https://doi.org/10.15407/ujpe70.6.361

B.I. LEV

Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine
(14-b, Metrolohichna Str., Kyiv 03143, Ukraine; e-mail: bohdan.lev@gmail.com)

CLIFFORD ALGEBRA AS A WAY TO QUANTUM GRAVITY

This article puts forward a novel hypothesis for a solution to the problem of quantization of gravity. The objective of this study is to demonstrate that the geometric representation of the wave function can be considered as a characteristic of the space-time manifold. In this approach, it is shown that the Dirac theory for the hydrogen atom and the Kepler dynamics for the planetary system describe analogous phenomena in the space-time. The states of these systems possess parameters that correspond to the permitted dynamic states of the space-time, thereby maintaining information regarding the corpuscular and wave nature. The proposed approach sheds a new light on the potential resolution of the problems of quantum gravity.

K e y w o r d s: Clifford algebra, wave function, test particle, space-time manifold.

1. Introduction

One of the approaches to quantum gravity [1–3] assumes that quantum theory can be represented in a geometric form that is compatible with the general theory of relativity. In this regard, the recent paper [3]) which proposes quantum gravity without metric quantization of a particular interest. That paper advanced a covariant extension of the Bohmian mechanics onto a curved space-time, where trajectories create a "hidden curvature" by replacing the metric superposition with a statistical ensemble. In such a case, gravitational effects arise from deterministic quantum trajectories.

As shown in the previous studies [4–14], quantum mechanics can be derived from the mathematical structure of the Clifford algebra without recourse to an external Hilbert space of wave functions. An interesting article in this direction is [15], where the formalism of geometric algebra is used to develop a theory that enhances conventional quantum mechanics. The use of Clifford's algebra in quantum mechanics [4,16] actually provides only an algebraic structure and leads to a quantum mechanical theory that contains no additional requirements; moreover, in the terms of such an approach one can find the basis for a common geometrical description as a different kind of interaction of a separate particle [17].

Clifford's space-time algebra provides a vivid example of an alternative formulation of the wave equation. The Dirac equation can be understood as a transfer rule for the wave function of any manifold, and it has a hidden geometric structure [17, 18]. This equation is used as one of the possible mathematical interpretations of quantum mechanics. Moreover, it does not depart from the principles of classical physics [19]. It is also essential that, in this case, it is easy to provide a geometrical representation of the generators for the gauge transformations. The appearance of fields of different natures is dictated by quantum fluctuations [20–22] has shown, and necessarily depends on the geometrical nature of the physical vacuum, that is, on what physical properties we attribute to it.

Citation: Lev B.I. Clifford algebra as a way to quantum gravity. *Ukr. J. Phys.* **70**, No. 6, 361 (2025). https://doi.org/10.15407/ujpe70.6.361.

In our interpretation, the transport equation for the wave function of a test particle on an arbitrary manifold is used as a prototype for the theory of space-time, based on the general physical principles of the equivalence of particle coupling constants and this manifold dynamics. As will be shown, the proposed algebraic approach is an effective means of describing the behavior of electrons in a hydrogen atom and planetary systems. The geometric representation of general relativity allows us to recover the standard quantum approach and can also be used to describe physical behavior on a large scale. The wave properties of macroscopic particles are manifested in the behavior of their "trajectories" in the observed space-time.

The main idea of the paper is to show that the quantum properties of the "test particle" can emerge from the properties of the space-time in which the dynamics takes place. The test particle exhibits corpuscular or wave properties on the dynamical manifold and depends on the scale. The geometric definition of the wave function shifts the focus, where the wave function is considered as a property of space-time and takes a probabilistic value according to the corresponding states of the manifold in which a "test particle" with the appropriate coupling coefficients is present. This approach, in our opinion, points a way to the quantization of the gravitational field. The quantization of the curved space-time in such a case is done through the quantization of possible trajectories of the motion of the test particle endowed with the appropriate physical properties.

2. Method

2.1. Geometrical presentation of a "Test particle"

First, it is necessary to note the standard definition of a test particle as an idealization of a physical object whose properties (spin, mass, charge, or size) are used to describe its dynamics on an arbitrary space-time manifold, which directly affects arbitrarily chosen properties on different scales. The primary concept is based on the correspondence between Dirac spinor matrices γ_μ and elements of the external algebra, and the definition of the state in terms of representations of the Clifford space-time algebra $\mathrm{Cl}_{1,3}$. It can be postulated that every elementary formation at any point can be described in terms of the Clifford number. Such a characteristic of the space-time manifold at the corresponding point is represented as the wave function of the test particle or excitation.

The characteristic of an excitation of the space-time manifold at an arbitrary point is represented by a complete geometric object consisting of the direct forms of the induced space of the Clifford algebra [18, 23–25]. In this case, the complete geometric object can be written as the direct sum of a scalar, a vector, a bi-vector, a three-vector, and a pseudo-scalar, i.e. $\Psi = S \oplus V \oplus B \oplus T \oplus P$, where the basis vector is represented by the Dirac matrix γ_μ. Another element of the symmetry is the change of the multiplication of the basis vectors to inverses in the representation of the Clifford numbers, which turns them into $\bar{\Psi} = S \oplus V \ominus B \ominus T \oplus P$. The ring structure is satisfied by the direct product in the symbolic notation given by $\Psi\Phi = \Psi \cdot \Phi + \Psi \wedge \Phi$, where $\Psi \cdot \Phi$ is an inner product or convolution that decreases the number of basis vectors, and $\Psi \wedge \Phi$ is an external product that increases the number of basis vectors.

The symplectic structure of the Clifford algebra follows from ground relation for two basis vector multiplication $\gamma_\mu\gamma_\nu = \gamma_\mu \cdot \gamma_\nu + \gamma_\mu \wedge \gamma_\nu$, where there is present simultaneously a scalars and an outer product for the basis vectors. The scalar product $\gamma_\mu \cdot \gamma_\nu = \frac{1}{2}(\gamma_\mu\gamma_\nu + + \gamma_\nu\gamma_\mu) = \eta_{\mu\nu}I$ can be defined as the Minkowski metric tensor in theEuclidean space (anticommutator) and the outer product $\gamma_\mu \wedge \gamma_\nu = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) = s_{\mu\nu}$ as the commutator of the basis matrices.

If we multiply each Clifford number by a fixed column u with 4 elements, where the first element is one and all others are zero, we get a Dirac bi-spinor with four elements $B = u\Psi$. Using this column, one can reproduce the spinor representation of each Clifford number. The complex conjugate bi-spinor can be obtained by multiplying the same Clifford number by a string u^+ whose first element is one and all others are zero $B^* = \bar{\Psi}u^+$ provided that $u^+u = 1$. There is a complete correspondence between the bi-spinors thus obtained, and the elements of the exterior algebra - the isomorphism [4].

