi^{(i)}_{p_z}(z)\hat{p}_z\Psi^{*(k)}_{p'_z}(z),\right)$$

the values of slowly varying multipliers (the transmission and reflection coefficients) are taken at the Fermi surface. Furthermore, the products of exponential functions containing the sum $p_z+p'_z\cong 2p_\mathrm{F}x$ and oscillating at atomic lengths are rejected, leaving only those products that contain the difference $p_z-p'_z=\frac{\xi-\xi'}{\mathrm{v}_\mathrm{F}x}$ and oscillate at lengths of an order of the coherence length $\xi_0\gg a$. In such a way we smooth out the current over atomic lengths and describe only its large-scale spatial variations. The described procedure of smoothing out at atomic lengths is also used, when calculating the matrix elements of the order parameter $\langle t,i|\hat{\Delta}(z)|j,t\rangle$.

3. Calculation of the Josephson Current Through a SISIS Tunnel Junction

Let us apply the quasiclassical equations (9) obtained in the previous section in order to describe the Josephson effect in a SISIS tunnel junction. In the theory of superconducting junctions, a model is widely used, in which a variation of the order parameter under the influence of the finite barrier transparency or the current is neglected, so that the absolute value of order parameter $\Delta(z)$ is assumed to be constant within each superconductor. If the parameter $\Delta(z)$ changes over the coherence length ξ_0, the application of this model does not result in a qualitative error, because the neglect of a $\Delta(z)$ variation gives rise to an uncertainty of order of 1 for the numerical multipliers in the expression for the current. Since the system may contain current states, the phase of the order parameter depends on the coordinate, and its gradient determines the superfluid velocity v_s, which is the velocity of condensate motion. In the dielectric interlayer (I), the density of the superfluid component sharply decreases, and the superfluid velocity has to drastically grow in order to provide the current continuity. As a consequence, the order parameter phase can change very rapidly, when crossing the barrier, over the lengths of an order of the barrier thickness.

Let $\Phi(z)$ be the order parameter phase, so that $\Delta(z)=|\Delta(z)|e^{i\Phi(z)}$. Let us put $\Phi(z)=\tilde{\Phi}(z)+\Lambda(z)$, where $\tilde{\Phi}(z)$ is the continuous phase component associated with the superfluid velocity $\mathrm{v_s}=\mathrm{grad}\,\tilde{\Phi}(z)$, and $\Lambda(z)$ is the phase component describing jumps at the barriers.

In this work, we confine the calculation to the current through the barriers. Therefore, in what follows, only the jump-like component of the order parameter phase will be taken into consideration:

$$\Lambda(z)=\frac{\varphi}{2}\left\{\theta\left(z-\frac{d}{2}\right)-\theta\left(-z-\frac{d}{2}\right)\right\},\quad \varphi=\mathrm{const}.$$

Hence, the following order parameter model will be dealt with:

$$\Delta(z)=\Delta\begin{cases}e^{-i\varphi/2}, & z<-d/2,\\ 1, & |z|<d/2,\\ e^{i\varphi/2}, & z>d/2,\end{cases}\qquad \Delta=\mathrm{const}. \tag{11}$$

On the basis of formulas (5), (6), and (11), taking the procedure of smoothing out over atomic lengths into account, we obtain the matrix elements for the order parameter,

$$\hat{\Delta}^{11}=\begin{cases}\hat{\Delta}_{\varphi/2}-2i\Delta R\sigma_x\sin\varphi/2, & z>d/2,\\ i\Delta\left(|C_2|^2+|C_3|^2\right)\sigma_y, & |z|<d/2,\\ \hat{\Delta}_{-\varphi/2}, & z<-d/2,\end{cases}$$

$$\hat{\Delta}^{12}=\begin{cases}2\Delta\sqrt{DR}\sin\varphi/2\sigma_x, & z>d/2,\\ i\Delta\left(C_2C_3^*+C_2^*C_3\right)\sigma_y, & |z|<d/2,\\ 0, & z<-d/2,\end{cases}$$

$$\hat{\Delta}^{21}=\begin{cases}2\Delta\sqrt{DR}\sin\varphi/2\sigma_x, & z>d/2,\\ i\Delta\left(C_2C_3^*+C_2^*C_3\right)\sigma_y, & |z|<d/2,\\ 0, & z<-d/2,\end{cases}$$

$$\hat{\Delta}^{2,2}=\begin{cases}\hat{\Delta}_{-\varphi/2}+2i\Delta R\sigma_x\sin\varphi/2, & z>d/2,\\ i\Delta\left(|C_2|^2+|C_3|^2\right)\sigma_y, & |z|<d/2,\\ \hat{\Delta}_{\varphi/2}, & z<-d/2.\end{cases}$$

Here, D and R are the coefficients of electron transmission and reflection, respectively, for the model of double δ-like barrier, and

$$\hat{\Delta}_\varphi=\Delta\begin{pmatrix}0 & e^{i\varphi}\\ -e^{-i\varphi} & 0\end{pmatrix}.$$