Now, let us determine the rule of comparing two Clifford numbers in different points of the manifold. An arbitrary transformation of the coordinate system may be set in terms of the basis deformations $e_\mu = R\gamma_\mu\tilde{R}$, where R is the Clifford number that describes arbitrary changes of the basis (including

arbitrary displacements and rotations), that do not violate its normalization, i.e., provided $\tilde{R}R = 1$. It is not difficult to verify that $e_\mu^2 = R\gamma_\mu\tilde{R}X\gamma_\mu\tilde{R} = R\gamma_\mu^2\tilde{R} = \gamma_\mu^2$, and this relation does not violate the normalization condition and commutation relations between the new matrices [26]. An arbitrary physical object must be represented by a mathematical object which can transformed during rotations and translations. The geometrical objects presented here have properties of the spinor transformation [27].

For an arbitrary basis, we may define, at every point of the space, a unique complete linearly independent form as a geometric entity that characterizes this point of the manifold. If this point of the manifold is occupied, then its geometric characteristics may be described by the coefficients of general representation. A product of arbitrary forms is given by a similar form with new coefficients, thus providing the ring structure. This approach makes it possible to consider the mutual relationship of fields of different physical natures [17, 28]. To determine the characteristics of the manifold implies to associate every point of the manifold with a Clifford number and to find its value. To define a transfer operation on an arbitrary manifold, we have to define a derivative operator, e.g., as given by $d = \gamma^\mu \frac{\partial}{\partial x_\mu} = \gamma^\mu \partial_\mu$ and represents changes along the curves passing through a given point in the space. The action of this operator for any Clifford number may be presented as $d\Psi = d \cdot \Psi + d \wedge \Psi$, where $d \cdot \Psi$ and $d \wedge \Psi$ may be referred to as the "divergence" and the "rotor" of the relevant Clifford number. Within the context of the definition of a differentiated variety, it is not enough to have one special coordinate system covering a variety whose topology differs from the topology of an open set in the Euclidean space.

2.2. Transfer rule and motion integrals at arbitrary space- time manifold

The assigning a specific geometric interpretation to the wave function of the "test particle" allows us to obtain precise transfer rules for an arbitrary space-time variety [17, 18], and, thus, facilitating the discovery of its nature. For the wave function as a geometric entity, the first structural equation can be written in standard form:

$$D\Psi = d\Psi + \Omega\Psi, \tag{1}$$

where the connectivity Ω include the influence of the space-time manifold. After this presentation of the transfer rule, several important remarks should be made. In such a form, the operator d and the connectivity Ω are scalar and are not changed by different symmetry transformations as coordinates by gauge transformations together. As for coordinate transformations, it is obvious, because there is only one part associated with such a transformation, that is

$$d^t = R\gamma^\mu\tilde{R}R\partial_\mu\tilde{R} = R\gamma^\mu\partial_\mu\tilde{R} = \gamma^\mu\partial_\mu = d.$$

There are two equivalent variants of calibration transformations $\Psi^c = R\Psi$. The first consists in equating the covariant derivation to zero, and then all the properties of the manifold appear as coefficients of connectivity of the wave function with this manifold. At the same time, the calibration relation, which is applied to the connectivity as a constant Ω, takes the form

$$\Omega^c = \gamma^\mu R\gamma_\mu\Omega\tilde{R} - \gamma^\mu\partial_\mu R\tilde{R}. \tag{2}$$

The second method can be used, if the wave function transfer rules are proportional to the same wave function with a selected scalar coefficient M which involves a separate connection with the manifold

$$D\Psi = d\Psi + \Omega^g\Psi - M\Psi. \tag{3}$$

Let has represent $\Omega^c = \gamma^\mu\Gamma_\mu$ as a product of two vectors, where Γ_μ can be called the vector-potential, and rewrite the previous equation in the form

$$D\Psi = d\Psi + \Omega^c\Psi - M\Psi = 0, \quad \gamma^\mu \cdot \nabla_\mu\Psi = M\Psi, \tag{4}$$

where $\nabla_\mu = \partial_\mu - \Gamma_\mu$ is the well-known covariant derivation. Then the calibration transformation valid for this equation takes the standard form

$$\Gamma_\mu^c = R\Gamma_\mu\tilde{R} - \partial_\mu R\tilde{R}. \tag{5}$$

The same transformation can be obtained in the first case, if we represent the general connectivity as $\Omega = \Omega^c - M$. In this sense, the scalar mass can also be considered as the coefficient of the coupling of a "test particle" with the space-time manifold. This proves the equivalence of the two approaches. The difference is only in that the first case allows us to use the transfer equation in the non-linear case where the general connectivity can be represented as a scalar

product of the same wave functions $\Omega = \Psi\tilde{\Psi}$ that were used in the articles [8, 10, 11]. In article [11], such representation of the connectivity was used to describe the vacuum state of the manifold and its predicted supersymmetric behavior. The representation of the connectivity of the space-time in the form $\Omega = \gamma^\mu \Gamma_\mu$ can always be done, if we assume that the general form of the connectivity has the same form as a wave function. Let us multiply this expression by $\gamma^\mu\gamma_\mu = 1$. Then it may be used as a vector-potential $\Gamma_\mu = \gamma_\mu\Omega$.

Now, we can demonstrate that the form of the transfer rule, the characteristic of a point of an arbitrary manifold 4, is entirely consistent with the Dirac equation in the geometric representation [18]. In the general case, the connectivity contains both the real and imaginary parts $\Gamma_\mu = \Gamma_\mu + iU_\mu$. For example, if there is only an imaginary part, it is enough to represent the general connectivity as $\Omega = \gamma^\mu\Gamma_\mu = i\gamma^\mu U_\mu = iq\gamma^\mu A_\mu$, where A_μ is the vector-potential of the electromagnetic field with the coefficient $q = \frac{e}{\hbar c}$ and $iM = \frac{mc}{\hbar}$ where m is the mass, e is the charge of the test particle, and $\hbar$ is the Planck constant. Then the transport equation for the wave function of an electron on the space-time turns into the well-known standard Dirac equation:

$$\gamma^\mu\left(i\hbar\partial_\mu - \frac{e}{c}A_\mu\right)\Psi = m_0 c\Psi \tag{6}$$

or canonical form [19]

$$\gamma^\mu\nabla_\mu\Psi = -i\frac{m_0 c}{\hbar}\Psi. \tag{7}$$

For the complete group of linear transformations $\Psi' = \Psi R$, where R defines the mapping elements and satisfies the condition $\tilde{R}R = 1$, the calibration transformation for the connectivity A_μ is defined, as previously:

$$A'_\mu = RA_\mu\tilde{R} - R\partial_\mu\tilde{R}. \tag{8}$$

The test particle (electron) affects the manifold only through its physical characteristics (mass, charge, spin) as coefficients of the general representation. Only in this approach, the dynamical equation for the wave function of the electron is represented as a parallel transfer rule on an arbitrary space-time manifold with the connectivity which describe the electromagnetic field. For this reason, the transfer rules 4 for the wave function can be called the Dirac-like equation.

It should be noted that, to determine the transfer rules, it is not necessary to introduce Planck's constant. It appears, only when we want to find a wave solution in the de Broglie presentation $\Psi \sim \exp i\frac{S}{\hbar}$, where S is the action, and Planck's constant h is a measure of the phase space that corresponds to a single state that is described by a given wave function. Accordingly, the size of the phase space which is celled belonging to one state directly depends on the scales of the momentum and coordinates. The scales of behavior of electrons and planets are significantly different, and, therefore, this plays a decisive role in the wave behavior of the test particle.

The symplectic structure of the Clifford algebra enables the immediate identification of all integrals of motion for such dynamics. The Clifford algebra allows for the simultaneous introduction of the scalar product and the commutator of these quantities due to the presence of inner and outer products. It was established in previous studies [12, 19] that the dynamic integrals of motion that are preserved for the Dirac equation and, in our case, with general transfer rules, have the form:

$$J_\mu = \bar{\Psi}\gamma_\mu\Psi, \quad J_5 = \bar{\Psi}\gamma_5\Psi, \tag{9}$$

$$J_{\mu\nu} = \bar{\Psi}s_{\mu\nu}\Psi, \quad J_{5\nu} = \bar{\Psi}\gamma_5\gamma_\nu\Psi, \tag{10}$$

where $s_{\mu\nu} = [\gamma_\mu\gamma_\nu] = \gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu$ is the commutator of the Dirac matrix and $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ product of all Dirac matrices. The quantity $\bar{\Psi}\gamma_0 = \Psi^*$; then the integral of motion $J_0 = \rho = \Psi^*\Psi$ is clearly identified as the probability of finding the test particle at the corresponding point on the trajectory. The integral of motion $J_\mu = \bar{\Psi}\gamma_\mu\Psi = \Psi^*\sigma_\mu\Psi$ corresponds to a probability flow on the space-time manifold, where σ_μ is the Pauli matrix in four dimension [19]. The conservation of the probability flow can be demonstrated by checking that $\partial^\mu J_\mu = 0$ is zero [13]. The integral of motion $J_{\mu\nu} = \bar{\Psi}s_{\mu\nu}\Psi$ is responsible for the conservation of the angular momentum, and the integral of motion $J_{5\nu} = \bar{\Psi}\gamma_5\gamma_\nu\Psi$ determines the Runge–Lenz vector, which is exactly conserved in a centrally symmetric field [29].

It should be noted that the obtained integrals of motion completely reproduce the very important hidden Lorentz covariance of quantum mechanics, fully investigated in the article [30]. In general, the commutators of two arbitrary matrices A and B can be written as

$$[\bar{\Psi}A\Psi, \bar{\Psi}B\Psi] = \bar{\Psi}(AB - BA)\Psi. \tag{11}$$

In the introduced notations for our integrals of motion J_{mu}, we can obtain

$$[J_\mu, J_\nu] = \rho\bar{\Psi}(\gamma_\mu\gamma_\nu - \gamma_\mu\gamma_\nu)\Psi = \rho\bar{\Psi}s_{\mu\nu}\Psi = \rho J_{\mu\nu} \quad (12)$$

and, for the integrals of motion $J_{\mu\nu}$, the following relation is valid:

$$[J_{\mu\nu}, J_{\rho\sigma}] = \rho\bar{\Psi}(s_{\mu\nu}s_{\rho\sigma} - s_{\rho\sigma}s_{\mu\nu})\Psi \quad (13)$$

or, in a more convenient form through the previously introduced integrals of motion,

$$[J_{\mu\nu}, J_{\rho\sigma}] = \rho\Big(\delta_{\mu\nu}J_{\rho\sigma} + \delta_{\nu\sigma}J_{\mu\rho} - \delta_{\mu\sigma}J_{\nu\rho} - \delta_{\nu\rho}J_{\mu\sigma}\Big), \quad (14)$$

which fully corresponds to the relations obtained in the article [30] and confirms the hidden Lorentzian covariance of the presented approach. Using the given integrals, one can always obtain a solution in the general case.

2.3. Clifford algebra in general relativity

As shown in the known articles [20–22] the problem of the appearance of a gravitational field can be solved by considering fluctuations of the vacuum. Thus, the Dirac equation with gravity appears as a prototype of interaction theory based on a generalized principle equivalence [21]. In the proposed geometric approach, this can be taken into account by means of nonlinear transformations of the coordinate system. The main difficulty of the theory of gravity can be overcome due to the quantum fluctuations of the vacuum, the ground state, in relation to which matter represents an excitation, the expected values of which are equal to zero, but its square gives the nonzero value. This is the simplest case of a field quantum theory that satisfies the equivalence requirement. As shown in [18] the Dirac equation in a geometric interpretation is, probably, a prototype of a more general equations including electromagnetism as well as another interactions. Such equation is linear in the term additional field, that being a necessary condition for the fulfillment of the equivalence claim. In the geometric interpretation, the Lagrange density for the transfer rule equations can be represented as

$$L = \bar{\Psi}D\Psi = \bar{\Psi}(\gamma^\mu \times \nabla_\mu - M)\Psi, \quad (15)$$

where operator $\nabla_\mu = \partial_\mu - \Gamma_\mu$. Such an image always occurs even in the case of a nonlinear dependence of the connectivity on the wave function itself. A scalar quantity $\Omega \sim \gamma^\mu\Gamma_\mu$ can always be represented as a scalar product of a unit vector and a "vector potential" of the unknown physical content. For our space-time additional "vector potentials" Γ_μ, the introduced Lagrangian density will be invariant not only with respect to unitary coordinate transformations but also under more general gauge transformations. As for the mass parameter M in this approach, it considered as a formal value, which will be discussed below. Now, we can introduce the conformal transformation of the field $\bar{\Phi} = \sqrt{g}\bar{\Psi}$ and $\Phi = \sqrt{g}\Psi$ where g, as usual, the magnitude of the determinant of the matrix tensor. In the new variables, the Lagrangian density can be present in the form