The substitution of the obtained matrix elements into the quasiclassical equations (9) makes it possible to calculate the Gor'kov Green's functions for a SISIS tunnel junction. In this work, we have analytically calculated the complete system of Green's functions for the indicated junction. The corresponding expressions are cumbersome. Therefore, for illustration, we present here only the formula for $\hat{G}^{11}_{\omega_n}(t,t')$ in the interval $z\geq d/2$:

$$\hat{G}^{11}_{\omega_n}(t,t')=\frac{1}{2i\tilde{\omega}_n}e^{-\tilde{\omega}_n|t-t'|}\times$$
$$\times\begin{pmatrix}\tilde{\omega}_n\mathrm{sign}(t-t')+\omega_n & i\Delta e^{i\frac{\varphi}{2}}+2\Delta R\sin\dfrac{\varphi}{2}\\ -i\Delta e^{-i\frac{\varphi}{2}}+2\Delta R\sin\dfrac{\varphi}{2} & \tilde{\omega}_n\mathrm{sign}(t-t')-\omega_n\end{pmatrix}+$$
$$+\,\alpha_1\begin{pmatrix}1 & \dfrac{\tilde{\omega}_n+\omega_n}{i\Delta}e^{i\frac{\varphi}{2}}\\ \dfrac{\tilde{\omega}_n-\omega_n}{i\Delta}e^{-i\frac{\varphi}{2}} & -1\end{pmatrix}e^{-\tilde{\omega}_n(t+t')}+$$
$$+\,\alpha_2\begin{pmatrix}1 & \dfrac{\tilde{\omega}_n+\omega_n}{i\Delta}e^{-i\frac{\varphi}{2}}\\ \dfrac{\tilde{\omega}_n-\omega_n}{i\Delta}e^{i\frac{\varphi}{2}} & -1\end{pmatrix}e^{-\tilde{\omega}_n(t+t')},$$

where $\tilde{\omega}_n=\sqrt{\omega_n^2+\Delta^2}$, and α_1 and α_2 are integration constants that are determined from the conditions of Green's function matching across the barriers.

On the basis of the expressions obtained for Green's functions and formula (10), the current through the junction can be calculated. The calculation gives the following expression for the Josephson current through the SISIS tunnel junction at $z = \pm d/2$:

$$j = \frac{\pi}{4}\, ev_{\mathrm{F}} N_{\mathrm{F}}\, \frac{\Delta D \sin\varphi}{\sqrt{1 - D\sin^2\frac{\varphi}{2}}}\, \tanh\frac{\Delta\sqrt{1 - D\sin^2\frac{\varphi}{2}}}{2T}. \qquad (12)$$

Here, $N_{\mathrm{F}} = \frac{3}{4}\frac{n}{E_{\mathrm{F}}}$ is the density of states at the Fermi surface.

Formula (12) describes the dependence of the Josephson current density at the SISIS tunnel junction edges on the coherent phase difference φ and the barrier parameters, as well as the temperature. In the case $D = 1$, formula (12) reads

$$j = \frac{\pi}{2}\, ev_{\mathrm{F}} N_{\mathrm{F}} \Delta \sin\varphi/2\, \tanh\frac{\Delta\cos\varphi/2}{2T}.$$

This relation agrees with the known formula for the SIS junction obtained in the work by Kulik and Omelyanchuk [17].

The critical current, $j_{\max}$, is determined from the extremum condition for expression (12) as a function of the phase difference. The phase difference, at which the current is maximum, depends on the barrier transparency, as well as on the junction parameters and the temperature. The corresponding value is determined from the relation

$$\varphi_{\max} = \arccos\left[1 - \frac{2}{D}\left(1 - x^2\right)\right],$$

where the parameter x is a solution of the transcendental equation

$$\sinh\left(\frac{\Delta}{T}x\right) = \frac{\Delta}{T}\,\frac{x^2\left(1 - x^2\right)\left(1 - D - x^2\right)}{1 - D - x^4}.$$

Let us calculate the critical current at the temperature $T = 2.5$ K for a tunnel junction on the basis of niobium (Nb), for which the superconducting gap $\Delta \simeq 3$ meV, and the critical temperature $T_{\mathrm{c}} \simeq 9.5$ K. In this case, $\Delta/T \simeq 14.2$. At $D = 1$, we have $x \simeq 0.23$, so that $\varphi_{\max} = 2.67$, and the corresponding critical current

$$j_{\max} \simeq 1.94\frac{\pi}{2} ev_{\mathrm{F}} N_{\mathrm{F}} \Delta.$$

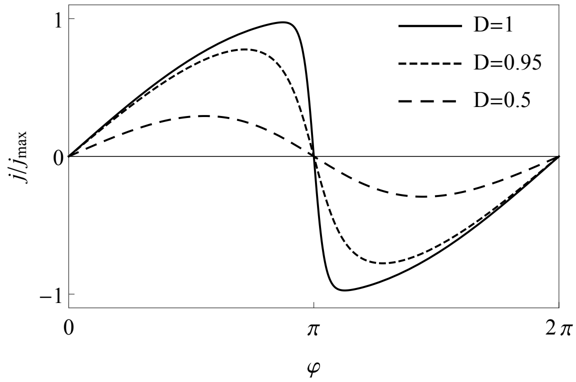

Fig. 2. Dependences of the tunnel current density on the phase difference for various transmission coefficients