$$L = \sqrt{g}\bar{\Phi}(\gamma^\mu \times \nabla_\mu - M)\Phi. \quad (16)$$

In this case, the coordinate transformations are not unitary. The important property of the Lagrangian density satisfies the equivalence claim in the sense that, at any space-time point, the first coordinate derivatives of the Dirac matrix can be made to vanish, which was shown in [8]. Minimizing point action with the new density Lagrangian [21, 28], we can obtain the equation $M\Phi = = \gamma^\mu \times \nabla_\mu\Phi$ and $M\bar{\Phi} = \gamma^\mu \times \nabla_\mu\bar{\Phi}$. Taking into account that $\gamma^\mu \times \nabla_\mu = \frac{1}{\sqrt{g}}\gamma^\mu \times \nabla_\mu\sqrt{g}$ and $(\gamma^\mu \times \nabla_\mu)^2 = = \sqrt{g}(\partial_\mu - \Gamma_\mu)\sqrt{g}(\partial_\nu - \Gamma_\nu)g^{\mu\nu} - \frac{R}{4}$, where R is the Einstein scalar curvature, we can obtain the action in final form

$$S = \int \sqrt{g}\bar{\Phi}\Big(\sqrt{g}(\partial_\mu - \Gamma_\mu)\sqrt{g}(\partial_\nu - \Gamma_\nu)g^{\mu\nu} - \frac{R}{4} - M^2\Big)\Phi. \quad (17)$$

Thus, taking the fluctuations of the spinor field in the Clifford algebra into account, it is possible to describe the motion of the test particle in the distorted Euclidean space-time. For the vacuum, this approach assumes an elastic deformation of the space-time due to the presence of a spinor field [18].

3. Results

3.1. Gravitation field in Clifford algebra

Let us now consider a very important question concerning a possible combination of quantum theory and general relativity. The preliminary definition of the transfer rules on an arbitrary manifold (space-time), as well as the existence of corresponding integrals of motion for the covariant transfer, already

contain such a possibility. It is enough to change the emphasis of the interpretation of the wave function itself. Let the physical object be a wave function with a given geometrical representation describe dynamical trajectory. That is, the wave function describes the test particle on the corresponding trajectory. On this trajectory, we can enter the corresponding coordinates in the form $x_\mu = J_\mu = \bar{\Psi}\gamma_\mu\Psi$, which correspond exactly to the first integrals of motion. Now, we can enter the definition of the interval on the corresponding trajectory $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}d\bar{\Psi}\gamma^\mu\Psi d\bar{\Psi}\gamma^\mu\Psi$.

If we use the transport equation for the wave function 4 and introduce a new definition of the metric tensor in terms of the introduced wave functions as $g^{\mu\nu} = \bar{\Psi}\gamma^\mu\gamma^\nu\Psi$, then we can obtain an interval on any trajectory in the form $ds^2 = \bar{\Psi}\Psi\Gamma^2 = B^*B\Gamma^2 = \rho\Gamma^2$, where only the real part of the space-time connection $\Gamma = \gamma_\mu\Gamma^\mu$ is present. As is seen, this is the vector potential of the gravitational field in Fock–Ivanenko presentation [8, 19] and ρ is probability density.

In the Clifford algebra, $\bar{\Psi}\Psi = \rho$ defines the probability density for a wave function which present in canonical form [7, 19] $\Psi = \rho^{\frac{1}{2}}R$, where R is the Clifford number and defines all possible transformations of the wave function, provided that $\bar{R}R = 1$. Now, let us introduce a new definition of the metric tensor in terms of the canonical form of the wave function $g^{\mu\nu} = \bar{\Psi}\gamma^\mu\gamma^\nu\Psi = \rho\bar{R}\gamma^\mu\gamma^\nu R = \rho\bar{R}\gamma^\mu\bar{R}R\gamma^\nu R = \rho e^\mu e^\mu$. In this interpretation, the metric tensor represents the probability of the corresponding deformed basis, which corresponds to the geometric representation of the general theory of relativity. With this definition of the metric tensor, it can take on both real and imaginary values. This defines a hidden non-classical geometry of the space-time. In such interpretation of the metric, the stochastic (quantum) behavior is hidden.

We may assume that if we consider the coordinate on the trajectory, where the test particle is located (see the articles [31, 32], where the spinor representation $x_\mu = J_\mu = \bar{\Psi}\gamma_\mu\Psi$ was used) and take the equation of a geodesic trajectory in the gravitational field, then we can obtain the equation for velocity $v_\mu = \frac{dx_\mu}{ds} = \frac{dJ_\mu}{ds} = \frac{d(\bar{\Psi}\gamma_\mu\Psi)}{ds}$ in the form

$$\frac{dv^\mu}{ds} + \Gamma^\mu_{\nu\lambda}v_\nu v_\lambda = \frac{d^2J^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\frac{dJ_\nu}{ds}\frac{dJ_\lambda}{ds} = 0. \qquad (18)$$

The publication of articles [31, 32], in which the spinor regularization of Kepler's motion was proposed, helped one to find the hidden symmetry of a hydrogen atom and reduce the problem to description of a harmonic oscillator. After that, we can use the quantization of the flow of possible trajectories. The observed trajectory may correspond to a bound state in the set of possible trajectories. For example, for closed periodic trajectories $\frac{d\Psi}{ds} = i\omega\Psi$ with frequency ω it corresponds to a hydrogen atom. For the Kepler motion, the above relation (18) holds exactly. For open trajectories $\frac{d\Psi}{ds} = \pm\omega\Psi$ corresponding to free motion in a gravitational field, we can obtain that $\rho = \gamma_\sigma\frac{\partial g_{\mu\nu}}{\partial x^\sigma}g^{\mu\nu}$ is determined only through the metric tensor of the space-time.

Now, it is necessary to consider the rules for the transformation of Clifford numbers in a non-Euclidean coordinate system. In order to address the issue of covariant derivation of the theory, it is essential to include the internal coupling of Clifford numbers within the context of an arbitrary Riemann geometry, where the Minkowski metric can be considered a relatively simple approximation. In the Fock–Ivanenko approach, as outlined in articles [8, 21, 28, 33], an analogy with electrodynamics was employed to define the essential properties required for the covariant derivation which is used in the context of Clifford numbers. The article [9] shows that a modification of the space-time metric allows one consider the geometrization of quantum mechanics in the de Broglie–Bohm formulation using non-Riemann Weyl structure. In this context, let us write a transfer rule on an arbitrary manifold represented by the "vector potential" field Γ_μ, due to which the Lagrange density will be invariant not only with respect to coordinates transformations, but also with respect to more general unitary transformations.

$$\gamma^\mu\nabla_\mu\Psi = \gamma^\mu(\partial_\mu - \Gamma_\mu)\Psi = M\Psi. \qquad (19)$$

In the general relativistic theory, the transport equation can be written in a similar way, if we introduce the Dirac matrices as functions of the space-time coordinates, which form a contravariant vector field. Anticommutators of these matrices must be multiples of the unit matrix $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}I$, where $g_{\mu\nu}$ is identified with the metric field. In addition, it is convenient to use the following: $2s_{\mu\nu} = [\gamma_\mu\gamma_\nu] = \gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu$ and $g = \det(-g_{\mu\nu})$. For general relativity theory, the covariant derivative of the metric tensor $\nabla_\lambda g_{\mu\nu} = 0$. The Dirac matrix γ_μ is usually not covariant with respect to R-transformations

$\gamma^r_\mu = R\gamma_\mu\tilde{R}$. The transformation of elements of the Clifford algebra must be modified in some way. We note that the present form can be obtained with the use of properties of the Clifford algebra. With the new field matrix presentation of the manifold Γ_μ, the covariant derivative of $\gamma\mu$ can be defined as [33]:

$$\nabla_\nu\gamma_\mu = \partial_\nu\gamma_\mu + [\gamma_\mu\Gamma_\nu], \tag{20}$$

where, for the introduced additional field, a gouge transformation is performed (5) which makes it possible to equate the covariant derivative of the Dirac matrix to zero. From well-known condition for the metric tensor $\nabla_\lambda g_{\mu\nu} = 0$, we can obtain [8]:

$$\Gamma^0_\mu = \frac{1}{8}\Big(\gamma^\alpha\gamma_{\mu,\alpha} - \gamma_{\mu,\alpha}\gamma^\alpha + \Gamma^\beta_{\mu,\nu}(\gamma_\beta\gamma^\nu - \gamma^\nu\gamma_\beta)\Big), \tag{21}$$

where $\Gamma^\beta_{\mu,\nu}$ Christopher's symbol. The index zero is just a reminder that we deal with a Minkowski background in an arbitrary system of coordinates. We can globally annihilate such connection by moving to an Euclidean coordinate system. At any spacetime point, the first coordinate derivatives of the γ_μ, which represent the field, can be made to vanish, while the γ_μ themselves become equal to the Dirac matrices. A direct consequence of the relation $\nabla_\mu\gamma_\nu = 0$ is that the covariant derivative of all γ_μ vanishes. This is a consequence of that the metric is Riemann: $\nabla_\lambda g_{\mu,\nu} = 0$. Although the condition that the covariant derivatives are equal to zero is sufficient to guarantee the Riemann structure, but the geometric representation of the Clifford number is optional.

3.2. Hidden quantization of a "test" particles trajectories

Article [8] considers the case where the dynamics is governed by the structure of the Clifford algebra under the condition of commutation:

$$\nabla_\mu\gamma_\nu = [U_\mu, \gamma_\nu], \tag{22}$$

where U_μ is an arbitrary element of the Clifford algebra which can be presented as the sum of vectors and pseudo-vectors $U_\mu = A_\mu + B_\mu\gamma_5$. We can obtain that $\nabla_\lambda g_{\mu,\nu} = [U_\lambda, \gamma_\mu]\,\gamma_\nu + \gamma_\mu\,[U_\lambda, \gamma_\nu] + [U_\lambda, \gamma_\nu]\,\gamma_\mu + + \gamma_\nu\,[U_\lambda, \gamma_\mu]$, and the use of anticommutators with all γ_μ implies that $\nabla_\lambda g_{\mu,\nu} = 0$. This holds for arbitrary vectors A_μ and B_μ. This provides a convenient equivalent way to describe the non-linear structure for choice of spinors U_μ. Thus, the internal connection takes the form $\Gamma_\mu = \Gamma^0_\mu - iU_\mu$, where Γ^0_μ is the "vector potential" in the Minkowski space. In such case, we can calculate the vector potential in small central gravitation field [8] and present the Hamiltonian in classical case in the well-known form:

$$H = E(\mathbf{p}) = mc^2 + \frac{\mathbf{p}^2}{2m} - \frac{GmM}{r}. \tag{23}$$

In Kepler's problem, the M is the Sun mass, the mass of the planet is m, and there is no first term. In this section, we will continue to calculation with the Hamiltonian without the first term.

The publication of articles [31, 32], in which the spinor regularization of Kepler's motion was proposed, helped us to find the hidden symmetry of a hydrogen atom and reduce the problem to description of a harmonic oscillator. In the future, we will use only the final formulas of the spinor representation of Kepler's dynamics. The motion of planets along their orbits around the Sun can be described by Newton's equation

$$m\ddot{\mathbf{r}} + \alpha\frac{\mathbf{r}}{r^2} = 0, \tag{24}$$

where $\alpha = GMm$ – coefficient of gravitation interaction between central mass M and test particle with mass m. Additional conditions for the solution of this problem are the law of conservation of the energy

$$E = m\dot{\mathbf{r}}^2 - \frac{\alpha}{r} \tag{25}$$

the momentum vector

$$\mathbf{M} = \frac{m}{r}(\mathbf{r}\times\dot{\mathbf{r}}) \tag{26}$$

and the Runge–Lenz vector

$$\mathbf{A} = \frac{1}{r}(\mathbf{M}\times\dot{\mathbf{r}}) + \alpha\frac{\mathbf{r}}{r}. \tag{27}$$

The last integral of motion involves a change in the position of the orbits in space, as well as the form of these orbits with constant energy.

An important point is that a new angular variable called "eccentric anomaly" was used to describe the positions of the planets on their orbits [31, 32]. With the help of this variable, using Kepler's laws, it is possible to accurately describe the flat trajectory of the orbit, as well as determine all possible integrals

of dynamic motion. If, instead of the radius-vector of a position of the planet on its orbit, we write its representation through the two-component spinor ψ and the Pauli matrix in the form $\boldsymbol{\sigma}$, $\mathbf{r} = \psi^* \sigma \psi$, then it is easy to check that the spinor equation

$$\ddot{\psi} + \omega^2 \psi = 0 \tag{28}$$

fully describes the dynamics of planets on Kepler orbits, where $\omega = \frac{a}{\tau}$, a is the value of the major axis of the ellipsoid, and τ is the period of rotation around the massive body at the center. The dot means the derivative with respect to the parameter $s = \frac{\tau\chi}{2ma}$, where χ is the eccentric anomaly of an elliptical orbit, which actually determines how an elliptical orbit differs from a circular one. For a circular orbit, this will be an angle indicating the orientation of the radius vector to an arbitrary point of the circular orbit.