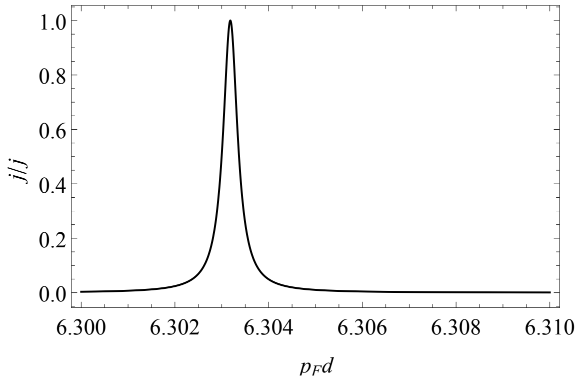

Fig. 3. Dependence of the current density on the distance between the barriers at $\kappa = 0.01$

At $D = 0.5$, the value of parameter x equals $x \simeq 0.84$. Then $\varphi_{\max} = 1.75$, and the corresponding critical current amounts to

$$j_{\max} \simeq 0.59\frac{\pi}{2} ev_{\mathrm{F}} N_{\mathrm{F}} \Delta.$$

Let us plot the dependences of the current density (12) on the characteristic junction parameters (all dependences are plotted for dimensionless quantities). In Fig. 2, the dependence of the tunnel current density on the order parameter phase difference between the junction egdes is shown. This dependence has a sawtooth behavior, when the value of transmission coefficient is close to 1. As the parameter D decreases, the analyzed dependence for the current density acquires a sinusoidal form [13]. Note also that the current density decreases with a reduction of the tunneling coefficient.

In Fig. 3, the dependence of the current density on the inner interlayer thickness is depicted. The dependence has a sharp resonance peak, which coincides with the maximum of the electron transmission coefficient through the double δ-like barrier. The current reaches a maximum, if the barrier thickness satisfies the equation

$$p_{\rm F}d_{\max} = \frac{1}{2}\left(-\arctan\frac{4\kappa}{4\kappa^2-1} + 2\pi n\right), \quad n \in N,$$

and a minimum, if

$$p_{\rm F}d_{\min} = \frac{1}{2}\left(\pi-\arctan\frac{4\kappa}{4\kappa^2-1} + 2\pi n\right), \quad n \in N.$$

Typical insulators used in tunnel junctions, e.g., Al_2O_3, have a thickness of about 10–20 nm. The potential barrier height in such insulators varies from 1 to 5 eV. Therefore, the coefficient α in potential (3) is of an order of $(1\div10)\times10^{-8}$ eV m. Since the Fermi energy for metals is about 2–10 eV, the parameter κ acquires values within an interval of $(5\div30)\times10^{-3}$. If the parameter $\kappa = 0.01$, the maximum values of current take place provided that $p_{\rm F}d \simeq 0.01999$, 3.16158, 6.30318, 9.44477, and so forth.

4. Conclusions

In this work, a microscopic theory of the stationary Josephson effect in an SISIS tunnel junction has been developed. The theory is based on the quasiclassical equations of superconductivity theory. Those equations are the Gor'kov equations for Matsubara Green's functions of a superconductor, but smoothed out over atomic lengths.

On the basis of the quasiclassical equations with the use of analytical methods, a formula for the Josephson current density through the junction is derived. The current density is shown to depend on the coherent phase difference between the superconductors, barrier parameters, and temperature. When the coefficient of barrier transmission D is close to 1, the current reveals a sawtooth dependence on the phase difference. This dependence gradually acquires a sinusoidal form, as the electron transmission coefficient decreases.

A formula for the critical current depending on the transmission coefficient is also derived. The critical current maximum is found to take place at $D = 1$.

While studying the dependence of the current on the inner interlayer thickness in a junction, resonance peaks in the current density are revealed. Those peaks coincide with the maxima of the electron transmission coefficient through the double δ-like barrier. An analytical relation is obtained for the interlayer thickness, at which the current is maximum.

Note that the developed theory can easily be modified for junctions with different geometries, e.g., $S_1IS_2IS_3$, SINIS, and so on.

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РОЗРАХУНОК СТРУМУ ДЖОЗЕФСОНА
У ДВОБАР'ЄРНОМУ ТУНЕЛЬНОМУ КОНТАКТІ

Р е з ю м е

У роботі аналітично розраховано струм Джозефсона у двобар'єрному тунельному контакті зі структурою SISIS. Обчислення виконано на основі квазікласичного наближення рівнянь мікроскопічної теорії надпровідності. Нами було розраховано функції Гріна для тунельного SISIS-контакту та одержано вираз для струму Джозефсона в моделі точкового контакту. Встановлена залежність тунельного струму від фази параметра впорядкування. Ми також проаналізували залежність величини струму від відстані між бар'єрами та показали наявність резонансних піків джозефсонівського струму.