The $O(4)$ algebra symmetry of the original problem becomes obvious, especially if we introduce a bispinor

$$\Psi = \sqrt{\frac{2m}{\omega}} \begin{pmatrix} \dot{\psi} \\ -i\omega\psi \end{pmatrix}$$

in which the previous equations transform to one equation

$$\dot{\Psi} = i\omega\gamma_5 \Psi, \tag{29}$$

where $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ is product all Dirac matrix. The presented solution in the bispinor form can be consider as Clifford number. Equation (29) is invariant with respect to transformations $\acute{\Psi} = \Psi \exp iQ$ where Q the four-by-four matrix which must commutes with γ_5.

The symplectic structure of the Clifford algebra allows one to identify all integrals of motion for such dynamics because it allows one to simultaneously introduce the scalar product and the commutator of these quantities due to the presence of inner and outer products. In our case, the dynamical integrals of motion (9) and (10), defined as with the integrals of motion of the Dirac equation as rules for transferring the Clifford number to the manifold [18]. The presence of detected integrals of motion allows one to "quantize" the corresponding adiabatic invariant. If we now pay attention to the integrals of motion, the value $\bar{\Psi}\gamma_5 = \Psi^*$ will then be clearly defined as the probability of finding the test particle in the corresponding orbit, $\rho = \Psi^*\Psi$. Such integral of the motion corresponds to the probability of finding the test particle (planet) at the corresponding point of the orbit, provide that the $\partial^\mu J_\mu = 0$ is zero. The first integral of the motion can be interpreted as the Hamiltonian of the test particle (planet). In addition to the unit matrix, the commutator of the gamma matrix $s_{\mu,\nu}$ also commutes with γ_5 and has the same integral of motion (9) and (10) as in the theory Dirac of a hydrogen atom. The application of these integrals allows for the derivation of a definitive solution to the problem.

If we now introduce a complex conjugate Dirac bispinor $\bar{\Psi} = \Psi^*\gamma_5 = \sqrt{\frac{2m}{\omega}}(-i\omega\psi^*, -\dot{\psi}^*)$, it is possible to represent the Hamiltonian in the form

$$H = \omega\bar{\Psi}\gamma_5\Psi, \tag{30}$$

with addition relation $\bar{\Psi}\Psi = 0$ and write the dynamic equation of motion in the form of a Poisson bracket

$$\dot{\Psi} = (\Psi, H). \tag{31}$$

The distinction with problem of an electron in the hydrogen atom lies in the fact that, in our case, the task is solved in terms of a function that describes the range of potential trajectories. These trajectories are shaped by the space-time generated by the central field. In our case, these are classical trajectories, but not without the possibility of space-time fluctuations that could alter the corresponding position of the planet. The problem of possible stable trajectories can be solved even at the expense of the quantization of the classical moment [33], and, thus, can be reduced to "quantum" analog.

Considering this larger invariant group will allow us to see, in a different light and understand why, that the Planck constant is not exactly the same for different cases. This is due primarily to the fact that the de Broglie presentation $\Psi \sim \exp i\frac{S}{h_{\text{eff}}}$, which defines wave properties, came to us from the understanding of the adiabatic invariant $\int pdq$, which presents the action S. For the "test part", which, in Kepler's problems, is a planet, the maximum momentum can be estimated as $p \sim Mc$, and the minimum excitation that can change position of the planet according to its size R. It follows that the real action takes on a very large value. The wave properties will manifest themselves, if the size of a cell in the phase space is of the order of the action itself, i.e., $h_{\text{eff}} = S$. There are two very important points here. The first is that, even for classical particles, there is Pauli's rule, according to which two classical particles of finite size

cannot be in the same state, since they cannot occupy the same spatial position, even if they have the same momentum, which is also unattainable due to fluctuations. Moreover, the symplectic structure of the mathematical description implies that the adiabatic invariant must be preserved, and cannot be equal to the usual, generally accepted Planck's quantum constant h, if we consider the corresponding values of the momentum and the coordinates of the test particle (planet). Accordingly, Planck's constant is contingent upon the dimensions and mass of the 'test particle, thus establishing the scale of the corresponding perturbations of the manifold that can alter the system's state.

Now, we can move to the quantum description of classical trajectories. We will try to present a "quantum version" of the solution to this problem. According to Dirac's approach, the transition from classical to quantum description is based on the replacement of classical Poisson brackets by commutation relations. It is the symplectic structure of the manifold that leads to the possibility of quantizing classical Hamiltonian systems [29, 34]. The quantum dynamical equation can, in our case, rewritten in the well-known form

$$ih_{\text{eff}}\dot{\Psi} = [\Psi, H], \tag{32}$$

where h_{eff} is the new effective "Plank constant.". The commutation conditions can be fully satisfied, when the function is represented in spinor form. We will try to, show that the energies of the planets are quantize according to the occupied orbits, and this can be shown in two ways. One is suggested in article [35], but we will continue to use the proposed approach with the geometrical description, when the obtained bispinor takes the form:

$$\begin{aligned}\Psi &= \sqrt{\frac{h_{\text{eff}}}{2}}\begin{pmatrix} u^+ + v \\ -u^+ + v\end{pmatrix},\\ \bar{\Psi} &= \sqrt{\frac{h_{\text{eff}}}{2}}(u - v^+, -u - v^+)\end{aligned} \tag{33}$$

but by giving a different physical meaning to the introduced quantity. Birth and annihilation operators of the corresponding components of the coordinates of the planet on the orbit, which can be numbered n. The Hamiltonian of the problem describing the motion of the corresponding planet in the corresponding elliptical orbit n can be obtained in the form

$$H = \omega\bar{\Psi}\gamma_5\Psi = \omega\Psi^+\Psi = h_{\text{eff}}\omega(u^+u + v^+v + 2). \tag{34}$$

Under symmetric conditions $\bar{\Psi}\Psi + \Psi\bar{\Psi} = \bar{h}((u^+u - -v^+v)) = 0$, we find that

$$H\,|\Psi\rangle = 2h_{\text{eff}}\omega(u^+u + 1)\,|\Psi\rangle = 2h_{\text{eff}}n\omega\,|\Psi\rangle. \tag{35}$$

If we take into account that the energy of the Keplerian motion is constant and equal to $E = -2\omega^2 m$, then we can obtain an expression for the energy levels for a planet with mass m in the corresponding orbit

$$E_n = -\frac{m(GMm)^2}{2h_{eff}^2 n^2} \tag{36}$$

around a centrally symmetric field. In this way, the proposed geometric representation of the "test particle" makes it possible to solve Kepler's problem both in the classical and in the quantum approaches.

The same can be done in the case of a hydrogen atom, if we assume the probability of the existence of electron orbits around the nucleus. Similarly, the entire mathematical procedure can be carried out for the classical hydrogen atom. In the non-relativistic case, a classical Hamiltonian for the hydrogen atom can be written in the well-known form

$$H = E(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} - \frac{Ze^2}{r}, \tag{37}$$

which will be used below for another representation in terms of spinors. Using geometric representations of the wave function at the level of finding solutions to the Dirac equation for a hydrogen atom, we can find all the necessary properties of a relativistic elementary particle. The exact solution of the Dirac equation in a warped space-time was proposed in [36]. In the simple approach proposed above for Kepler's problem, we can determine the expression for the energy levels for electrons with mass m on the corresponding orbit as

$$E_n = -\frac{m(Ze^2)^2}{2h^2n^2} \tag{38}$$

which completely to reproduce the well-known results of quantum mechanics, where h is the usual Planck constant. This means that, from the quantization of the orbits, it is possible to obtain the corresponding energy values of electrons of the hydrogen atom. Now,

the square of the wave function determines the probability of being in the corresponding orbit. It is not possible to determine at which, but such the interpretation also corresponds to our knowledge about the quantum behavior of the atom.

4. Discussion

Of course, the classical Hamiltonian for an electron in the hydrogen atom, as well as the Hamiltonian for Kepler's problem, could be written immediately using simple textbooks. But, for the consistency of the proposed approach, we have derived these expressions from the transfer rules for the corresponding geometric object describing the physical situation. This is done to show that the mathematical structure does not allow any other kind of Hamiltonian in the proposed consideration. The corresponding form of the Hamiltonian allows the representation of a manifold due to the characteristics of the "test particle" that scans this manifold. Therefore, it is possible to describe the dynamics of a physical system in terms of the geometric representation of a test particle and to study the dynamics of the space-time.

Indeed, the transformation that takes the Dirac equation to its canonical form as a wave function transfer rule does not depend on Planck's constant. In fact, this independence is a general fact. Thus, the normal representations (or canonical forms) of the Dirac equations are better representations for expressing the generalized de Broglie relations in a curved space-time. The presented results indicate that the Dirac equation in the geometric representation of general relativity is a transfer equation on an arbitrary manifold. As was previously discussed, the complete set of coordinate transformations associated with the structural equation exists only in the Clifford number representation of the wave function. The initial structural equation for the wave function is found to be identical to the Dirac equation. As was demonstrated by numerous authors, including [4] and [19], the solutions to this equation for definite integrals of motion are identical to those obtained in the spinor representation.

From all the above, it can be assumed that such a representation of the wave function contains both corpuscular and wave properties. It seems possible to attribute such properties primarily to the manifold on which the corresponding phenomenon is considered. The stationary states of the test particle correspond to the stationary states of the manifold. At the same time, it is not very important which quantum or classical interpretation we attribute to it. What is important is that this wave function describes a manifold, which means that the dual corpuscular and wave nature of the manifold is built into the behavior of the geometric representation. The true value of Planck's quantum constant is determined by scales of motion, since the dynamics of the "test particle" can only be affected by perturbations or fluctuations comparable to the size of the test particles themselves.

The articles [35, 37] showed that the orbits of planets and satellites around the large central mass in our solar system are quantized. Verification of the predictions of the Titius–Bode law for different Kepler's multi-planetary systems is being carried out. The word "quantized" is usually applied to the physics on the subatomic scale. According to the above results, the orbits in the gravitational system are quantized; that is, the distance, period, and velocity can only have certain discrete values. To describe this, it is necessary to use not only the features of the "test particle", but also those of the space-time in which its dynamics takes place.

The wave function of a"test particle" on the hypersurface of constant energy in the phase space in all cases can be written in the form of a de Broglie wave. Definition of the introduced coefficients follows from the fact that we are looking for a squared complex wave function which module that does not exceed unity. In addition, we need to have a representation of all possible states in the phase space, and this is possible, if the phase of this complex function is represented in this phase space. Such a function is action, the minimum of which determines the classical trajectory. Finally, the number of states in the phase space will depend on the size of the cell to which only one state can be attributed. The size of a unit cell in the phase space is determined by Planck's constant h. Then the general form of the wave function can be given as: $\Psi \sim \rho^{\frac{1}{2}} \exp(\frac{i}{h} S)$, where $S = \int (H(p,q)dt - pdq)$ action is written in terms of the coordinates q and impulses p. All the necessary attributes of such form of the wave function are present in the geometric representation [18, 19, 26]. In addition, only such representation contains all the necessary detailed definitions of the relationship be-

tween the energy and momentum in the relativity theory.

An estimation of the value of the effective "Planck constant" can be made as $h_{\text{eff}} = mcR$, which facilitates the application of this definition on various scales. For specific values of the electron mass and the size of the hydrogen atom, the effective constant $h_{\text{eff}} = h$ aligns with that introduced by Planck. Consequently, it can be concluded that the effective quantization constant is contingent on the scale of the objects under investigation. From whence, we can obtain the relation for effective Planck's constants $\frac{h_{\text{eff}}}{h} \sim \frac{R^4}{R^4_{\text{atom}}}$, which, for the values of the radius of the orbit R of the Earth and that of the electron R_{atom} in a hydrogen atom, gives $\frac{h_{\text{eff}}}{h} \sim 10^{80}$.

We can also estimate the value of the wavelength of a perturbation $\lambda_g = \frac{h_{\text{eff}}}{m_E c}$, analogous to the Compton length for a hydrogen atom $\lambda_c = \frac{h}{m_E c}$, and compare these two quantities. The ratio between the Planck constant for gravitational perturbations and the Compton length of a hydrogen atom can be estimated as $\frac{h_{\text{eff}}}{h} = \frac{\lambda_g m_E}{\lambda_c m_e}$. This value can be called the scale factor. If we take the wavelength of a gravitational perturbation of the order of an astronomical unit $\lambda_g \sim 149 \times 10^9$ m and the values of the masses of the electron $m_e \sim 9.1 \times 10^{-31}$ rg and the Earth, $m_E \sim \times 10^{24}$ kg, for the Compton wavelength $\lambda_c \sim 2.4 \times 10^{-12}$ m, we obtain $h_{\text{eff}} \sim 10^{78}$ h, which agrees with the estimates in the previous work [35, 38].

It is necessary to make an important remarks on the transport equation on an arbitrary manifold, since, in the general case, the connectivity can be represented also by the Clifford number, and individual components can and should have different response coefficients for fields of different natures. But, the mathematical structure of such a representation [17] is still far from complete and needs a more consistent development. Even the simplest implementation gives non-trivial results, because it contains more information about the possible nature of the manifold and the wave function of a "test particle". This requires a more careful explanation and, probably, will be done in the future.

This work was supported by the National Research Foundation of Ukraine through the project No. 2023.03/0165, "Quantum correlations of electromagnetic radiation".

1. J. Oppenheim. A postquantum theory of classical gravity. *Phys.l Rev. X* **13**, 041040 (2023).
2. B.J. Wolk. Quantum gravity through geometric algebra. *J. Phys. A: Math. Theor.* **57**, 015402 (2024).
3. M. Hatifi. Quantum gravity without metric quantization: From hidden variables to hidden spacetime curvatures. arXiv:2502.08421v1 [gr-qc] 12 Feb 2025.
4. D. Hestenes. Clifford algebra and the interpretation of quantum mechanics. In: *Clifford Algebras and Their Applications in Mathematical Physics.* Edited by J.S.R. Chisholm, A.K. Commons (Reidel, 1986).
5. D. Hestenes. Spacetime physics with geometric algebra. *American J. Phys.* **71**, 691 (2003).
6. B.F. Schutz. *Geometrical Methods of Mathematical Physics* (Cambridge University Press, 1982).
7. A. Harvey-Tremblay. The observer, defined as a measure space of halting programs. Is a complete and constructive formulation of physics. scholar.archive.org, (2022).
8. M. Novello. Spinor theory of gravity. arXiv.gr-qc/0609033 (gr-qc).
9. M. Novello, E. Bittencourt. Metric relativity and the dynamical bridge: Highlights of Riemannian geometry in physics. *Brazilian J. Phys.* **45**, 756 (2015).
10. B.I. Lev. Geometric interpretation of the origin of the universe. *J. Modern Phys.* **13**, 89 (2022).
11. B.I. Lev. Application of Clifford algebra to describe the early universe. *Mathematics* **12**, 3396 (2024).
12. C. Doran, A. Lasenby. *Geometric Algebra for Physicists* (Cambridge University Press, 2003).
13. M. Arminjon, F. Reifler. Equivalent forms of Dirac equations in curved spacetimes and generalized de Broglie relations. *Brazilian J. Phys.* **43**, 64 (2013).
14. G. Horton, C. Dewdney, A. Nesteruk. Time-like owns of energy-momentum and particle trajectories for the Klein–Gordon equation. *J. Phys. A: Math. Gen.* **33**, 7337 (2000).
15. A. Soiguine. *The Geometric Algebra Lift of Qubits and Beyond* (LAMBERT Academic Publishing, 2020).
16. I.M. Benn, R.W. Tucker. Clifford analysis of exterior forms and Fermi-Bose symmetry. *J. Phys. A: Math. Gen.* **16**, 4147 (1999).
17. B.I. Lev. A probable approach to the geometrization of interaction. *Mod. Phys. Lett. A* **3** (10), 1025 (1988), ib. **4**, erratum, (1989).
18. B.I. Lev. *Research and Applications Towards Mathematics and Computer Science.* Vol. 4, Chap. 11 [ISBN: 978-81-19491-72-8].
19. C. Daviau, J. Bertrand. *New Insights in the Standard Model of Quantum Physics in Clifford Algebra* HAL Id: hal-00907848 (2013).
20. A.D. Sakharov. Vacuum quantum fluctuations n curved space and the theory of gravitations. *Dok. Akad. Nauk. SSSR* **177**, 70 (1967) [Sov. Phys. Dokl. **12**, 1040 (1968)].
21. O. Klein. Generalization of Einstein principle of equivalence so as to embrace the field equations of gravitation. *Phys. Scr.* **9**, 69 (1974).

22. S.L. Adler. Einstein gravity as a symmetry-breaking effect in quantum field theory. *Rev. Mod. Phys.* **54**, 3 (1982).
23. P. Cameron. Impedance approach to the chiral anomaly (2014). rxiv.org/pdf/1402.0064vA.pdf.
24. P. Cameron. Generalized quantum impedances: A background independent model for the unstable particle spectrum (2012). http://dx.doi.org/10.13140/RG.2.2.25702.84800.
25. B.I. Lev. Geometrical presentation of elementary particle wave function. *Low Temp. Phys.* **48**, 938 (2022).
26. G. Kasanova. *Vector Algebra* (Presses Universitaires de France, 1976).
27. M.J. Hadley. Spin-1/2 in classical general relativity. *Class. Quantum Grav.* **17**, 4187 (2000).
28. A.G. Klein. Schrödinger inviolate: Neutron optical searches for violations of quantum mechanics. *Physics B* **151**, 44, (1988).
29. A.C. Bakai, Yu.G. Stepanovskij. *Adiabatic Invariant* (Naukova Dunka, 1981).
30. P. Nandi, F.G. Scholtz. The hidden Lorentz covariance of quantum mechanics. *Annals Phys.* **464**, 169643 (2024).
31. P. Kustaanheimo. Spinor regularization of the Kepler motion. *Ann. Univ. Turku. Ser. A* **73**, 1 (1964).
32. P. Kustaanheimo, E. Stiefel. Perturbation theory of Kepler motion based on spinor regularization. *J. Reine, Angew. Math.* **218**, 204 (1965).
33. B.E. Laurent. A variational principle and conservation theorems in connection with the generally relativistic Dirac equation. *Phys. Scripta* **16** (25), 263 (1960).
34. K. Bliokh. Geometric amplitude, adiabatic invariants, quantization, and strong stability of Hamiltonian systems. *J. Math. Phys.* **43** (1), 25 (2002).
35. R. Louise. Loi de titius-bode et formalisme ondulatoire. *Moon Planets* **26**, 389 (1982).
36. M.D. de Oliveira, Alexandre G.M. Schmidt. Exact solutions of Dirac equation on a static curved space-time. *Annals of Physics* **401**, 21 (2019).
37. J. Llibre, C. Pinol. A gravitational approach to the Titius–Bode law. *J.: Astronomical Journal* **93**, 1272 (1987).
38. C.X. Huang, G.A. Bakos. Testing the Titius–Bode law predictions for Kepler multiplanet systems. *MNRAS* **442**, 674 (2014).

Received 14.05.25

Б.І. Лев

АЛГЕБРА КЛІФОРДА
ЯК ШЛЯХ ДО КВАНТОВОЇ ГРАВІТАЦІЇ

У статті висувається нова гіпотеза щодо розв'язування проблеми квантування гравітації. Метою дослідження є демонстрація того, що геометричне представлення хвильової функції можна розглядати як характеристику просторово-часового многовиду. У цьому підході показано, що теорія Дірака для атома водню та динаміка Кеплера для планетної системи описують аналогічні явища у просторі-часі. Стани цих систем мають параметри, що відповідають дозволеним динамічним станам простору-часу, тим самим зберігаючи інформацію щодо корпускулярної та хвильової природи. Запропонований підхід проливає нове світло на потенційне вирішення проблем квантової гравітації.

Ключові слова: алгебра Кліфорда, хвильова функція, пробна частинка, просторово-часовий многовид.